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Exact dynamic stiffness matrix for a thin-walled beam-column of doubly asymmetric cross-section

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Abstract. Bernoulli-Euler beam theory is used to develop an exact dynamic stiffness matrix for the flexural-torsional coupled motion of a three-dimensional, axially loaded, thin-walled beam of doubly asymmetric cross-section. This is achieved through solution of the differential equations governing the motion of the beam including warping stiffness. The uniform distribution of mass in the member is also accounted for exactly, thus necessitating the solution of a transcendental eigenvalue problem. This is accomplished using the Wittrick-Williams algorithm. Finally, examples are given to confirm the accuracy of the theory presented, together with an assessment of the effects of axial load and loading eccentricity.

Keywords: coupled flexural-torsional motion; thin-walled beams; exact dynamic stiffness matrix; transcendental eigenvalue problem.

1. Introduction

Thin-walled beams are widely used in the civil, mechanical and aerospace industries. As such, they represent an important class of structural elements. However, in many applications their cross-sections are not doubly symmetric, with the result that their flexural and torsional motions become coupled in one or more planes. The resulting motion has been described in the literature using a wide variety of solution techniques with various levels of sophistication. In this short review, attention will be focused on those methods that have a direct bearing on the work that follows.

An early example of an exact dynamic stiffness matrix for a straight, uniform, bending-torsion beam element was developed by Hallauer and Liu (1982). This was achieved using Bernoulli-Euler-Saint Venant theory and restricting the motion to a single plane. Dokumaci (1987) subsequently used the same assumptions to determine the coupled bending-torsion natural frequencies of a cantilever beam. However, it was Bishop *et al.* (1989) who extended the foregoing theory to include the effect of warping stiffness and was able to demonstrate that its omission led to considerable errors in the coupled frequencies of open section beams. Over the same period, Friberg (1985) and Leung (1991) presented numerical procedures for developing an exact dynamic stiffness matrix of a thin-walled beam based on Vlasov beam theory.

Banerjee (1989), using the simplest theory and considering only a singly asymmetric cross-section,

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appears to have been the first to derive explicit expressions for the stiffness elements and later included the effect of shear deformation and rotary inertia (Banerjee and Williams 1992). In a later paper, Banerjee *et al*. (1996) formulated an exact dynamic stiffness matrix for a Bernoulli-Euler thin-walled beam including the effects of warping torsion, but the member was again restricted to planar motion. This restriction was later removed in a paper by Tanaka and Bercin (1999).

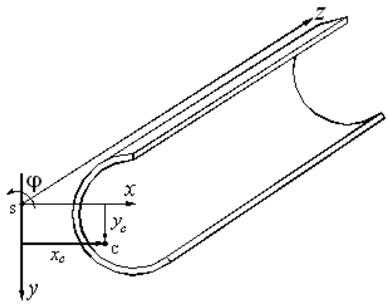
The papers mentioned so far do not allow for the effect of a static axial load in the member. This problem was addressed by Li *et al.* (2004a), who extended Tanaka and Bercin's work by including the effect of a static axial force and re-casting the solution in the form of a transfer matrix. Subsequently, Li *et al.* (2004b) extended this work to include the effects of rotary inertia and shear deflection, but limited the flexural motion to a single plane. However, both papers suffer from the inherent difficulty associated with the transfer matrix method, in that it becomes increasingly less well conditioned as the higher natural frequencies are sought. In addition, the Wittrick-Williams root finding algorithm cannot be applied in such cases, leading to the possibility that close or coincident natural frequencies could be missed.

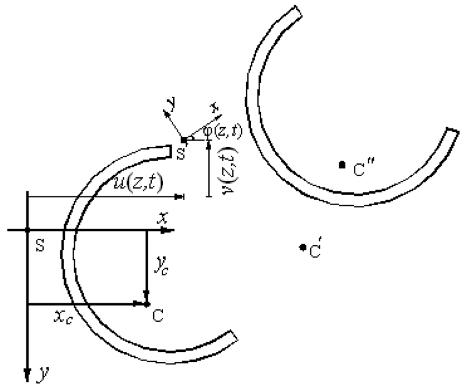
Recently, Rafezy and Howson (2006a) established a dynamic stiffness matrix for a three-dimensional shear-torsion beam with doubly asymmetric cross-section. Such beams have the unusual theoretical property that they allow for coupled shearing and torsional deformation, but not bending deformation. This formulation is relatively simple and can be used very efficiently in the approximate determination of the lower natural frequencies of three-dimensional, multi-storey framed structures (Rafezy and Howson 2003, 2008, 2009, Rafezy *et al.* 2007), including those that are doubly asymmetric on plan and which may contain step changes in member properties at one or more storey levels. The same authors also developed an exact dynamic stiffness matrix for the flexural motion of a three dimensional, bi material beam of doubly asymmetric cross-section (Rafezy and Howson 2006b). In this case, the beam consists of a thin walled outer layer that encloses and works compositely with its shear sensitive core material. The effect of warping stiffness was considered, but not the effect of axial force.

In this paper, the equations governing the flexural-torsional coupled motion of a three-dimensional, axially loaded, thin-walled beam of doubly asymmetric cross-section are developed in such a way as to allow for the effects of warping torsion and the presence of a static axial load. This leads to a twelfth order differential Equation whose solution is posed in the form of a dynamic stiffness matrix that can be used to create more complex structures. The uniform distribution of mass in the member is accounted for exactly and thus necessitates the solution of a transcendental eigenvalue problem. This is accomplished using the Wittrick-Williams algorithm, which enables the required natural frequencies to be converged upon to any required accuracy with the certain knowledge that none have been missed.

2. Theory

Fig. 1(a) shows a straight and uniform, thin-walled beam of length L with a doubly asymmetric cross-section. The x and y axes are aligned with the principle axes of inertia and have their origin at the shear centre, denoted by S. The z axis runs through S and coincides with the elastic axis (i.e., the loci of points joining the shear centre of each cross-section). In similar fashion, the centroid of the cross-section, denoted by C, lies on the mass axis (i.e., the loci of points joining the centroid of each cross-section), which corresponds to the line of points (x_c, y_c). The bending translation in the x





(a) (b)

Fig. 1 (a) co-ordinate system and notation for a three-dimensional thin-walled beam of length L with doubly asymmetric cross-section, (b) typical displacement configuration of a cross-section

and y directions, respectively, and the torsional rotation of the shear centre about the z axis are denoted by $u(z, t)$, $v(z, t)$ and $\varphi(z, t)$, where z and t denote distance from the origin and time. A constant axial force P, that can be compressive (positive) or tensile (negative), is assumed to act through the centroid of the cross-section.

During vibration, the displacement of the shear and mass centres at any time t in the x-y plane can be determined as the result of a pure translation followed by a pure rotation, see Fig. 1(b). During the translation phase the shear centre S moves to S' and the mass centre C moves to C'. During the rotation phase, the mass centre rotates about S', thus moving from C' to C''. The resulting translations, (u_c, v_c) of the mass centre in the x and y directions, are

$$u_c(z,t) = u(z,t)+y_c\varphi(z,t)\,, \quad v_c(z,t) = v(z,t)-x_c\varphi(z,t) \tag{1a,b}$$

More generally, it is clear that the displacements of a typical point (x_i, y_i) on the cross-section are given by Eq. (1) when $c = i$.

The coupled equations of motion that stem from the three orthogonal planes can now be developed from Figs. 1 and 2. In the x-z and y-z planes, this is achieved by equating the resultant shear force on the element to the corresponding mass accelerations. In the x-y plane, the resultant torsional moment about the shear centre S is equated to the sum of the moments of the mass accelerations about the same point. This yields

$$\frac{\partial Q_x(z,t)}{\partial z} = m\left(\frac{\partial^2 u(z,t)}{\partial t^2}+y_c\frac{\partial^2 \varphi(z,t)}{\partial t^2}\right) \tag{2a}$$

$$\frac{\partial Q_y(z,t)}{\partial z} = m\left(\frac{\partial^2 v(z,t)}{\partial t^2}-x_c\frac{\partial^2 \varphi(z,t)}{\partial t^2}\right) \tag{2b}$$

$$\frac{\partial T(z,t)}{\partial z} = m\left(r_m^2\frac{\partial^2 \varphi(z,t)}{\partial z^2}+y_c\frac{\partial^2 u(z,t)}{\partial t^2}-x_c\frac{\partial^2 v(z,t)}{\partial t^2}\right) \tag{2c}$$

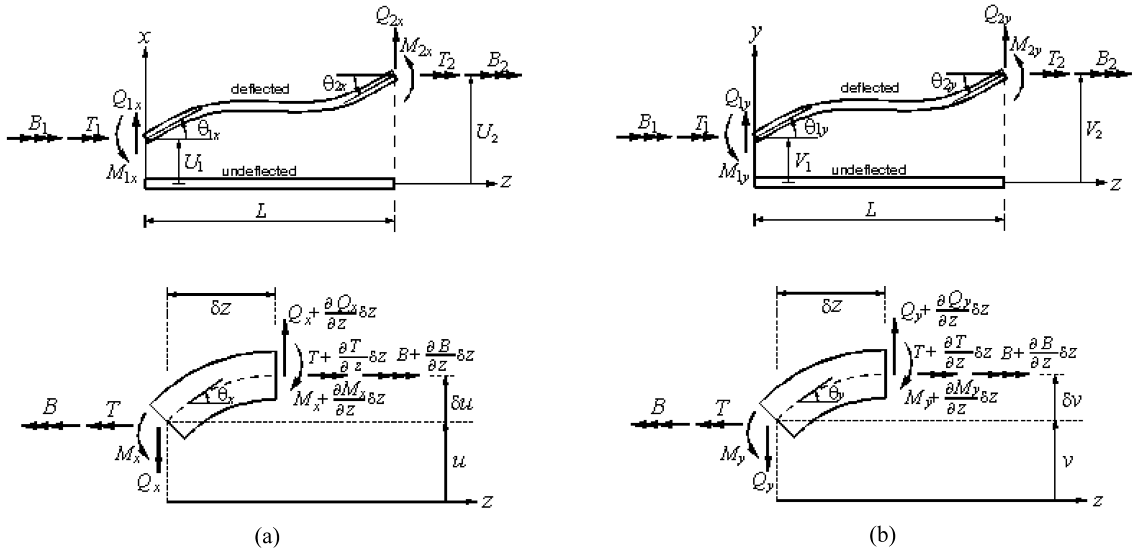


Fig. 2 Positive sign convention for forces and displacements (a) member and element convention in the x-z plane, (b) member and element convention in the y-z plane

where m is the mass/unit length of the member and r_m is the polar mass radius of gyration of the cross-section about the z-axis. The shear forces $Q_x(z,t)$, $Q_y(z,t)$ and the torsional moment $T(z,t)$ about S can be obtained from the appropriate stress/strain relationships and the sign conventions of Fig. 2 as

$$Q_x(z,t) = -EI_x\frac{\partial^3 u(z,t)}{\partial z^3} - P\left(\frac{\partial u(z,t)}{\partial z} + y_c\frac{\partial \varphi(z,t)}{\partial z}\right) \tag{3a}$$

$$Q_y(z,t) = -EI_y\frac{\partial^3 v(z,t)}{\partial z^3} - P\left(\frac{\partial v(z,t)}{\partial z} - x_c\frac{\partial \varphi(z,t)}{\partial z}\right) \tag{3b}$$

$$T(z,t) = -EI_w\frac{\partial^3 \varphi(z,t)}{\partial z^3} + GJ\frac{\partial \varphi(z,t)}{\partial z} - P\left(r_m^2\frac{\partial \varphi(z,t)}{\partial z} + y_c\frac{\partial u(z,t)}{\partial z} - x_c\frac{\partial v(z,t)}{\partial z}\right) \tag{3c}$$

in which EI_x and EI_y are the flexural rigidity of the thin-walled section in the x-z and y-z planes, respectively, P is a constant compressive axial load and GJ and EI_w are the Saint-Venant and warping torsion rigidity about S, where I_w is the warping moment of inertia or warping constant.

Substituting Eq. (3) into Eq. (2) yields the required equations of motion as

$$EI_x\frac{\partial^4 u(z,t)}{\partial z^4} + P\left(\frac{\partial^2 u(z,t)}{\partial z^2} + y_c\frac{\partial^2 \varphi(z,t)}{\partial z^2}\right) + m\frac{\partial^2 u(z,t)}{\partial t^2} + my_c\frac{\partial^2 \varphi(z,t)}{\partial t^2} = 0 \tag{4a}$$

$$EI_y\frac{\partial^4 v(z,t)}{\partial z^4} + P\left(\frac{\partial^2 v(z,t)}{\partial z^2} - x_c\frac{\partial^2 \varphi(z,t)}{\partial z^2}\right) + m\frac{\partial^2 v(z,t)}{\partial t^2} - mx_c\frac{\partial^2 \varphi(z,t)}{\partial t^2} = 0 \tag{4b}$$

$$EI_w\frac{\partial^4\varphi(z,t)}{\partial z^4}-GJ\frac{\partial^2\varphi(z,t)}{\partial z^2}+P\left(r_m^2\frac{\partial^2\varphi(z,t)}{\partial z^2}+y_c\frac{\partial^2 u(z,t)}{\partial z^2}-x_c\frac{\partial^2 v(z,t)}{\partial z^2}\right)$$
$$+my_c\frac{\partial^2 u(z,t)}{\partial t^2}-mx_c\frac{\partial^2 v(z,t)}{\partial t^2}+mr_m^2\frac{\partial^2\varphi(z,t)}{\partial t^2}=0 \tag{4c}$$

Assuming harmonic motion, the instantaneous displacements can be written as

$$u(z,t)=U(z)\sin\omega t \quad v(z,t)=V(z)\sin\omega t \quad \varphi(z,t)=\Phi(z)\sin\omega t \tag{5a,b,c}$$

where $U(z), V(z)$ and $\Phi(z)$ are the amplitudes of the sinusoidally varying displacements.

Substituting Eq. (5) into Eq. (4) and re-writing in non-dimensional form gives

$$U''''(\xi)-\alpha_x^2U''(\xi)+y_c\alpha_x^2\Phi''(\xi)-\beta_x^2\omega^2U(\xi)-y_c\omega^2\beta_x^2\Phi(\xi)=0 \tag{6a}$$

$$V''''(\xi)+\alpha_y^2V''(\xi)-x_c\alpha_y^2\Phi''(\xi)-\beta_y^2\omega^2V(\xi)+x_c\omega^2\beta_y^2\Phi(\xi)=0 \tag{6b}$$

$$\Phi''''(\xi)+(\alpha_\varphi^2-\gamma_\varphi^2)\Phi''(\xi)+y_c\frac{\alpha_x^2}{\gamma_x^2}U''(\xi)-x_c\frac{\alpha_y^2}{\gamma_y^2}V''(\xi)-\omega^2\beta_\varphi^2\Phi(\xi)-y_c\omega^2\frac{\beta_x^2}{\gamma_x^2}U(\xi)+x_c\omega^2\frac{\beta_y^2}{\gamma_y^2}V(\xi)=0 \tag{6c}$$

where

$$\alpha_x^2=\frac{PL^2}{EI_x},\quad \alpha_y^2=\frac{PL^2}{EI_y}\quad\text{and}\quad \alpha_\varphi^2=\frac{Pr_m^2}{EI_w}L^2 \tag{7a,b,c}$$

$$\beta_x^2=\frac{mL^4}{EI_x},\quad \beta_y^2=\frac{mL^4}{EI_y}\quad\text{and}\quad \beta_\varphi^2=r_m^2\frac{mL^4}{EI_w} \tag{7d,e,f}$$

$$\gamma_x^2=\frac{EI_w}{EI_x},\quad \gamma_y^2=\frac{EI_w}{EI_y},\quad \gamma_\varphi^2=\frac{GJ}{EI_y}L^2\quad\text{and}\quad \xi=\frac{z}{L} \tag{7g,h,i,j}$$

Eqs. (6) can be re-written in the following matrix form

$$\begin{bmatrix} D^4+\alpha_x^2D^2-\omega^2\beta_x^2 & 0 & y_c\alpha_x^2D^2-y_c\omega^2\beta_x^2 \\ 0 & D^4+\alpha_y^2D^2-\omega^2\beta_y^2 & -x_c\alpha_y^2D^2+x_c\omega^2\beta_y^2 \\ y_c\frac{\alpha_x^2}{\gamma_x^2}D^2-y_c\omega^2\frac{\beta_x^2}{\gamma_x^2} & -x_c\frac{\alpha_y^2}{\gamma_y^2}D^2+x_c\omega^2\frac{\beta_y^2}{\gamma_y^2} & D^4+(\alpha_\varphi^2-\gamma_\varphi^2)D^2-\omega^2\beta_\varphi^2 \end{bmatrix}\begin{bmatrix} U(\xi) \\ V(\xi) \\ \Phi(\xi) \end{bmatrix}=\mathbf{0} \tag{8}$$

in which $D=d/d\xi$.

Eq. (8) can be combined into one equation by eliminating either U, V or Φ to give the twelfth-order differential equation

$$\begin{vmatrix} D^4+\alpha_x^2D^2-\omega^2\beta_x^2 & 0 & y_c\alpha_x^2D^2-y_c\omega^2\beta_x^2 \\ 0 & D^4+\alpha_y^2D^2-\omega^2\beta_y^2 & -x_c\alpha_y^2D^2+x_c\omega^2\beta_y^2 \\ y_c\frac{\alpha_x^2}{\gamma_x^2}D^2-y_c\omega^2\frac{\beta_x^2}{\gamma_x^2} & -x_c\frac{\alpha_y^2}{\gamma_y^2}D^2+x_c\omega^2\frac{\beta_y^2}{\gamma_y^2} & D^4+(\alpha_\varphi^2-\gamma_\varphi^2)D^2-\omega^2\beta_\varphi^2 \end{vmatrix}W(\xi)=0 \tag{9}$$

where $W=U$, V or Φ.

The solution of Eq. (9) is found by substituting the trial solution $W(\xi) = e^{a\xi}$ to yield the characteristic equation

$$\begin{vmatrix} \tau^2+\alpha_x^2\tau-\omega^2\beta_x^2 & 0 & y_c\alpha_x^2\tau-y_c\omega^2\beta_x^2 \\ 0 & \tau^2+\alpha_y^2\tau-\omega^2\beta_y^2 & -x_c\alpha_y^2\tau+x_c\omega^2\beta_y^2 \\ y_c\dfrac{\alpha_x^2}{\gamma_x^2}\tau-y_c\omega^2\dfrac{\beta_x^2}{\gamma_x^2} & -x_c\dfrac{\alpha_y^2}{\gamma_y^2}\tau+x_c\omega^2\dfrac{\beta_y^2}{\gamma_y^2} & \tau^2+(\alpha_\varphi^2-\gamma_\phi^2)\tau-\omega^2\beta_\varphi^2 \end{vmatrix} W(\xi)=0 \tag{10}$$

where $\tau = a^2$.

Eq. (10) is a sixth order Equation in τ and it can be proven (Rafezy and Howson 2006b, Appendix A) that it always has three negative and three positive real roots. Let these six roots be $\tau_1, \tau_2, \tau_3, -\tau_4, -\tau_5$ and $-\tau_6$, where τ_j $(j = 1, 6)$ are all real and positive. Therefore the twelve roots of Eq. (10) can be obtained as

$$\alpha,-\alpha \quad \beta,-\beta \quad \gamma,-\gamma \quad \mathrm{i}\delta,-\mathrm{i}\delta \quad \mathrm{i}\eta,-\mathrm{i}\eta \quad \mathrm{i}\mu,-\mathrm{i}\mu \tag{11a}$$

where

$$\alpha=\sqrt{\tau_1}, \quad \beta=\sqrt{\tau_2}, \quad \gamma=\sqrt{\tau_3}, \quad \delta=\sqrt{\tau_4}, \quad \eta=\sqrt{\tau_5} \quad \text{and} \quad \mu=\sqrt{\tau_6} \tag{11b}$$

and $\mathrm{i} = \sqrt{-1}$.

It follows that the solution of Eq. (9) is of the form

$$W(\xi) = C_1\cosh\alpha\xi + C_2\sinh\alpha\xi + C_3\cosh\beta\xi + C_4\sinh\beta\xi + C_5\cosh\gamma\xi + C_6\sinh\gamma\xi + C_7\cos\delta\xi + C_8\sin\delta\xi + C_9\cos\eta\xi + C_{10}\sin\eta\xi + C_{11}\cos\mu\xi + C_{12}\sin\mu\xi \tag{12}$$

Eq. (12) represents the solution for $U(\xi), V(\xi)$ and $\Phi(\xi)$, since they are related via Eq. (9). Hence they can be written individually as

$$U(\xi) = C_1^u\cosh\alpha\xi + C_2^u\sinh\alpha\xi + C_3^u\cosh\beta\xi + C_4^u\sinh\beta\xi + C_5^u\cosh\gamma\xi + C_6^u\sinh\gamma\xi + C_7^u\cos\delta\xi + C_8^u\sin\delta\xi + C_9^u\cos\eta\xi + C_{10}^u\sin\eta\xi + C_{11}^u\cos\mu\xi + C_{12}^u\sin\mu\xi \tag{13a}$$

$$V(\xi) = C_1^v\cosh\alpha\xi + C_2^v\sinh\alpha\xi + C_3^v\cosh\beta\xi + C_4^v\sinh\beta\xi + C_5^v\cosh\gamma\xi + C_6^v\sinh\gamma\xi + C_7^v\cos\delta\xi + C_8^v\sin\delta\xi + C_9^v\cos\eta\xi + C_{10}^v\sin\eta\xi + C_{11}^v\cos\mu\xi + C_{12}^v\sin\mu\xi \tag{13b}$$

$$\Phi(\xi) = C_1\cosh\alpha\xi + C_2\sinh\alpha\xi + C_3\cosh\beta\xi + C_4\sinh\beta\xi + C_5\cosh\gamma\xi + C_6\sinh\gamma\xi + C_7\cos\delta\xi + C_8\sin\delta\xi + C_9\cos\eta\xi + C_{10}\sin\eta\xi + C_{11}\cos\mu\xi + C_{12}\sin\mu\xi \tag{13c}$$

The relationship between the constants C_j^u, C_j^v and C_j $(j = 1, 12)$ also follows from Eq. (9) as

$$C_{2j-1}^u = t_j^u C_{2j-1} \quad \text{and} \quad C_{2j}^u = t_j^u C_{2j} \quad (j = 1, 6) \tag{14a,b}$$

$$C_{2j-1}^v = t_j^v C_{2j-1} \quad \text{and} \quad C_{2j}^v = t_j^v C_{2j} \quad (j = 1, 6) \tag{14c,d}$$

where

$$t_j^u = \frac{-y_c\alpha_x^2\tau_j + y_c\omega^2\beta_x^2}{\tau_j^2 + \alpha_x^2\tau_j - \omega^2\beta_x^2} \quad (j=1,2,3), \quad t_j^u = \frac{y_c\alpha_x^2\tau_j + y_c\omega^2\beta_x^2}{\tau_j^2 - \alpha_x^2\tau_j - \omega^2\beta_x^2} \quad (j=4,5,6) \tag{15a,b}$$

$$t_j^v = \frac{x_c\alpha_y^2\tau_j - x_c\omega^2\beta_y^2}{\tau_j^2 + \alpha_y^2\tau_j - \omega^2\beta_y^2} \quad (j=1,2,3), \quad t_j^v = \frac{-x_c\alpha_y^2\tau_j - x_c\omega^2\beta_y^2}{\tau_j^2 - \alpha_y^2\tau_j - \omega^2\beta_y^2} \quad (j=4,5,6) \tag{15c,d}$$

Following the sign convention of Figs. 2(a) and 2(b), expressions for the bending rotations $\theta_x(\xi)$, $\theta_y(\xi)$ and the gradient of the twist $\Phi'(\xi)$ are easily established as

$$\theta_x(\xi) = \frac{1}{L}\frac{dU(\xi)}{d\xi}, \; \theta_y(\xi) = \frac{1}{L}\frac{dV(\xi)}{d\xi} \text{ and } \Phi'(\xi) = \frac{1}{L}\frac{d\Phi(\xi)}{d\xi} \tag{16a,b,c}$$

The corresponding bending moments $M_x(\xi), M_y(\xi)$ and the bi-moment $B(\xi)$ are likewise easily determined from the appropriate stress/strain relationships as

$$M_x(\xi) = \frac{-EI_x}{L^2}\frac{d^2U(\xi)}{d\xi^2}, \; M_y(\xi) = \frac{-EI_y}{L^2}\frac{d^2V(\xi)}{d\xi^2} \quad \text{and} \quad B(\xi) = \frac{-EI_w}{L^2}\frac{d^2\Phi(\xi)}{d\xi^2} \tag{17a,b,c}$$

Substituting Eqs. (5), (7j) and (13) into Eq. (3) yields the equations for the lateral shear forces and torsional moment as

$$Q_x(\xi) = \frac{-EI_x}{L^3}\frac{d^3U(\xi)}{d\xi^3} - \frac{P}{L}\left(\frac{dU(\xi)}{d\xi} + y_c\frac{d\Phi(\xi)}{d\xi}\right) \tag{17d}$$

$$Q_y(\xi) = \frac{-EI_y}{L^3}\frac{d^3V(\xi)}{d\xi^3} - \frac{P}{L}\left(\frac{dV(\xi)}{d\xi} - x_c\frac{d\Phi(\xi)}{d\xi}\right) \tag{17e}$$

$$T(\xi) = \frac{-EI_w}{L^3}\frac{d^3\Phi(\xi)}{d\xi^3} + \frac{GJ}{L}\frac{d\Phi(\xi)}{d\xi} - \frac{P}{L}\left(r_m^2\frac{d\Phi(\xi)}{d\xi} + y_c\frac{dU(\xi)}{d\xi} - x_c\frac{dV(\xi)}{d\xi}\right) \tag{17f}$$

The nodal displacements and forces can now be defined in the member co-ordinate system of Figs. 2(a) and 2(b), as follows

$$\text{At} \quad \xi=0 \quad U=U_1,\ \theta_x=\theta_{1x},\ V=V_1,\ \theta_y=\theta_{1y},\ \Phi=\Phi_1,\ \Phi'=\Phi_1' \tag{18a}$$

$$\text{At} \quad \xi=1 \quad U=U_2,\ \theta_x=\theta_{2x},\ V=V_2,\ \theta_y=\theta_{2y},\ \Phi=\Phi_2,\ \Phi'=\Phi_2' \tag{18b}$$

$$\text{At} \quad \xi=0 \quad Q_x=-Q_{1x},\ M_x=M_{1x},\ Q_y=-Q_{1y},\ M_y=M_{1y},\ T=-T_1,\ B=B_1 \tag{18c}$$

$$\text{At} \quad \xi=1 \quad Q_x=Q_{2x},\ M_x=-M_{2x},\ Q_y=Q_{2y},\ M_y=-M_{2y},\ T=T_2,\ B=-B_2 \tag{18d}$$

Then the nodal displacements can be determined from Eqs. (13) and (16) as

$$\begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{E}_1\mathbf{R}_1 & \mathbf{E}_2\mathbf{R}_2 \\ \mathbf{E}_1\mathbf{C}_h & \mathbf{E}_2\mathbf{C} & \mathbf{E}_1\mathbf{S}_h & \mathbf{E}_2\mathbf{S} \\ \mathbf{E}_1\mathbf{R}_1\mathbf{S}_h & -\mathbf{E}_2\mathbf{R}_2\mathbf{S} & \mathbf{E}_1\mathbf{R}_1\mathbf{C}_h & \mathbf{E}_2\mathbf{R}_2\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{C}_o \\ \mathbf{C}_e \end{bmatrix} \tag{19}$$

where

$$\mathbf{d}_1 = \begin{bmatrix} U_1 \\ V_1 \\ \Phi_1 \\ \theta_{1x} \\ \theta_{1y} \\ \Phi_1' \end{bmatrix}, \ \mathbf{d}_2 = \begin{bmatrix} U_2 \\ V_2 \\ \Phi_2 \\ \theta_{2x} \\ \theta_{2y} \\ \Phi_2' \end{bmatrix}, \mathbf{C}_o = \begin{bmatrix} C_1 \\ C_3 \\ C_5 \\ C_7 \\ C_9 \\ C_{11} \end{bmatrix}, \mathbf{C}_e = \begin{bmatrix} C_2 \\ C_4 \\ C_6 \\ C_8 \\ C_{10} \\ C_{12} \end{bmatrix}, \ \mathbf{E}_1 = \begin{bmatrix} t_1^u & t_2^u & t_3^u \\ t_1^v & t_2^v & t_3^v \\ 1 & 1 & 1 \end{bmatrix}, \ \mathbf{E}_2 = \begin{bmatrix} t_4^u & t_5^u & t_6^u \\ t_4^v & t_5^v & t_6^v \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{R}_1 = \frac{1}{L}\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \quad \mathbf{R}_2 = \frac{1}{L}\begin{bmatrix} \delta & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \mu \end{bmatrix} \quad \mathbf{C}_h = \begin{bmatrix} \cosh\alpha & 0 & 0 \\ 0 & \cosh\beta & 0 \\ 0 & 0 & \cosh\gamma \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} \cos\delta & 0 & 0 \\ 0 & \cos\eta & 0 \\ 0 & 0 & \cos\mu \end{bmatrix}, \ \mathbf{S}_h = \begin{bmatrix} \sinh\alpha & 0 & 0 \\ 0 & \sinh\beta & 0 \\ 0 & 0 & \sinh\gamma \end{bmatrix}, \ \mathbf{S} = \begin{bmatrix} \sin\delta & 0 & 0 \\ 0 & \sin\eta & 0 \\ 0 & 0 & \sin\mu \end{bmatrix} \tag{20}$$

Hence the vector of constants $[\mathbf{C}_o \ \mathbf{C}_e]^T$ can be determined from Eq. (19) as

$$\begin{bmatrix} \mathbf{C}_o \\ \mathbf{C}_e \end{bmatrix} = \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{E}_1\mathbf{R}_1 & \mathbf{E}_2\mathbf{R}_2 \\ \mathbf{E}_1\mathbf{C}_h & \mathbf{E}_2\mathbf{C} & \mathbf{E}_1\mathbf{S}_h & \mathbf{E}_2\mathbf{S} \\ \mathbf{E}_1\mathbf{R}_1\mathbf{S}_h & -\mathbf{E}_2\mathbf{R}_2\mathbf{S} & \mathbf{E}_1\mathbf{R}_1\mathbf{C}_h & \mathbf{E}_2\mathbf{R}_2\mathbf{C} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix} \tag{21}$$

In similar fashion the vector of nodal forces can be determined from Eqs. (17) and (18) as

$$\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{M}_1 & \mathbf{M}_2 & \mathbf{0} & \mathbf{0} \\ -\mathbf{Q}_1\mathbf{S}_h & \mathbf{Q}_2\mathbf{S} & -\mathbf{Q}_1\mathbf{C}_h & -\mathbf{Q}_2\mathbf{C} \\ -\mathbf{M}_1\mathbf{C}_h & -\mathbf{M}_2\mathbf{C} & -\mathbf{M}_1\mathbf{S}_h & -\mathbf{M}_2\mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{C}_o \\ \mathbf{C}_e \end{bmatrix} \tag{22}$$

where

$$\mathbf{p}_1 = \begin{bmatrix} Q_{1x} \\ Q_{1y} \\ T_1 \\ M_{1x} \\ M_{1y} \\ B_1 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} Q_{2x} \\ Q_{2y} \\ T_2 \\ M_{2x} \\ M_{2y} \\ B_2 \end{bmatrix}, \tag{23a,b}$$

$$\mathbf{Q}_1 = \begin{bmatrix} t_1^u(\alpha^3 B_x + \alpha p) + y_c \alpha p & t_2^u(\beta^3 B_x + \beta p) + y_c \beta p & t_3^u(\gamma^3 B_x + \gamma p) + y_c \gamma p \\ t_1^v(\alpha^3 B_y + \alpha p) - x_c \alpha p & t_2^v(\beta^3 B_y + \beta p) - x_c \beta p & t_3^v(\gamma^3 B_y + \gamma p) - x_c \gamma p \\ t_1^u y_c \alpha p - t_1^v x_c \alpha p + \alpha p r_m^2 + \alpha^3 E_o - \alpha F_o & t_2^u y_c \beta p - t_2^v x_c \beta p + \beta p r_m^2 + \beta^3 E_o - \beta F_o & t_3^u y_c \gamma p - t_3^v x_c \gamma p + \gamma p r_m^2 + \gamma^3 E_o - \gamma F_o \end{bmatrix}$$

$$\mathbf{Q}_2 = \begin{bmatrix} t_4^u(-\delta^3 B_x + \delta p) + y_c \delta p & t_5^u(-\eta^3 B_x + \eta p) + y_c \eta p & t_6^u(-\mu^3 B_x + \mu p) + y_c \mu p \\ t_4^v(-\delta^3 B_y + \delta p) - x_c \delta p & t_5^v(-\eta^3 B_y + \eta p) - x_c \eta p & t_6^v(-\mu^3 B_y + \mu p) - x_c \mu p \\ t_4^u y_c \delta p - t_4^v x_c \delta p + \delta p r_m^2 - \delta^3 E_o - \delta F_o & t_5^u y_c \eta p - t_5^v x_c \eta p + \eta p r_m^2 - \eta^3 E_o - \eta F_o & t_6^u y_c \mu p - t_6^v x_c \mu p + \mu p r_m^2 - \mu^3 E_o - \mu F_o \end{bmatrix}$$

$$\mathbf{M}_1 = \begin{bmatrix} -t_1^u(\alpha^2 A_x) & -t_2^u(\beta^2 A_x) & -t_3^u(\gamma^2 A_x) \\ -t_1^v(\alpha^2 A_y) & -t_2^v(\beta^2 A_y) & -t_3^v(\gamma^2 A_y) \\ -\alpha^2 D_o & -\beta^2 D_o & -\gamma^2 D_o \end{bmatrix}, \quad \mathbf{M}_2 = \begin{bmatrix} t_4^u(\delta^2 A_x) & t_5^u(\eta^2 A_x) & t_6^u(\mu^2 A_x) \\ t_4^v(\delta^2 A_y) & t_5^v(\eta^2 A_y) & t_6^v(\mu^2 A_y) \\ \delta^2 D_o & \eta^2 D_o & \mu^2 D_o \end{bmatrix} \quad \text{(23c,d,e,f)}$$

where

$$A_x = \frac{EI_x}{L^2}, \quad A_y = \frac{EI_y}{L^2}, \quad B_x = \frac{EI_x}{L^3}, \quad B_y = \frac{EI_y}{L^3} \quad \text{(24a,b,c,d)}$$

$$p = \frac{P}{L}, \quad D_o = \frac{EI_w}{L^2}, \quad E_o = \frac{EI_w}{L^3}, \quad F_o = \frac{GJ}{L} \quad \text{(24e,f,g,h,i)}$$

Thus the required stiffness matrix can be developed by substituting Eq. (21) into Eq. (22) to give

$$\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{M}_1 & \mathbf{M}_2 & \mathbf{0} & \mathbf{0} \\ -\mathbf{Q}_1\mathbf{S}_h & \mathbf{Q}_2\mathbf{S} & -\mathbf{Q}_1\mathbf{C}_h & -\mathbf{Q}_2\mathbf{C} \\ -\mathbf{M}_1\mathbf{C}_h & -\mathbf{M}_2\mathbf{C} & -\mathbf{M}_1\mathbf{S}_h & -\mathbf{M}_2\mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{E}_1\mathbf{R}_1 & \mathbf{E}_2\mathbf{R}_2 \\ \mathbf{E}_1\mathbf{C}_h & \mathbf{E}_2\mathbf{C} & \mathbf{E}_1\mathbf{S}_h & \mathbf{E}_2\mathbf{S} \\ \mathbf{E}_1\mathbf{R}_1\mathbf{S}_h & -\mathbf{E}_2\mathbf{R}_2\mathbf{S} & \mathbf{E}_1\mathbf{R}_1\mathbf{C}_h & \mathbf{E}_2\mathbf{R}_2\mathbf{C} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix} \quad (25)$$

or

$$\mathbf{p} = \mathbf{kd} \quad (26)$$

The stiffness relationship of Eq. (26) is general and can be used in the normal way to assemble more complex forms. The required natural frequencies of the resulting structure are determined by evaluating its overall dynamic stiffness matrix at a trial frequency ω^* and using the Wittrick-Williams algorithm to establish how many natural frequencies have been exceeded by ω^*. This clearly provides the basis for a convergence procedure that can yield the required natural frequencies to any desired accuracy. The corresponding mode shapes can then be recovered by any appropriate method (Howson 1979).

3. Wittrick-Williams algorithm

The Wittrick-Williams algorithm (Williams and Wittrick 1970, Wittrick and Williams 1971) has been available for over thirty years and has received considerable attention. The algorithm states that

$$J = J_0 + s\{\mathbf{K}\} \tag{27}$$

where J is the number of natural frequencies of the structure exceeded by some trial frequency, ω^*, J_o is the number of natural frequencies that would still be exceeded if all members were clamped at their ends so as to make $\mathbf{D} = \mathbf{0}$ and $s\{\mathbf{K}\}$ is the sign count of the dynamic structure stiffness matrix $\mathbf{K}$. $s\{\mathbf{K}\}$ is defined in reference (Wittrick and Williams 1971) and is equal to the number of negative elements on the leading diagonal of the upper triangular matrix obtained from $\mathbf{K}$, when $\omega = \omega^*$, by the standard form of Gauss elimination without row interchanges.

From the definition of J_o, it can be seen that

$$J_o = \sum J_m \tag{28}$$

where J_m is the number of natural frequencies of a member, with its ends clamped, which have been exceeded by ω^*, and the summation extends over all members of the structure. In some cases it is possible to determine the value of J_m for an individual member symbolically using a direct approach (Howson 1979) that gives an analytical expression for J_m. However this is impractical in the present case due to the algebraic complexity. Instead, J_m is evaluated using an argument based on Eq. (27) that applies the Wittrick-Williams algorithm (Wittrick and Williams 1971) in reverse. The procedure corresponds to the one originally proposed by Howson and Williams (1973) and is described as follows.

Consider an element that has been isolated from its neighbours by clamping its ends. Treating this members as a complete structure, it is evident that the required value of J_m could be evaluated if its natural frequencies were known. Unfortunately this simple structure can rarely be solved easily. We therefore seek to establish a different set of boundary conditions (other than clamped-clamped) which admit a simple solution from which the solution for the clamped-clamped case can be deduced. This is most easily achieved in the present case by imposing what will be defined as simply supported boundary conditions, i.e., at

$$\xi = 0 \quad \text{and} \quad \xi = 1, \; U = V = \Phi = 0 \quad \text{and} \quad M_x = M_y = B = 0 \tag{29}$$

The stiffness relationship for a single member subject to these boundary conditions can then be obtained by deleting appropriate rows and columns from Eq. (26) to leave

$$\begin{bmatrix} M_{1x} \\ M_{1y} \\ B_1 \\ M_{2x} \\ M_{2y} \\ B_2 \end{bmatrix} = \begin{bmatrix} K_{4,4} & K_{4,5} & K_{4,6} & K_{4,10} & K_{4,11} & K_{4,12} \\ K_{5,4} & K_{5,5} & K_{5,6} & K_{5,10} & K_{5,11} & K_{5,12} \\ K_{6,4} & K_{6,5} & K_{6,6} & K_{6,10} & K_{6,11} & K_{6,12} \\ K_{10,4} & K_{10,5} & K_{10,6} & K_{10,10} & K_{10,11} & K_{10,12} \\ K_{11,4} & K_{11,5} & K_{11,6} & K_{11,10} & K_{11,11} & K_{11,12} \\ K_{12,4} & K_{12,5} & K_{12,6} & K_{12,10} & K_{12,11} & K_{12,12} \end{bmatrix} \begin{bmatrix} \theta_{1x} \\ \theta_{1y} \\ \Phi_1' \\ \theta_{2x} \\ \theta_{2y} \\ \Phi_2' \end{bmatrix} \tag{30}$$

or

$$\mathbf{p}_{ss} = \mathbf{k}_{ss}\mathbf{d}_{ss} \tag{31}$$

where the $K_{i,j}$ are the remaining elements of $\mathbf{k}$ with their original row i and column j subscripts and $\mathbf{k}_{ss}$ is the required 6×6 matrix for this simple one member structure.

Application of the Wittrick-Williams algorithm to this simple structure gives

$$J_{ss} = J_m + s\{\mathbf{k}_{ss}\} \quad \text{or} \quad J_m = J_{ss} - s\{\mathbf{k}_{ss}\} \tag{32,33}$$

where J_{ss} is the number of natural frequencies of the simply supported member that lie below the trial frequency ω^*, J_m is the required number of clamped-clamped natural frequencies of the member lying below ω^*, $s\{\mathbf{k}_{ss}\}$ is the number of negative elements on the leading diagonal of $\mathbf{k}_{ss}^{\Delta}$, where $\mathbf{k}_{ss}^{\Delta}$ is the upper triangular matrix obtained by applying the usual form of Gauss elimination to $\mathbf{k}_{ss}$.

The evaluation of $s\{\mathbf{k}_{ss}\}$ is clearly straightforward and the problem thus lies in determining J_{ss}.

Based on Eqs. (13), (16) and (17a,b,c), the boundary conditions of Eq. (30) are satisfied by assuming solutions for the displacements $U(\xi)$, $V(\xi)$ and $\Phi(\xi)$ of the form

$$U(\xi) = C_i\sin(i\pi\xi), \quad V(\xi) = D_i\sin(i\pi\xi) \quad \text{and} \quad \Phi(\xi) = E_i\sin(i\pi\xi) \quad (i = 1,2,3,\ldots) \tag{34a,b,c}$$

where C_i, D_i and E_i are constants.

Substituting Eq. (34) into Eq. (8) gives

$$\begin{bmatrix} (i\pi)^4 - \alpha_x^2(i\pi)^2 - \omega^2\beta_x^2 & 0 & -y_c\alpha_x^2(i\pi)^2 - y_c\omega^2\beta_x^2 \\ 0 & (i\pi)^4 - \alpha_y^2(i\pi)^2 - \omega^2\beta_y^2 & x_c\alpha_y^2(i\pi)^2 + x_c\omega^2\beta_y^2 \\ -y_c\dfrac{\alpha_x^2}{\gamma_x^2}(i\pi)^2 - y_c\omega^2\dfrac{\beta_x^2}{\gamma_x^2} & x_c\dfrac{\alpha_y^2}{\gamma_y^2}(i\pi)^2 + x_c\omega^2\dfrac{\beta_y^2}{\gamma_y^2} & (i\pi)^4 + (\gamma_\varphi^2 - \alpha_\varphi^2)(i\pi)^2 - \omega^2\beta_\varphi^2 \end{bmatrix} \begin{bmatrix} C_i \\ D_i \\ E_i \end{bmatrix} = 0 \tag{35}$$

in which ω represents the coupled natural frequencies of the member with simply supported ends. The non-trivial solution of Eq. (35) is obtained when

$$\begin{vmatrix} (i\pi)^4 - \alpha_x^2(i\pi)^2 - \omega^2\beta_x^2 & 0 & -y_c\alpha_x^2(i\pi)^2 - y_c\omega^2\beta_x^2 \\ 0 & (i\pi)^4 - \alpha_y^2(i\pi)^2 - \omega^2\beta_y^2 & x_c\alpha_y^2(i\pi)^2 + x_c\omega^2\beta_y^2 \\ -y_c\dfrac{\alpha_x^2}{\gamma_x^2}(i\pi)^2 - y_c\omega^2\dfrac{\beta_x^2}{\gamma_x^2} & x_c\dfrac{\alpha_y^2}{\gamma_y^2}(i\pi)^2 + x_c\omega^2\dfrac{\beta_y^2}{\gamma_y^2} & (i\pi)^4 + (\gamma_\varphi^2 - \alpha_\varphi^2)(i\pi)^2 - \omega^2\beta_\varphi^2 \end{vmatrix} = 0 \tag{36}$$

Eq. (36) is a cubic equation in ω^2 and yields three positive values of ω for each value of i. It is then possible to calculate J_{ss} by substituting progressively larger values of i until all of those natural frequencies lying below ω^* have been accounted for. Once J_{ss} is known, J_m can be calculated from Eq. (33).

4. Numerical results

The foregoing stiffness theory can be used to calculate the natural frequencies of single, or more complex, thin-walled beam-columns that can be assembled in the usual way. The following examples have been chosen to confirm the accuracy of the theory and provide an insight into its range of application.

Example 1. This example considers a thin-walled beam with semi-circular cross-section that has been investigated by other researchers and for which comparative results are available in the literature (Li *et al.* 2004a). The properties of the cross-section are as follows:

$$I_x = 1.77 \times 10^{-8}\ \text{m}^4,\quad I_y = 9.26 \times 10^{-8}\ \text{m}^4,\quad J = 1.64 \times 10^{-9}\ \text{m}^4,\quad r_m^2 = 6 \times 10^{-4}\ \text{m}^2$$
$$x_c = 0.0155\ \text{m},\quad y_c = 0.0\ \text{m},\quad I_w = 1.52 \times 10^{-12}\ \text{m}^6,\quad m = 0.835\ \text{kg/m}$$
$$L = 0.82\ \text{m},\quad E = 68.9 \times 10^9\ \text{Nm}^{-2},\quad G = 26.5 \times 10^9\ \text{Nm}^{-2}$$

The first ten bending-torsion coupled natural frequencies have been calculated for $P = 0$ and $P = 1790$ N for various boundary conditions. The results are compared with others available in the literature in Tables 1-3. It can be seen that there is very good agreement.

The natural frequencies, denoted by asterisks in column seven of Table 2, were also determined in a paper by Leung (1991), as shown in Table 3.

Table 1 Natural frequencies (Hz) when P = 0. C = Clamped, F = Free and S = Simple support

Freq. No.	C-C		F-F		C-F		S-S	
	Li (2004a)	Proposed method	Li (2004a)	Proposed method	Li (2004a)	Proposed method	Li (2004a)	Proposed method
1	198.81	198.814	202.38	202.384	31.80	31.8052	89.27	89.2783
2	202.38	202.384	233.95	233.959	63.79	63.7923	150.44	150.446
3	425.04	425.046	322.89	322.895	137.68	137.688	320.32	320.324
4	557.87	557.878	557.87	557.878	199.31	199.319	357.11	357.113
5	618.09	618.094	575.57	575.572	278.35	278.359	365.81	365.813
6	695.63	695.638	684.22	684.222	484.77	484.776	604.13	604.130
7	999.31	999.320	857.91	857.914	558.09	558.099	803.50	803.503
8	1093.66	1093.66	1093.66	1093.66	663.84	663.840	885.01	885.015
9	1365.73	1365.73	1141.21	1141.21	768.35	768.356	1106.59	1106.59
10	1688.57	1688.57	1505.76	1505.76	1076.36	1076.36	1217.97	1217.97

Table 2 Natural frequencies (Hz) when P=1790 N. C = Clamped, F = Free and S = Simple support

Freq. No.	C-C		F-F		C-F		S-S	
	Li (2004a)	Proposed method	Li (2004a)	Proposed method	Li (2004a)	Proposed method	Li (2004a)	Proposed method
1	196.55	196.555	192.23	192.235	25.01	25.0141*	84.69	84.6968
2	199.91	199.912	232.02	232.024	61.31	61.3199*	147.77	147.773
3	420.89	420.891	317.53	317.536	136.15	136.159*	319.07	319.077
4	554.53	554.534	549.93	549.932	192.62	192.626*	352.62	352.621
5	616.77	616.774	569.08	569.087	275.03	275.037	361.42	361.429
6	690.47	690.475	680.40	680.404	479.40	479.401	598.16	598.164
7	992.45	992.452	850.78	850.784	552.47	552.478	799.02	799.027
8	1090.00	1090.01	1086.74	1086.74	661.37	661.373	877.78	877.781
9	1357.95	1357.95	1131.86	1131.86	761.75	761.759	1105.14	1105.15
10	1687.35	1687.35	1495.49	1495.50	1068.29	1068.30	1209.76	1209.77

Table 3 The first four C-F natural frequencies (Hz) for the problem of Example 1, as given by Leung (1991)

Freq. No.	Vlasov Theory (Four elements)	Exact (Unreferenced source)
1	25.01	25.01
2	61.59	61.28
3	137.22	136.00
4	194.28	192.40

Example 2. This example considers the beam studied by Tanaka and Bercin (1999). It is a uniform thin-walled beam with doubly asymmetric cross-section. The material and geometric properties of the cross-section in the co-ordinate system of Figs. 2(a) and 2(b) are as follows:

$$EI_x = 73480\ \text{Nm}^2,\ EI_y = 16680\ \text{Nm}^2,\ GJ = 10.81\ \text{Nm}^2,\ r_m^2 = 3.0303 \times 10^{-3}\ \text{m}^2,$$
$$x_c = 0.02316\ \text{m},\ y_c = 0.02625\ \text{m},\ EI_w = 26.34\ \text{Nm}^4,\ m = 1.947\ \text{kg/m},\ L = 1.5\ \text{m}.$$

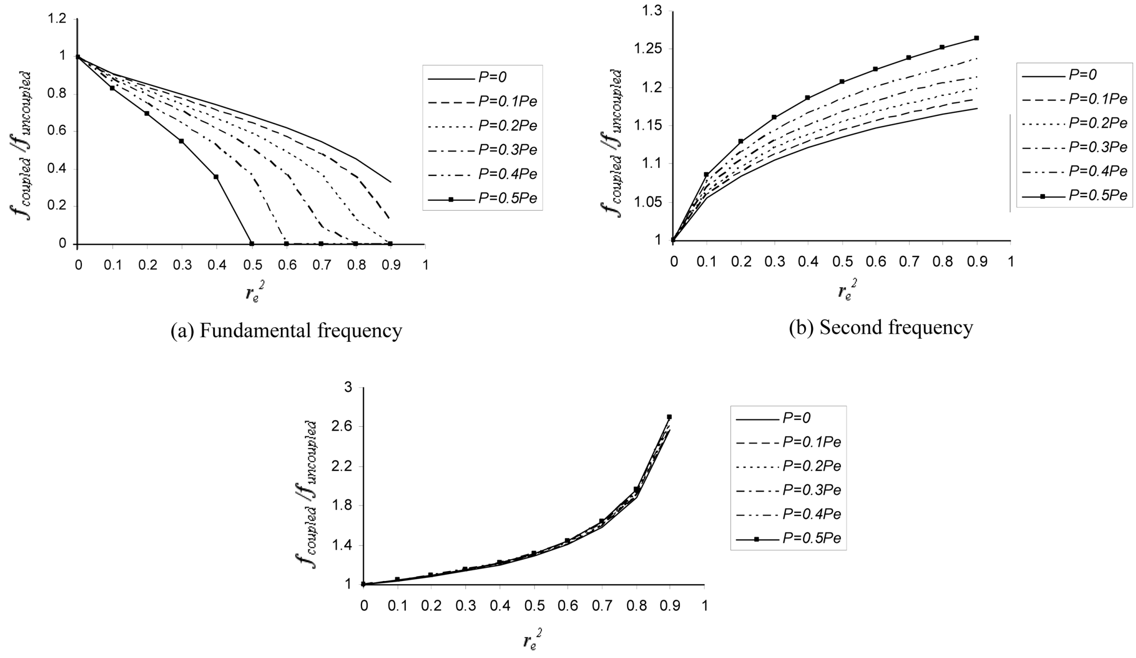
The results presented by Tanaka and Bercin are compared with those of the present study in Table 4.

The same beam with C-F boundary conditions is now used in a small parametric study to examine the effects of eccentricity and compressive axial force on the first three natural frequencies of the structure. The eccentricity is measured by a non-dimensional parameter r_e^2 where $r_e^2 = (x_c^2 + y_c^2) / r_m^2$. The eccentricity parameter represents a measure of the mass centre offset from the shear centre and is equal to zero in the case of twofold symmetry. It is clear that r_e must lie in the range $0 \le r_e \le 1$. The applied axial load is non-dimensionalised by the fundamental elastic critical load when $x_c = y_c = 0$, i.e., $P_e = \pi^2 EI_y/(4L^2)$. The frequency results are presented non-dimensionally in Fig. 3 by $f_{coupled}/f_{ucoupled}$, the ratio of the coupled natural frequency to the uncoupled natural frequency of the beam.

It can be seen from Figs. 3(a)-3(c) that as the eccentricity increases, the first natural frequency decreases and the second and third frequencies increase. Furthermore, Figs. 3(a) and 3(b) show that when the beam-column is subjected to progressively larger axial forces, the eccentricity's influence on the natural frequencies becomes substantially greater. It is interesting to note that the relationship between the third uncoupled and coupled natural frequencies is not affected to any large extent by increasing the axial load. This is due to the fact that it corresponds to a torsional mode in which the secondary effect of axial load on the internal forces of the beam-column is small.

Table 4 Comparative results for the natural frequencies (Hz) for the beam of Example 2. *C* = Clamped, *F* = Free and *S* = Simple support

Freq. No.	C-C		F-F		C-F		S-S	
	Tanaka and Bercin (1999)	Proposed method	Tanaka and Bercin (1999)	Proposed method	Tanaka and Bercin (1999)	Proposed method	Tanaka and Bercin (1999)	Proposed method
1	96.94	98.7229	20.04	20.3449	17.03	17.1688	41.48	44.7131
2	168.81	169.437	99.57	101.271	27.58	27.3135	74.12	75.1476
3	268.46	270.907	169.71	170.319	59.25	59.1020	164.11	164.879



(a) Fundamental frequency

(b) Second frequency

(c) Third frequency

Fig. 3 Ratio of $f_{coupled}/f_{uncoupled}$ for the first three natural frequencies for various eccentricities and compressive loads

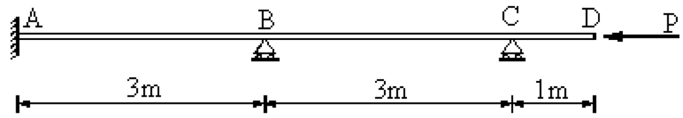


Fig. 4 The singly asymmetric, continuous channel section of Example 3

Table 5 Comparison of the natural frequencies (Hz) determined by Banerjee *et al.* (1996) and the proposed method for the structure of Fig. 4

Freq. No	P = 0		P = 45 × 10^4 N
	Banerjee (1996)	Proposed method	Proposed method
1	6.127	6.07128	5.38549
2	19.34	19.2648	10.2623
3	32.71	32.7813	24.1901

Example 3. This example examines the continuous channel section shown in Fig. 4 that was studied by Banerjee *et al.* (1996). The material and geometric properties of the member are given below and comparative results are shown in Table 5.

$$EI = 0.1704 \times 10^7\,\text{Nm}^2,\ GJ = 0.314 \times 10^4\,\text{Nm}^2,\ r_m^2 = 7.6206 \times 10^{-3}\,\text{m}^2,$$
$$x_c = 0.05626\ \text{m},\ y_c = 0.0\ \text{m},\quad EI_w = 0.1337 \times 10^4\,\text{Nm}^4,\ m = 17.61\ \text{kg/m}$$

5. Conclusions

An analytical method for determining the natural frequencies of an axially loaded, thin-walled, Bernoulli-Euler beam with doubly asymmetric cross-section has been developed. This has been achieved by formulating the governing differential equations, which include the effects of warping torsion and distributed mass, and then solving them in closed form to yield an exact dynamic stiffness matrix. The resulting matrix can be used to establish more complex beam systems in the usual way. The application of such theory necessitates the solution of a transcendental eigenvalue problem. This has been accommodated in the present case by use of the Wittrick-Williams algorithm, which enables convergence upon any required frequency to any desired accuracy with the certain knowledge that none have been missed. Examples have also been presented that confirm the accuracy of the theory and a small parametric study has been undertaken. The latter has highlighted a number of interesting features that require further investigation.

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