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Low-discrepancy sampling for structural reliability sensitivity analysis

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Abstract. This study presents an innovative method to estimate the reliability sensitivity based on the low-discrepancy sampling which is a new technique for structural reliability analysis. Two advantages are contributed to the method: one is that, by developing a general importance sampling procedure for reliability sensitivity analysis, the partial derivative of the failure probability with respect to the distribution parameter can be directly obtained with typically insignificant additional computations on the basis of structural reliability analysis; and the other is that, by combining various low-discrepancy sequences with the above importance sampling procedure, the proposed method is far more efficient than that based on the classical Monte Carlo method in estimating reliability sensitivity, especially for problems of small failure probability or problems that require a large number of costly finite element analyses. Examples involving both numerical and structural problems illustrate the application and effectiveness of the method developed, which indicate that the proposed method can provide accurate and computationally efficient estimates of reliability sensitivity.

Keywords: reliability sensitivity; failure probability; importance sampling; Quasi-Monte Carlo; low-discrepancy sampling.

1. Introduction

For design under uncertainty, probabilistic sensitivity analysis methods have been developed to provide an important insight into the probabilistic behavior of a complex model so that one can make informed decisions on minimizing the variability of a system or optimizing a system's performance with an acceptable risk (Du and Chen 2002, Enevoldsen and Sorensen 1994). Various probabilistic sensitivity analysis methods have different meanings, and the definition of a probabilistic sensitivity differs between application fields and users. Probability-based sensitivity methods have a long and storied history with respect to first and second order reliability methods. In general, the reliability sensitivity refers to the partial derivative of the failure probability with respect to probability distribution parameters of the random variables. It can be used to identify and rank the distribution parameters of the design variables and to adjust a design to achieve reliability-based objectives. It is therefore important to develop an applicable method for reliability sensitivity analysis.

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There are a number of approaches to assess the reliability sensitivity, including the first and second order reliability methods, the simulation method and the finite-difference method (Ditlevsen and Madsen 1996, Karamchandani and Cornell 1992, Bjerager and Krenk 1989, L'Ecuyer and Perron 1994, Melchers and Ahammed 2004, Wu 1994, Wu and Mohanty 2006, Sues and Cesare 2005, Au 2005, Lu *et al.* 2008, Song *et al.* 2009). Although there are many reasons for choosing one method over another, the most widely employed method is the Monte Carlo (MC) simulation method for complex problems. But, the MC method demonstrates a poor computational efficiency in evaluating problems of small failure probability or problems that require a large number of costly finite element analyses in each sampling cycle. The challenge is to minimize the computational cost for achieving the required accuracy of reliability sensitivity, and many efforts have been made in this direction recently. Wu (1994) proposed a reliability sensitivity method based on the CDF of the structural response variable, the normalized reliability sensitivity coefficient is expressed as an expectation of the partial derivative of the PDF, wherein the sampling-based method can be used to compute the reliability sensitivity. Sues and Cesare (2005) applied the most probable point system simulation (MPPSS) and the sampling-based reliability sensitivity method to system reliability problems. Au (2005) presented a method of reliability-based design sensitivity analysis by efficient simulation, and this simulation is in fact a subset simulation on the basis of Markov Chain Monte Carlo (MCMC). Lu *et al.* (2008) and Song *et al.* (2009) provided two reliability sensitivity methods that are based on the line sampling and the subset sampling, which are new techniques for evaluating high-dimensional structural reliability problems. Although the above methods require much fewer samples than crude MC, in some cases their samples required can still be considered as large, especially when the computation of each sample is very costly as it may involve expensive nonlinear finite element solutions.

Although the reliability sensitivity analysis and the reliability analysis serve different aims, in practice, the implementation of the reliability analysis and that of the reliability sensitivity analysis are closely connected on both a conceptual and a computational level (Lu *et al.* 2008, Helton *et al.* 2006). Thus the available reliability sensitivity methods are mostly based on the corresponding reliability analysis methods. Therefore, this paper presents an innovative method to estimate the reliability sensitivity based on the low-discrepancy sampling which is a quite new approach for structural reliability analysis (Shinoda 2007, Dai and Wang 2009, Bucher 2009). The reason why the low-discrepancy sampling method is used to construct the novel reliability sensitivity method is that it can often achieve a given accuracy with far fewer samples and effectively decrease the total computational cost when compared with the traditional MC method.

This paper is organized as follows: Section 2 gives a brief description of MC and Quasi-Monte Carlo (QMC) integration algorithms and the construction of the low-discrepancy sequences. Section 3 presents a comprehensive illustration of the proposed low-discrepancy sampling method for estimating the reliability sensitivity. In Section 3.1, a general importance sampling (IS)-based procedure, which can directly estimate the reliability sensitivity rather than the sensitivity coefficient in Wu (1994), is derived in detail. Then the proposed method in which the low-discrepancy sequences are combined with the above IS technique is innovatively developed for reliability sensitivity analysis in Section 3.2. After the examples verify the accuracy and efficiency of the proposed method in comparison with MC and IS methods in Section 4, Section 5 concludes with a summary of the main advantages of the proposed methodology.

2. Formulation of the low-discrepancy sampling

2.1 Quasi-Monte Carlo integration

To gain an insight into the QMC approach, the traditional MC method is briefly reviewed herein. Suppose $f(\cdot)$ is an integrable function defined on the s-dimensional unit cube $C^s = [0, 1]^s$. Consider the integral

$$I(f) = \int_{C^s} f(\mathbf{x}) d\mathbf{x} \tag{1}$$

The MC integration draws a random point sequence $P_N = \{\mathbf{x}_i, i = 1, \ldots, N\}$ in C^s and then approximates the integral in Eq. (1) through

$$\hat{I}(f, P_N) = \frac{1}{N} \sum_{i=1}^{N} f(\mathbf{x}_i) \tag{2}$$

By the *Strong Law of Large Numbers* the estimate $\hat{I}(f, P_N)$ stochastically converges to $I(f)$ with probability one as $N \to \infty$. Moreover, the *Central Limit Theorem* guarantees that $\hat{I}(f, P_N)$ is asymptotically normally distributed when the sample size N is large enough. The convergence rate for MC integration is, in average, of the order $O(N^{-1/2})$, regardless of the integral dimensionality s.

The QMC approach aims to improve the MC approximation in terms of faster convergence rate and less computational cost. The key idea is to use the deterministic uniformly distributed sequences known as low-discrepancy sequences, instead of the MC random samples. The reason behind this is due to the famous *Koksma-Hlawka inequality*

$$|I(f) - \hat{I}(f, P_N)| \leq V(f) D(P_N) \tag{3}$$

where $V(f)$ is the bounded total variation of function $f(\cdot)$ over C^s in the sense of Hardy and Krause (Fang and Wang 1994). The quantity $D(P_N)$ which measures the evenness of spread of the point sequence P_N, is defined as

$$D(P_N) = \sup_{\mathbf{x} \in C^s} |F_N(\mathbf{x}) - F(\mathbf{x})| \tag{4}$$

where $F(\mathbf{x})$ is the uniform distribution in C^s and $F_N(\mathbf{x})$ is the empirical distribution of P_N. $D(P_N)$ in Eq. (4) is known as the discrepancy of P_N in analytical number theory. Further details of the above concepts can be found in Fang and Wang (1994), Niederreiter (1992).

2.2 Low-discrepancy sequences

There are a few well known and commonly used low-discrepancy sequences. The following only briefly introduces four kinds of such sequences which have been successfully used for structural reliability analysis in Dai and Wang (2009). More details on the construction of low-discrepancy sequences can also be found in Fang and Wang (1994), Niederreiter (1992).

2.2.1 Good lattice point (GLP) set

Assume $(N; h_1, \ldots, h_s)$ to be a vector with integral components satisfying $1 \leq h_i < N$, $h_i \neq h_j (i \neq j)$,

$s < N$ and the greatest common divisor $(n, h_i) = 1$ for $i = 1, \ldots, s$. Then, let

$$\begin{cases} q_{ki} \equiv kh_i (\mathrm{mod}\ N) \\ \varphi_{ki} = (2q_{ki} - 1)/2N \end{cases} \quad k = 1, \ldots, N,\ i = 1, \ldots, s \tag{5}$$

where the operation mod confines q_{ki} to the range between 1 and N.

The point set $P_N = \{\varphi_k = (\varphi_{k1}, \ldots, \varphi_{ks}), k = 1, \ldots, N\}$ is known as lattice points with the *generating vector* $(N; h_1, \ldots, h_s)$. Once P_N has the smallest discrepancy among all possible generating vectors, then P_N is the GLP set. (Fang and Wang 1994, Hua and Wang 1981) give the methods for finding the best *generating vectors* and tabulate the corresponding results for different numbers s and N. It is noted that the shortcoming of GLP set is that it is only available for the dimensions of 2-18 and for a few numbers of sample points given in Fang and Wang (1994).

2.2.2 Hua-Wang (H-W) point set

Let $\gamma = (\gamma_1, \ldots, \gamma_s) \in R^s$. If the first N terms of the set $P = \{(\{\gamma_1 k\}, \ldots, \{\gamma_s k\}), k = 1 \ldots\}$ has the discrepancy $D(P_N) \le c(\gamma, \varepsilon) N^{-1+\varepsilon}$, $(N = 1, 2, \ldots)$ then P is called a good point (GP) set and γ is a good point. The most effective method for generating good point γ is suggested by Hua and Wang (1981) with

$$\gamma = \left(\left\{2\cos\frac{2\pi}{p}\right\}, \left\{2\cos\frac{4\pi}{p}\right\}, \ldots, \left\{2\cos\frac{2\pi s}{p}\right\}\right) \tag{6}$$

where p is a prime and $p \ge 2s + 3$. Thus, the sequence P obtained in this way is called H-W point set.

2.2.3 Halton sequence

Let m be a prime number, and then any natural number k has a unique m-digits representation

$$k = b_0 + b_1 m + b_2 m^2 + \ldots + b_r m^r \tag{7}$$

where $b_i \in \{0, 1, \ldots, m-1\}$ for $i = 0, 1, \ldots, r$, and $m^r \le k < m^{r+1}$. Define the base-m radical inverse function $\phi_m(k)$ as

$$\phi_m(k) = b_0 m^{-1} + b_1 m^{-2} + \ldots + b_r m^{-r-1} \tag{8}$$

Notice that for every k, $\phi_m(k) \in [0, 1]$.

Let $p_i (1 \le i \le s)$ be s distinct prime numbers, and then the s-dimensional sequence $P = \{\varphi_k = (\phi_{p_1}(k), \ldots, \phi_{p_s}(k))$, $k = 1, 2, \ldots\}$ is called Halton sequence.

2.2.4 Hammersley sequence

Halton sequence has many variants which have smaller discrepancies, and one of them is the Hammersley sequence. Let $s \ge 2$ and $p_1, \ldots, p_{s-1}$ be $(s-1)$ distinct prime numbers, and then the sequence $P = \{\varphi_k = (2k - 1/2N, \phi_{p_1}(k), \ldots, \phi_{p_{s-1}}(k)), k = 1 \ldots N\}$ is called Hammersley sequence.

3. Low-discrepancy sampling-based reliability sensitivity analysis

As pointed out above, Wu (1994) firstly developed a sampling-based reliability sensitivity method, in which the normalized reliability sensitivity coefficient was suggested to compute reliability sensitivity. This pioneer work demonstrates the potential of the sampling technique in its application to reliability sensitivity analysis, and many efforts have been done followed his work. However, a further research needs to be done in this direction in order to pursue a more general and efficient use of the sampling technique for assessing the reliability sensitivity. Therefore, this study firstly presents a new IS-based procedure to directly estimate the partial derivative of the failure probability with respect to the distribution parameter based on Wu's work and then develops a method in which the low-discrepancy sequences are combined with the above IS technique for efficiently estimating reliability sensitivity. Generally, the proposed methodology can obtain lower error and improved convergence due to the low-discrepancy sequences.

3.1 Reliability sensitivity analysis by importance sampling

3.1.1 General distributions

In structural reliability analysis, the probability of failure P_F is defined as

$$P_F = \int_{G(\mathbf{x}) \le 0} f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} = \int_{R^s} I_F(\mathbf{x}) f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} \tag{9}$$

where $\mathbf{x}$ is the vector of random variables described by the joint probability density function $f_{\mathbf{x}}(\mathbf{x})$; $G(\mathbf{x})$ is the function that defines the limit state such that $G(\mathbf{x}) \le 0$ represents the failure domain F; s is the dimension of the problem and $I_F(\mathbf{x})$ denotes the indicator function of F.

Structural reliability sensitivity is defined as the partial derivative of the failure probability with respect to probability distribution parameters of the basic random variables. Based on this definition, one can take the partial derivative of both sides of Eq. (9) in order to compute the partial derivative of P_F with respect to parameter of the i-th random variable θ_i

$$\frac{\partial P_F}{\partial \theta_i} = \frac{\partial}{\partial \theta_i} \int_{R^s} I_F(\mathbf{x}) f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} = \int_{R^s} I_F(\mathbf{x}) \frac{\partial f_{\mathbf{x}}(\mathbf{x})}{\partial \theta_i} d\mathbf{x} \tag{10}$$

where θ_i represents a parameter of the i-th random variable, e.g., mean, standard deviation, shape or scale factor. If the random variables are independent, the joint probability density function can be written as a product of the individual random variables and Eq. (10) becomes

$$\frac{\partial P_F}{\partial \theta_i} = \int_{R^s} I_F(\mathbf{x}) \frac{\partial f_{x_i}(x_i)}{\partial \theta_i} \frac{1}{f_{x_i}(x_i)} f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} = E_f\left[\frac{I_F(\mathbf{x})}{f_{x_i}(x_i)} \frac{\partial f_{x_i}(x_i)}{\partial \theta_i}\right] \tag{11}$$

The expected value in Eq. (11) can be approximated using the MC method as

$$\frac{\partial \hat{P}_F}{\partial \theta_i} \approx \frac{1}{N} \sum_{j=1}^{N} \frac{I_F(\mathbf{x}_j)}{f_{x_{i,j}}(x_{i,j})} \frac{\partial f_{x_{i,j}}(x_{i,j})}{\partial \theta_i} \tag{12}$$

where $x_{i,j}$ are samples that drawn from $f_{\mathbf{x}}(\mathbf{x})$ and N is the number of the samples.

The samples simulated from the PDF $f(\mathbf{x})$, for the crude MC method, are mostly located in the safe region for problems of small failure probability, and this will lead to the poor computational efficiency in estimating the reliability sensitivity. It has to introduce the IS density function $h(\mathbf{x})$, as the manner that is in structural reliability analysis, to address this problem. Thus, the Eq. (11) can be re-expressed as

$$\frac{\partial P_F}{\partial \theta_i} = \int_{R^s} I_F(\mathbf{x}) \frac{\partial f_{x_i}(x_i)}{\partial \theta_i} \frac{1}{f_{x_i}(x_i)} \frac{f_{\mathbf{x}}(\mathbf{x})}{h(\mathbf{x})} h(\mathbf{x}) d\mathbf{x} = E_h\left[\frac{I_F(\mathbf{x}) f_{\mathbf{x}}(\mathbf{x})}{h(\mathbf{x}) f_{x_i}(x_i)} \frac{\partial f_{x_i}(x_i)}{\partial \theta_i}\right] \tag{13}$$

where $h(\mathbf{x})$ is an auxiliary density function intended to produce samples in the region that contributes most to the integral (9). Then the estimate of the reliability sensitivity becomes

$$\frac{\partial \hat{P}_F}{\partial \theta_i} = \frac{1}{N} \sum_{j=1}^{N} \frac{I_F(\mathbf{x}_j)}{h(\mathbf{x}_j)} \frac{f_{\mathbf{x}}(\mathbf{x}_j)}{f_{x_{i,j}}(x_{i,j})} \frac{\partial f_{x_{i,j}}(x_{i,j})}{\partial \theta_i} \tag{14}$$

where the samples $x_{i,j}$ are now drawn from $h(\mathbf{x})$. The variance of the estimator in Eq. (14) can be derived approximately as

$$\mathrm{Var}\left(\frac{\partial \hat{P}_F}{\partial \theta_i}\right) = \frac{1}{N-1}\left[\frac{1}{N} \sum_{j=1}^{N} \left(\frac{I_F(\mathbf{x}_j)}{h(\mathbf{x}_j)} \frac{f_{\mathbf{x}}(\mathbf{x}_j)}{f_{x_{i,j}}(x_{i,j})} \frac{\partial f_{x_{i,j}}(x_{i,j})}{\partial \theta_i}\right)^2 - \left(\frac{\partial \hat{P}_F}{\partial \theta_i}\right)^2\right] \tag{15}$$

It is well known that the choice of the IS density $h(\mathbf{x})$ in structural reliability analysis is a crucial factor affecting the efficiency of the IS method. One of the popular choices for the IS density is to center it at design point because the design point has the highest probability density among other points on the failure domain. In this study, the IS density function is taken as the multivariate Gaussian distribution, with random variable means shifted to the design point although this point may not have the highest probability density in reliability sensitivity analysis. The appealing character of this choice is that if IS technique is used, the failure probability and the reliability sensitivity estimates are computed with respect to the same IS density function and, therefore, the same samples can be used for both. As a result, reliability sensitivities can be obtained with typically insignificant additional computations on the basis of structural reliability analysis.

3.1.2 Independently normal distributions

In general, the independently normal random variables are extremely important for structural reliability and reliability sensitivity analysis, because any non-normal correlated random variable can be transformed into independently normal one by applying Rosenblatt's transformation or Nataf's transformation (Melchers 1999). Consider a normal random variable x_i, the partial derivative of $f_{x_i}(x_i)$ with respect to its distribution parameter μ_i and σ_i can be easily obtained as

$$\frac{\partial f_{x_i}(x_i)}{\partial \mu_i} = \frac{f_{x_i}(x_i)}{\sigma_i} \frac{x_i - \mu_i}{\sigma_i} \tag{16}$$

$$\frac{\partial f_{x_i}(x_i)}{\partial \sigma_i} = \frac{f_{x_i}(x_i)}{\sigma_i}\left[\left(\frac{x_i - \mu_i}{\sigma_i}\right)^2 - 1\right] \tag{17}$$

where μ_i and σ_i is the mean value and the standard deviation of the variable x_i, respectively. Substituting Eqs. (16) and (17) into Eq. (14), the reliability sensitivities of P_F with respect to μ_i and σ_i can be obtained by Eqs. (18) and (19)

$$\frac{\partial \hat{P}_F}{\partial \mu_i} = \frac{1}{N} \sum_{j=1}^{N} \frac{I_F(\mathbf{x}_j) f_{\mathbf{x}}(\mathbf{x}_j)}{h(\mathbf{x}_j)} \frac{x_{i,j} - \mu_i}{\sigma_i} \frac{}{\sigma_i} \tag{18}$$

$$\frac{\partial \hat{P}_F}{\partial \sigma_i} = \frac{1}{N}\sum_{j=1}^{N} \frac{I_F(\mathbf{x}_j) f_{\mathbf{x}}(\mathbf{x}_j)}{h(\mathbf{x}_j) \quad \sigma_i}\left[\left(\frac{x_{i,j}-\mu_i}{\sigma_i}\right)^2 - 1\right] \tag{19}$$

where the samples $x_{i,j}$ are drawn from $h(\mathbf{x})$. The variance of the estimator in Eqs. (18) and (19) can be derived approximately as Eqs. (20) and (21)

$$\mathrm{Var}\left(\frac{\partial \hat{P}_F}{\partial \mu_i}\right) = \frac{1}{N-1}\left[\frac{1}{N}\sum_{j=1}^{N}\left(\frac{I_F(\mathbf{x}_j) f_{\mathbf{x}}(\mathbf{x}_j)}{h(\mathbf{x}_j) \quad \sigma_i}\frac{x_{i,j}-\mu_i}{\sigma_i}\right)^2 - \left(\frac{\partial \hat{P}_F}{\partial \mu_i}\right)^2\right] \tag{20}$$

$$\mathrm{Var}\left(\frac{\partial \hat{P}_F}{\partial \sigma_i}\right) = \frac{1}{N-1}\left\{\frac{1}{N}\sum_{j=1}^{N}\left[\frac{I_F(\mathbf{x}_j) f_{\mathbf{x}}(\mathbf{x}_j)}{h(\mathbf{x}_j) \quad \sigma_i}\left(\left(\frac{x_{i,j}-\mu_i}{\sigma_i}\right)^2 - 1\right)\right]^2 - \left(\frac{\partial \hat{P}_F}{\partial \sigma_i}\right)^2\right\} \tag{21}$$

3.2 Reliability sensitivity analysis by low-discrepancy sampling

Although the developed IS procedure above requires fewer samples than crude MC in estimating the reliability sensitivity, one may still try to adapt it to QMC procedure to further improve the computational efficiency. The following focuses on the use of the low-discrepancy sequences in conjunction with the above IS technique for efficient reliability sensitivity analysis.

As mentioned in Section 2, the QMC method can be described in simple terms as the deterministic version of the MC method, and the integration rule for the QMC method are also taken from the appropriate MC estimate. Consider the integration in Eq. (1), QMC method approximates it through

$$\hat{I}(f, P_N) = \frac{1}{N}\sum_{i=1}^{N} f(\varphi_i) \tag{22}$$

where φ_i is the i-th element of a low-discrepancy sequence. In contrast to Eq. (2), one can clearly see that the only difference between the two methods is that the random samples in the MC method are replaced by the low-discrepancy sequences.

The concept of the proposed low-discrepancy sampling method is to draw samples of the vector of the low-discrepancy sequence from a distribution $h(\varphi)$ which is concentrated in the 'important region' of the random variable space, and this can be easily done by suitably transforming the uniform low-discrepancy sequences. Thus, Eq. (13) can be re-written as

$$\frac{\partial P_F}{\partial \theta_i} = \int_{R^s} I_F(\varphi)\frac{\partial f_{\varphi_i}(\varphi_i)}{\partial \theta_i}\frac{1}{f_{\varphi_i}(\varphi_i)}\frac{f(\varphi)}{h(\varphi)}h(\varphi)d\varphi = E_h\left[\frac{I_F(\varphi)}{h(\varphi)}\frac{f(\varphi)}{f_{\varphi_i}(\varphi_i)}\frac{\partial f_{\varphi_i}(\varphi_i)}{\partial \theta_i}\right] \tag{23}$$

and using an estimator of the form

$$\frac{\partial \hat{P}_F}{\partial \theta_i} = \frac{1}{N}\sum_{j=1}^{N}\frac{I_F(\varphi_j)}{h(\varphi_j)}\frac{f(\varphi_j)}{f_{\varphi_{i,j}}(\varphi_{i,j})}\frac{\partial f_{\varphi_{i,j}}(\varphi_{i,j})}{\partial \theta_i} \tag{24}$$

where the sample points $\{\varphi_j\}_{j=1}^{N}$ are generated as low-discrepancy sequences such as GLP, H-W, Halton or Hammersley sequence. The variance of $\partial \hat{P}_F / \partial \theta_i$ is given by

$$\mathrm{Var}\left(\frac{\partial \hat{P}_F}{\partial \theta_i}\right) = \frac{1}{N-1}\left[\frac{1}{N}\sum_{j=1}^{N}\left(\frac{I_F(\varphi_j)}{h(\varphi_j)}\frac{f(\varphi_j)}{f_{\varphi_{i,j}}(\varphi_{i,j})}\frac{\partial f_{\varphi_{i,j}}(\varphi_{i,j})}{\partial \theta_i}\right)^2 - \left(\frac{\partial \hat{P}_F}{\partial \theta_i}\right)^2\right] \tag{25}$$

and the coefficient of variation of the reliability sensitivity estimator is given by

$$\mathrm{Cov}\left(\frac{\partial \hat{P}_F}{\partial \theta_i}\right) = \left|\sqrt{\mathrm{Var}\left(\frac{\partial \hat{P}_F}{\partial \theta_i}\right)}\Big/\left(\frac{\partial \hat{P}_F}{\partial \theta_i}\right)\right| \tag{26}$$

It is known that random number sampling in the MC method is prone to clustering: for any sampling there are always empty areas as well as regions in which random points are wasted due to clustering. As new points are added randomly, they do not necessarily fill the gaps between already sampled points. A higher rate of convergence and more precise integral estimate can be obtained by using low-discrepancy sequences instead of pseudo random numbers. Therefore, the proposed low-discrepancy sampling method can greatly improve the computational efficiency of the MC method for reliability sensitivity analysis.

4. Numerical examples

The proposed method, in which the low-discrepancy sequences are combined with the developed IS procedure, is applied to five examples involving both numerical and structural problems to investigate its accuracy and efficiency in estimating the reliability sensitivity. These examples, which cover a wide variety of possible limit state functions of varying complexity, were developed to compare different methods with respect to the following aspect: space dimension (number of RVs), probability level (value of failure probability), nonlinear problem and system problem. For comparison purposes, the results calculated by the MC-based method in Wu (1994) are referred to exact ones, denoted as MC. In all cases, the comparison is made with respect to the exact results, the results given by the presented IS method (denoted as MC+IS), and the results given by the proposed low-discrepancy sampling method (denoted as QMC/GLP+IS, QMC/H-W+IS, QMC/Halton+IS and QMC/Hammersley+IS). It is noted that the computational cost of estimating the failure probability or the reliability sensitivity is governed by the number of structural analyses that have to be carried out. Therefore, the comparison criterion used to assess the computational efficiency of different methods in this study is the number of sample points that must be carried out in order to achieve the same level of accuracy.

4.1 Example 1: a multi-dimensional case

The limit state function for the first example, which was also studied in Engelund and Rackwitz (1993), Nie and Ellingwood (2004) is an n-dimensional hyperplane

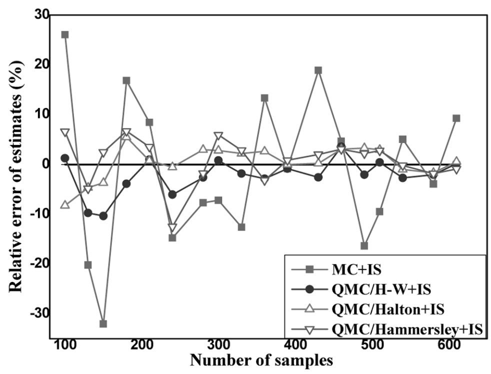
$$g_1 = \beta S^{-1/2} - \sum_{i=1}^{s} U_i \tag{27}$$

where U_i, $i = 1, 2, \ldots, s$ are independent standard normal distributed variables. The example was calculated for $\beta = 3.0$ and $\beta = 4.0$ corresponding to $s = 2$ and $s = 20$, respectively. The purpose is to investigate the performance of the proposed method for different probability levels and different number of RVs. The similar results for different combinations of β and s were obtained and only the results for the case $\beta = 4.0$ and $s = 2$ are shown as follows for the sake of simplification in the paper.

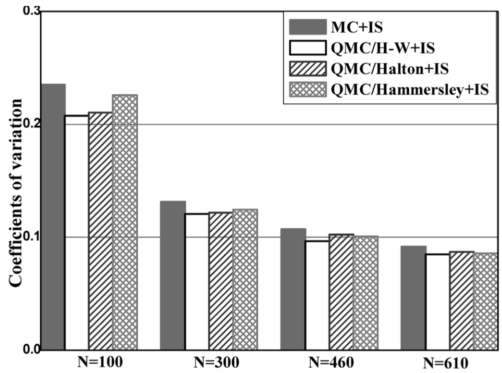
The results of the structural reliability sensitivity are shown in Table 1. For the sake of comparison with GLP procedure, the number of samples was selected as $N = 610$. It can be seen

Table 1 The reliability and sensitivity results of Example 1

	MC	MC+IS	QMC/GLP +IS	QMC/H-W +IS	QMC/Halton +IS	QMC/ Hammersley+IS
$\partial P_F/\partial \mu_1$	9.461×10^{-5}	10.337×10^{-5}	9.506×10^{-5}	9.448×10^{-5}	9.408×10^{-5}	9.549×10^{-5}
$\partial P_F/\partial \mu_2$	9.463×10^{-5}	10.127×10^{-5}	9.520×10^{-5}	9.689×10^{-5}	9.625×10^{-5}	9.571×10^{-5}
$\partial P_F/\partial \sigma_1$	2.678×10^{-4}	3.104×10^{-4}	2.685×10^{-4}	2.612×10^{-4}	2.643×10^{-4}	2.710×10^{-4}
$\partial P_F/\partial \sigma_2$	2.679×10^{-4}	2.515×10^{-4}	2.694×10^{-4}	2.749×10^{-4}	2.766×10^{-4}	2.719×10^{-4}
P_F	3.167×10^{-5}	3.463×10^{-5}	3.184×10^{-5}	3.205×10^{-5}	3.188×10^{-5}	3.200×10^{-5}
Sample size	10^6	610	610	610	610	610



(a) Relative error of estimates of $\partial \hat{P}_F/\partial \mu_1$



(b) Coefficients of variation of $\partial \hat{P}_F/\partial \mu_1$

Fig. 1 Relative error of estimates and coefficients of variation as a function of number of samples for example 1

that the results calculated by the presented method including the MC+IS method are in good agreement with the exact ones with much fewer samples than the MC method. In particular, different versions of QMC+IS procedures give noticeably smaller error estimates than MC+IS method with the same number of samples, indicating that the proposed method can achieve higher accuracy for linear limit state function that contains different number of RVs. In addition, it is found that the value between $\partial \hat{P}_F/\partial \mu_1$ and $\partial \hat{P}_F/\partial \mu_2$, $\partial \hat{P}_F/\partial \sigma_1$ and $\partial \hat{P}_F/\partial \sigma_2$ are similar, indicating that U_1 and U_2 have the comparable effect on P_F. Thus, the significance of the distribution parameter with respect to the failure probability can be reflected by the results of the reliability sensitivity. It is also noted that when failure probability is computed by the low-discrepancy sampling method, the reliability sensitivity can be addressed with only a little extra computational effort on the basis of getting structural reliability results.

Fig. 1(a) and (b) shows the estimates of the reliability sensitivity $\partial \hat{P}_F/\partial \mu_1$ relative to the exact value (relative error of estimates) and the coefficients of variation of these estimates as a function of the number of samples for different methods, respectively. It can be seen that, with comparable coefficients of variation, the results computed by different versions of QMC+IS procedures quickly and stably converge to the exact solution with the increasing of the number of samples. The relative error of estimates are continuously less than 5% as long as the number of samples N reaches 300 for all QMC+IS procedures, while the relative error of estimate produced by MC+IS is still nearly 10% although $N = 610$. The above observations reveal that the low-discrepancy sequences lead to the merits in expediting convergence and increasing solution stability for reliability sensitivity analysis.

4.2 Example 2: a highly non-linear case

The second example has been discussed by several researchers including Kim and Na (1997), Lu *et al.* (2008). The limit state function is

$$g_2 = \exp(0.4U_1 + 7) - \exp(0.3U_2 + 5) - 200 \tag{28}$$

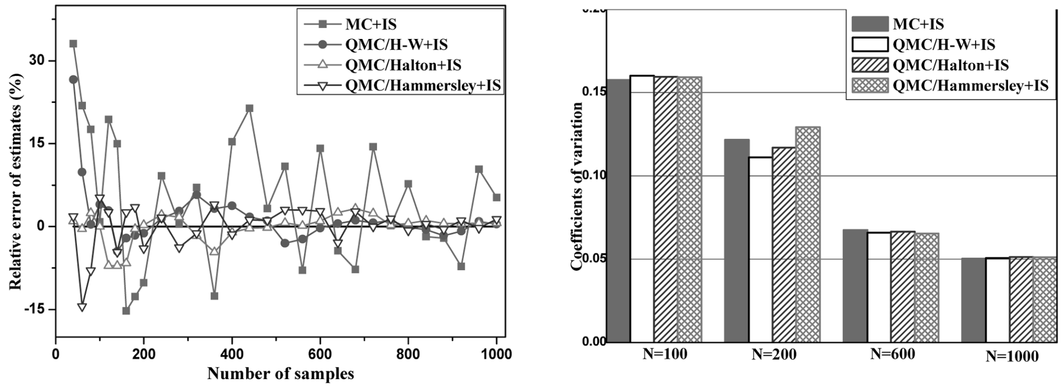
where U_i, $i = 1, 2$ are independently standard normal variables. The objective of this example is to demonstrate the accuracy and efficiency of the proposed method for solving a problem with high nonlinear behavior.

The results of the reliability sensitivity are shown in Table 2. It can be observed that this example gives almost the same results as the previous one, the reliability sensitivity results obtained by the presented method, including the MC+IS method, are consistent with the exact solutions calculated by the MC method with much fewer sample points. On the other hand, the proposed method gives the noticeably smaller error estimates when compared with MC+IS with the same number of samples. It can be also seen that the value of $\partial \hat{P}_F / \partial \mu_1$ is negative, implying that the increase of the mean value μ_1 can lead to the decrease of the failure probability. While the value of $\partial \hat{P}_F / \partial \mu_2$ is positive, indicating that the increase of the variability of variable U_2 can lead to the decrease of the reliability. Thus, the influential tendency of the distribution parameter with respect to the failure probability can be judged by the sign of reliability sensitivity estimator.

Fig. 2(a) and (b) shows the relative error of estimates of $\partial \hat{P}_F / \partial \sigma_1$ and the coefficients of variation of these estimates as a function of the number of samples for different methods, respectively. As the observation described in Fig. 1, the results obtained by MC+IS procedure strongly oscillate in the neighborhood of the exact solution while QMC+IS method make the results converge the exact solution more quickly and stably. It is obvious that the low-discrepancy sampling method requires much fewer sample points than MC+IS method to achieve the same level of accuracy, indicating that the efficiency of the proposed method is much higher than that of the IS method. For instance, QMC/Halton+IS procedure needs 400 samples to decrease the relative error of estimates within 3%, whereas MC+IS more than 1000 samples.

Table 2 The reliability and sensitivity results of Example 2

	MC	MC+IS	QMC/GLP +IS	QMC/H-W +IS	QMC/Halton +IS	QMC/ Hammersley+IS
$\partial P_F / \partial \mu_1$	-0.01033	-0.00876	-0.01011	-0.01022	-0.01033	-0.01001
$\partial P_F / \partial \mu_2$	0.00385	0.00306	0.00378	0.00376	0.00379	0.00362
$\partial P_F / \partial \sigma_1$	0.02598	0.02237	0.02548	0.02573	0.02602	0.02538
$\partial P_F / \partial \sigma_2$	0.00414	0.00265	0.00393	0.00353	0.00377	0.00364
P_F	0.003689	0.003087	0.003605	0.003640	0.003683	0.003548
Sample size	2.5×10^6	987	987	987	987	987



(a) Relative error of estimates of $\partial \hat{P}_F / \partial \sigma_1$ (b) Coefficients of variation of $\partial \hat{P}_F / \partial \sigma_1$

Fig. 2 Relative error of estimates and coefficients of variation as a function of number of samples for example 2

4.3 Example 3: one-story one-bay frame

The one-story one-bay elastoplastic frame, shown in Fig. 3 is used to test whether or not the proposed method can handle the system problem. The four potential failure modes of the system are defined by four linear limit state functions as follows (Song *et al.* 2009, Zhao and Ang 2003)

$$
\begin{aligned}
g_3^{(1)} &= 2M_1 + 2M_3 - 4.5S \\
g_3^{(2)} &= 2M_1 + M_2 + M_3 - 4.5S \\
g_3^{(3)} &= M_1 + M_2 + 2M_3 - 4.5S \\
g_3^{(4)} &= M_1 + 2M_2 + M_3 - 4.5S
\end{aligned} \tag{29}
$$

Since this is a series system, the limit state function g_3 of the structural system can be defined as the minimum of the above, i.e.

$$
g_3 = \min\{g_3^{(1)}, g_3^{(2)}, g_3^{(3)}, g_3^{(4)}\} \tag{30}
$$

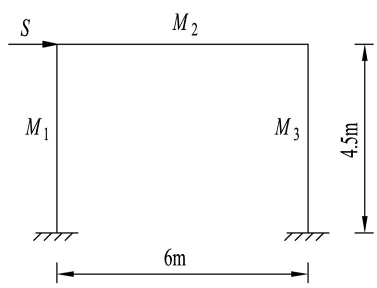


Fig. 3 One-story one-bay elastoplastic frame of example 3

Table 3 The reliability and sensitivity results of Example 3

	MC	MC+IS	QMC/GLP +IS	QMC/H-W +IS	QMC/Halton +IS	QMC/ Hammersley+IS
$\partial P_F/\partial \mu_{M_1}$	-0.03818	-0.03729	-0.03817	-0.03709	-0.03848	-0.03659
$\partial P_F/\partial \mu_{M_2}$	-0.02390	-0.02508	-0.02317	-0.02426	-0.02155	-0.02399
$\partial P_F/\partial \mu_{M_3}$	-0.03819	-0.03770	-0.03773	-0.03539	-0.03730	-0.03752
$\partial P_F/\partial \mu_S$	0.11305	0.11219	0.11279	0.11380	0.11365	0.11390
$\partial P_F/\partial \sigma_{M_1}$	0.01748	0.01294	0.01391	0.01840	0.02043	0.01633
$\partial P_F/\partial \sigma_{M_2}$	0.01962	0.01052	0.01807	0.01684	0.02321	0.02077
$\partial P_F/\partial \sigma_{M_3}$	0.01762	0.01550	0.01618	0.01486	0.01914	0.01902
$\partial P_F/\partial \sigma_S$	0.23148	0.23035	0.23145	0.23331	0.23267	0.23329
P_F	0.01813	0.01792	0.01806	0.01823	0.01822	0.01826
Sample size	10^7	12004	12004	12004	12004	12004

where the yield moment capacity $M_i\ (i = 1, 2, 3)$ and the lateral load S are independently normal random variables with means and standard deviations given by $\mu_{M_i} = 5.2872$ kN·m, $\sigma_{M_i} = 0.1492$ kN·m $(i = 1, 2, 3)$, $\mu_S = 3.8378$ kN, $\sigma_S = 0.3853$ kN. The different branches have comparable contribution to the system failure probability. The reliability sensitivity results of example 3 are shown in Table 3.

4.4 Example 4: beam-cable system

Consider the simple elastoplastic beam-cable system shown in Fig. 4. The limit state functions of the potential failure modes are listed below (Lu *et al.* 2008, Zhao and Ang 2003)

$$\begin{aligned} g_4^{(1)} &= 6M - \omega L^2/2 \\ g_4^{(2)} &= F_1 L + 2F_2 L - 2\omega L^2 \\ g_4^{(3)} &= M + F_2 L - \omega L^2/2 \\ g_4^{(4)} &= 2M + F_1 L - \omega L^2 \end{aligned} \tag{31}$$

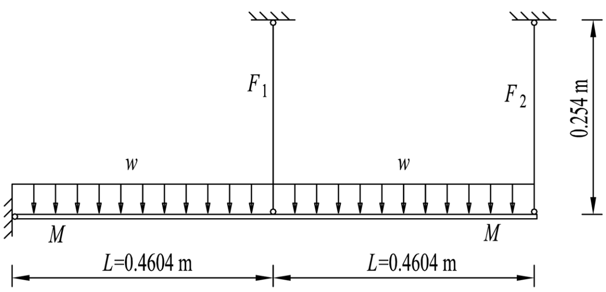


Fig. 4 Beam-cable system of example 4

Table 4 The reliability and sensitivity results of Example 4

	MC	MC+IS	QMC/H-W +IS	QMC/Halton +IS	QMC/ Hammersley+IS
$\partial P_F/\partial \mu_{F_1}$	-3.426 × 10^{-5}	-3.824× 10^{-5}	-3.204 × 10^{-5}	-3.290 × 10^{-5}	-3.397 × 10^{-5}
$\partial P_F/\partial \mu_{F_2}$	-6.944 × 10^{-5}	-7.853 × 10^{-5}	-7.085 × 10^{-5}	-6.171 × 10^{-5}	-6.669 × 10^{-5}
$\partial P_F/\partial \mu_M$	-2.657 × 10^{-6}	-1.844 × 10^{-6}	-2.890 × 10^{-6}	-2.283 × 10^{-6}	-2.171 × 10^{-6}
$\partial P_F/\partial \mu_\omega$	0.00115	0.00104	0.00110	0.00105	0.00110
$\partial P_F/\partial \sigma_{F_1}$	5.006 × 10^{-5}	5.838 × 10^{-5}	4.075 × 10^{-5}	5.008 × 10^{-5}	5.170 × 10^{-5}
$\partial P_F/\partial \sigma_{F_2}$	1.007 × 10^{-4}	1.261 × 10^{-4}	1.054 × 10^{-4}	0.869 × 10^{-4}	1.004 × 10^{-4}
$\partial P_F/\partial \sigma_M$	8.185 × 10^{-6}	4.349 × 10^{-6}	6.123 × 10^{-6}	6.290 × 10^{-6}	6.479 × 10^{-6}
$\partial P_F/\partial \sigma_\omega$	0.00348	0.00324	0.00325	0.00321	0.00334
P_F	1.428 × 10^{-4}	1.570 × 10^{-4}	1.365 × 10^{-4}	1.302 × 10^{-4}	1.373 × 10^{-4}
Sample size	5 × 10^7	5000	5000	5000	5000

where M, F_1, F_2 and ω are normal random variables with mean values of $\mu_M = 1.356$ kN·m, $\mu_{F_1} = 266.9$ kN, $\mu_{F_2} = 133.4$ kN, $\mu_\omega = 29.2$ kN/m and coefficients of variation are Cov(M) = Cov(F_i) = 0.1 $(i = 1, 2)$, Cov(ω) = 0.2. The reliability sensitivity results of example 4 are shown in Table 4. The GLP procedure was not considered in this example for the limitations of its generating vector.

4.5 Example 5: 10-bar truss structure

A ten-bar truss structure, which has been widely studied by Kang *et al.* (2010), Rahman and Wei (2006), is used to examine the effectiveness of the proposed method for problems where closed-form failure functions are not available. The Young's modulus of the material is 68.96 GPa. Two concentrated forces of 444.8 kN are applied at nodes 2 and 3, as shown in Fig. 5. The cross-sectional area X_i, $i = 1, \ldots, 10$ for each bar follows normal distribution with mean value $\mu = 3.2 \times 10^{-4}\text{m}^2$ and standard deviation $\sigma = 3.2 \times 10^{-4}\text{m}^2$. According to the loading condition, the maximum stress $[\sigma_3(X_1, \ldots, X_{10})]$ occurs at element 3, as shown in Fig. 5, thus the limit state function of the truss structure can be expressed as

$$g_5 = \sigma_{\text{allow}} - |\sigma_3(X_1, \ldots, X_{10})| \tag{32}$$

where the permissible stress σ_{allow} is limited to 172.4 Mpa. For the purpose of simplicity, only the reliability sensitivity results for element 1, 3, 5, 7 and 8 are listed in Table 5 because the results for the rest elements are similar with them. Also, the GLP procedure was not considered in due to the limitations of its generating vector.

In examples 3, and 4, the structural systems are series ones with multiple failure modes. While in example 5, the limit state function is implicit and the finite element analysis has to be carried out to compute the failure probability and the reliability sensitivity. From the results in Tables 3, 4 and 5, it can be seen that the results calculated by the proposed method are in good agreement with the exact ones with far fewer samples. In addition, it can be also found that the results calculated from QMC+IS procedures lie within the smaller error bound when compared with those from MC+IS

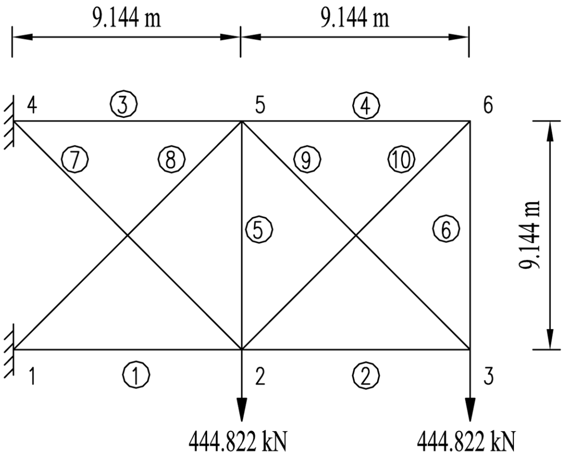


Fig. 5 10-bar truss structure of example 5

Table 5 The reliability and sensitivity results of Example 5

	MC	MC+IS	QMC/H-W +IS	QMC/Halton +IS	QMC/ Hammersley+IS
$\partial P_F/\partial \mu_{A_1}$	-1.378×10^{-3}	-1.483×10^{-3}	-1.577×10^{-3}	-1.391×10^{-3}	-1.444×10^{-3}
$\partial P_F/\partial \mu_{A_3}$	-1.151×10^{-2}	-1.211×10^{-2}	-1.178×10^{-2}	-1.162×10^{-2}	-1.141×10^{-2}
$\partial P_F/\partial \mu_{A_5}$	2.341×10^{-4}	3.529×10^{-4}	1.897×10^{-4}	2.318×10^{-4}	2.783×10^{-4}
$\partial P_F/\partial \mu_{A_7}$	-1.992×10^{-3}	-2.182×10^{-3}	-2.145×10^{-3}	-2.039×10^{-3}	-1.799×10^{-3}
$\partial P_F/\partial \mu_{A_8}$	1.515×10^{-3}	1.403×10^{-3}	1.624×10^{-3}	1.577×10^{-3}	1.274×10^{-3}
$\partial P_F/\partial \sigma_{A_1}$	8.948×10^{-4}	1.020×10^{-3}	9.216×10^{-4}	8.185×10^{-4}	9.485×10^{-4}
$\partial P_F/\partial \sigma_{A_3}$	5.469×10^{-2}	5.598×10^{-2}	5.560×10^{-2}	5.420×10^{-2}	5.343×10^{-2}
$\partial P_F/\partial \sigma_{A_5}$	-3.322×10^{-5}	-1.778×10^{-5}	-2.466×10^{-5}	-3.359×10^{-5}	-2.125×10^{-5}
$\partial P_F/\partial \sigma_{A_7}$	1.734×10^{-3}	1.691×10^{-3}	1.947×10^{-3}	1.979×10^{-3}	1.420×10^{-3}
$\partial P_F/\partial \sigma_{A_8}$	8.405×10^{-4}	6.924×10^{-4}	8.001×10^{-4}	8.916×10^{-4}	7.592×10^{-4}
P_F	7.660×10^{-7}	7.993×10^{-7}	7.758×10^{-7}	7.738×10^{-7}	7.561×10^{-7}
Sample size	5×10^5	2000	2000	2000	2000

method. This again proves the advantage of using the proposed method for reliability sensitivity analysis. Another benefit of the proposed method is that the reliability sensitivity can be addressed with only a little extra computational effort on the basis of getting structural reliability results.

5. Conclusions

The low-discrepancy sampling for structural reliability analysis is successfully applied to establish a method for efficient reliability sensitivity analysis. Two innovative features of the proposed methodology are: Firstly, the partial derivative of the failure probability with respect to the distribution parameter can be obtained with typically insignificant additional computations on the basis of structural reliability analysis by developing a general importance sampling method for reliability sensitivity analysis. Secondly, the advantages of the low-discrepancy sampling for structural reliability analysis are propagated to that for reliability sensitivity analysis by combining the low-discrepancy sequences with the above importance sampling technique. Examples illustrate that the proposed method gives noticeably higher accuracy than MC or IS method with the same number of samples to estimate the reliability sensitivity. That is, for the given accuracy, the proposed method needs far fewer samples and thus decreases the total simulation effort with a remarkable stability when compared with traditional method. Therefore, the low-discrepancy sampling method qualifies as a comprehensive tool in reliability sensitivity analysis.

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