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Response of a completely free beam on a tensionless Pasternak foundation subjected to dynamic load

Z. Celep*[1], K. Güler[1a] and F. Demir[2b]

[1]*Department of Structural and Earthquake Engineering, Faculty of Civil Engineering, Istanbul Technical University, 34469 Maslak, Istanbul, Turkey*
[2]*Department of Civil Engineering, Faculty of Engineering, Süleyman Demirel University, 32260 Isparta, Turkey*

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Abstract. Static and dynamic responses of a completely free elastic beam resting on a two-parameter tensionless Pasternak foundation are investigated by assuming that the beam is symmetrically subjected to a uniformly distributed load and concentrated load at its middle. Governing equations of the problem are obtained and solved by paying attention on the boundary conditions of the problem including the concentrated edge foundation reaction in the case of complete contact and lift-off condition of the beam in a two-parameter foundation. The nonlinear governing equation of the problem is evaluated numerically by adopting an iterative procedure. Numerical results are presented in figures to demonstrate the non-linear behavior of the beam-foundation system for various values of the parameters of the problem comparatively by considering the static and dynamic loading cases.

Keywords: elastic beam; two-parameter foundation; lift-off.

1. Introduction

In recent years considerable attention has been given to the response of elastic beams on an elastic foundation which is one of the structural engineering problems of theoretical and practical interest. A large number of studies have been devoted to the subject. In these studies a number of foundation models having various degrees of sophistication have been used to capture the complex behavior of the soil. The simplest model for the soil is the one-parameter Winkler model which represents the soil as a system of closely spaced but mutually independent linear springs. In the model, the foundation reaction is assumed to be proportional to the vertical displacement of the foundation at the same point. However, the Winkler model has various shortcomings due to the independence of the springs. Because the springs are assumed to be independent and unconnected to each other, no interaction exists between the springs. When loading displays a discontinuity, similar discontinuity will appear on the foundation surface as well. The soil outside the loading area does not contribute to the foundation response. In order to take care of these shortcomings and to improve the model,

*Corresponding author, Professor, E-mail: celep@itu.edu.tr
[a]Professor
[b]Associate Professor

two-parameter models have been proposed. Pasternak model is one of the simplest two-parameter models used commonly. This model can be visualized as a system of closely spaced linear springs coupled to each other with elements which transmit a shear force proportional to the slope of the foundation surface. The model can be seen as a membrane having a surface tension laid on a system of elastic springs as well. Due to connection of spring the continuity of the foundation surface is maintained. However, a discontinuity in slope of the displacement can appear at a edge of the beam or the plate resting on the foundation. The analytical aspects of the continuously supported structures and corresponding boundary conditions on various soil models have been discussed by Kerr (1964, 1976). It is pointed out that the intuitive approach in the boundary conditions may lead to the incorrect formulation of the boundary conditions for the case of a two-parameter foundation model.

Response of structural elements resting on the one- and two-parameter foundation is usually analyzed by assuming that the foundation supports compressive as well as tensile stresses. Although this assumption simplifies the analysis considerably, it is questionable or not valid for many supporting media including the soil. In order to increase the validity of the model, tensionless foundation models which can support compressive reactions only are introduced. In this model separation between the foundation and the structural element takes place in order to avoid the tensile stresses. However, this assumption complicates the analysis and makes it highly non-linear, since the region of contact and separation is not known in advance. As a result, only a limited number of studies dealing with the tensionless foundation are published.

Response of structural elements such as, rings and beams resting on tensionless Winkler foundation is considered. There are various studies dealing with the static problems (Tsai *et al*. 1967, Weisman 1971, Lin *et al*. 1987, Hsu 2006) and the dynamic problems (Celep *et al*. 1989, 1990 and Coşkun *et al*. 1999). The papers dealing with the rectangular and circular plates on tensionless Winkler foundation can be considered as extension of the beam problem. There are various studies dealing with the plates subjected to static loads (Weisman 1970, Dempsey *et al*. 1984, Celep *et al*. 1988, Hong *et al*. 1999, Silva *et al*. 2001) and to dynamic loads (Celep *et al*. 2003, 2004, Güler *et al*. 1995). Generally, in order to investigate lift-off occurrence from the foundation and separation condition, completely free beams and plates are considered in many studies and solutions are obtained by applying approximate numerical techniques to the nonlinear governing equation of the problem by employing the coordinate functions which satisfy the corresponding boundary conditions. In this way the nonlinear problem is reduced to the iterative solution of the system of the nonlinear algebraic equations, since the contact region is not known in advance. On the other hand, when the load is a function of time, then the contact region appears as a function of time as well. After the initial configuration of the contact region is found, the numerical analysis is carried out by adopting step-wise integration in the time domain by updating the contact region continuously.

There are various studies dealing with the structural elements resting on the conventional two-parameter foundation assuming continuous contact. When the boundary of the beam or the plate is fixed, then the boundary condition does not pose any new aspect that of the Winkler foundation. However, when a free end of the beam or plate is considered, then the corresponding boundary condition includes an additional concentrated load due to the membrane stiffness of the two-parameter foundation, which is discussed in detail by Kerr *et al.* (1964, 1991). When a tensionless two-parameter Pasternak model is considered, the solution gets more complicated due to the free edge conditions even in the static problems. Another major difficulty lays in the definition of the contact zone. Due to these reasons the number of the publications on a two-parameter foundation model that reacts in compression only is very limited. Rings, beams and plates resting on a two-parameter foundation are investigated by Güler (2004), Celep *et al*. (2005, 2007), Coşkun *et al*. (2008) and Ma *et al*. (2009). Nonlinear oscillations take place, when the external loads depend on time. Celep *et al*. (1984, 2007) and Coşkun (2003) studied dynamic response of beams and circular plates subjected to time dependent loading.

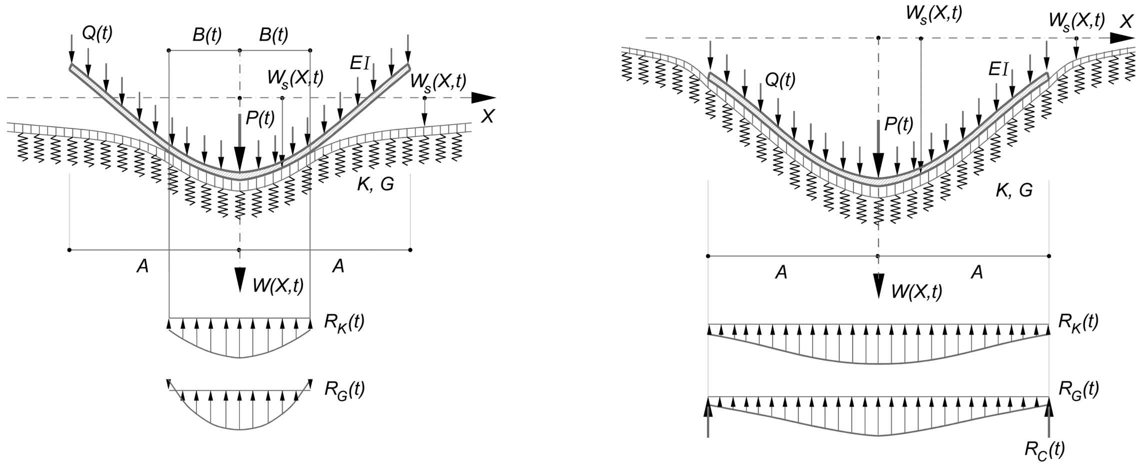
In the present study a completely free beam resting on a two-parameter foundation, i.e., Pasternak model is studied. The beam is assumed to be subjected to a concentrated load at the middle and a uniformly distributed load which depend on time. The problem investigated in the present study is symmetric and similar to the study of Coşkun (2003), however the present treatment is more straightforward. The study is carried out by assuming symmetric lift-off region and it depends on time, when oscillation takes place. The nonlinear governing equation of the problem is discretized by using Galerkin's approximation technique. Numerical results are obtained and presented to assess the validity of the procedure adopted and to illustrate the response of the beam. In the analytical and numerical investigations special attention is paid on the boundary conditions at the free end of the beam, on the contact-lift-off condition and on the global force equilibrium in the static dynamic cases including inertia force. Moreover the time variations of the displacements and that of the extent of the contact region are obtained and illustrated comparatively for various values of the parameters of the problem.

2. Statement of the problem

Consider a completely free elastic beam of length $2A$ and bending stiffness EI, on a two-parameter elastic foundation having a Winkler (spring) stiffness K and a shear (membrane) stiffness G, as Fig. 1 shows. The beam is assumed to be subjected to a concentrated load $P(t)$ applied at the middle of the beam and a uniformly distributed load $Q(t)$. These two loads are assumed to depend on time. Since the geometry and the loading of the problem is symmetric with respect to the middle axis of the beam, it is obvious that the displacement configuration of the beam and that of the foundation surface in the contact and lift-off regions will reflect the same symmetry. For the two-parameter foundation model, it is assumed that foundation reaction $P_f(X)$ and the foundation displacement $W_s(X)$ are related to each other according to

$$P_f(X) = P_k(X) + P_g(X) = KW_s(X) - G\frac{\partial^2 W_s(X)}{\partial X^2} \tag{1}$$

As Eq. (1) shows, the two-parameter foundation can be visualized as spaced closely linear springs of stiffness K and a membrane having a surface tension G. Consequently, the response of the foundation consists of two parts. The first one $P_k(X)$ is proportional to the vertical displacement directly. On the other hand, the second part $P_g(X)$ is proportional to the second derivative of the surface displacement. In the present case, it is assumed that the foundation reacts in compression only. In other word, a contact between the beam and the foundation develops only, when the interaction is maintained by compression, i.e., $P_f(X, t) \geq 0$. In order to avoid tensile reaction, the beam lifts off the foundation. The separation point is not necessarily point of zero displacement in the Pasternak foundation, as it is in the Winkler foundation. When a load is static, the separation point between the contact and lift-off regions is an unknown parameter to be determined. In case of



(a) Partial contact (b) Complete contact

Fig. 1 Completely free beam on a tensionless Pasternak foundation

dynamic load, the position of the separation point depends on time to be determined as one of the main parameters of the problem.

Fig. 1(a) shows the beam on the two-parameter foundation, where a partial contact develops. The beam has a contact region for $-B \le X \le B$ and a lift-off region for $-A \le X \le -B$ and $B \le X \le A$. The surface of the foundation is divided into two regions. The first one is the stress free surface of the foundation and its displacement $W_s(X, t)$ is controlled by the equation

$$G\frac{\partial^2 W_s}{\partial X^2} - KW_s = 0 \tag{2}$$

obtained from Eq. (1). On the other hand the displacement of the beam is controlled by

$$EI\frac{\partial^4 W_b}{\partial X^4} + \left[KW_b - G\frac{\partial^2 W_b}{\partial X^2}\right] - H(X,t) - P(t)\delta(X) - Q(t) = -M\frac{\partial^2 W_b}{\partial t^2} \tag{3}$$

where $W_b(X, t)$ is the displacement of the beam, EI is the bending rigidity of the cross section of the beam, M is the mass of the beam per unit length and $\delta(X)$ is the Dirac's delta function. In addition to the regular terms, the above equation includes the concentrated load $P(t)$ at the middle of the beam, the foundation reaction distributed on the contact region ($-B \le X \le B$). However, when separation takes place, this edge reaction vanishes as stated by Kerr (1964). $H(X, t)$ is an auxiliary function known as the contact function and is defined as

$$H(X,t) = 1 \quad \text{for} \quad -B \le X \le B$$

$$H(X,t) = 0 \quad \text{for} \quad -A \le X \le -B \quad \text{and} \quad B \le X \le A \tag{4}$$

which reflects the symmetry of the problem as well. When the complete contact develops, as Fig. 1(b) shows, the edge reactions

$$R_c(t) = G\left[\frac{\partial W_b}{\partial X} - \frac{\partial W_s}{\partial X}\right] \quad \text{for} \quad X = \pm A \tag{5}$$

come into being at the two ends of the beam. It is worth to note that the edge reactions have not been considered and treated properly in various studies. Often the edge reaction is avoided by assuming that the foundation is defined under the beam only and does not extend beyond the ends of the beam. Consequently, the governing equation of the problem is analyzed under the beam only and the differential equation for the free foundation surface is omitted. However, an additional foundation spring under the edge of the beam is defined to represent the edge reaction. In this case the edge reaction depends only on the edge displacement. Furthermore, the edge reaction can be avoided as well, when a fixed support at the end of the beam is assumed. In the present study the edge reactions as defined in Eq. (5) are taken into consideration in the analysis. In that case the governing Eq. (1) can be modified as follows

$$EI\frac{\partial^4 W_b}{\partial X^4}+\left[KW_b-G\frac{\partial^2 W_b}{\partial X^2}\right]H(X,t)-P(t)\delta(X)-Q(t)+$$

$$+G\left[\frac{\partial W_b}{\partial X}-\frac{\partial W_s}{\partial X}\right]\delta(X-A)H(X=A,t)-G\left[\frac{\partial W_b}{\partial X}-\frac{\partial W_s}{\partial X}\right]\delta(X+A)H(X=-A,t)=-M\frac{\partial^2 W_b}{\partial t^2} \quad (6)$$

For simplicity and convenience in the mathematical formulation, the following non-dimensional parameters are introduced

$$\tau^2 = EIt^2/(MA^4) \quad k = KA^4/(EI) \quad g = GA^2/(EI) \quad p = PA^2/(EI)$$
$$q = QA^3/(EI) \quad b = B/A \quad x = X/A \quad \lambda = k/g = KA^2/G \quad (7)$$

By using these nondimensional parameters, the two governing equations of the problem (2, 6) can be rewritten as

$$gw_s''-kw_s = 0 \quad (8)$$

$$w_b^{iv}+(kw_b-gw_b'')H(x,\tau)-p(\tau)\delta(x)-q(\tau)+g(w_b'-w_s')\delta(x-1)H(x=1,\tau)-$$
$$-g(w_b'-w_s')\delta(x+1)H(x=-1,\tau) = -\ddot{w}_b \quad (9)$$

Eq. (8) can be solved very easily for the left hand side of the beam as

$$w_s(x,\tau) = Ce^{-\lambda x} \quad (10)$$

where C represents the integration constant to be determined by using boundary conditions of the problem. The second integration constant in the solution (10) is eliminated by considering that the displacement has to be finite and approaches to zero for larger values of x. Due to the tensionless character of the foundation, the boundary between the contact and lift-off regions is not known in advance and it depends on the parameters of the problem. Therefore the problem is highly non-linear and the closed solution of Eq. (9), including the free end conditions of the beam, is very difficult, if not impossible. In the present study Galerkin's method is adopted for the solution. The displacement function of the beam is assumed to be a linear combination of the axially symmetric free vibration mode shapes of the completely free beam including a rigid vertical translation as follows

$$W_b(X,t) = W_o(X,t) + \sum_{n=1}^{\infty} W_n(X,t) = A w_b(x,\tau) = A\left[T_o(\tau) + \sum_{n=1}^{\infty} T_n(\tau) w_n(x)\right] \tag{11}$$

where $T_o(\tau)$ and $T_n(\tau)$ are the time dependent parameters of the series and $w_n(x)$ are the symmetric free vibration mode shapes of completely free beam defined as

$$w_n(x) = \cosh\lambda_n(1-x) + \cos\lambda_n(1-x) - \frac{\cosh 2\lambda_n - \cos 2\lambda_n}{\sinh 2\lambda_n - \sin 2\lambda_n}[\sinh\lambda_n(1-x) + \sin\lambda_n(1-x)] \tag{12}$$

and λ_n are the roots of the equation

$$\cosh 2\lambda \cos 2\lambda = 1 \tag{13}$$

and the first five roots are

$$\lambda_n = 2.365020,\ 3.926602,\ 5.497803,\ 7.068582,\ 8.639379 \tag{14}$$

which correspond to the symmetric mode shapes. The integration constant C in Eq. (10) can be obtained in terms of $T_o(\tau)$ and $T_n(\tau)$ by using the displacement continuity at the lift-off point. Considering the symmetry of the problem, this can be expressed as

$$W_b(X=B,t) = W_s(X=B,t) \tag{15}$$

or in terms of the non-linear parameters

$$w_b(w=b,\tau) = T_o(\tau) + \sum_{n=1}^{\infty} T_n(\tau) w_n(b) = w_s(w=b,\tau) = Ce^{-\lambda b} \tag{16}$$

This equation is also valid for $b = 1$, when the complete contact develops. By substituting the displacement function (11) into the governing equation of the problem (9) and by using the following identity

$$w_n^{iv} = \lambda_n^4 w_n \tag{17}$$

the following non-dimensional equation is obtained for the unknown parameters $T_o(\tau)$ and $T_n(\tau)$

$$\sum_{n=1}^{\infty} \lambda_n^4 w_n(x) T_n + H(x,\tau)k\left[T_o + \sum_{n=1}^{\infty} w_n(x) T_n\right] - H(x,\tau)g\sum_{n=1}^{\infty} w_n''(x) T_n - p(\tau)\delta(x) - q(\tau) +$$
$$+2g\left[\sum_{n=1}^{\infty} w_n'(x) T_n + \lambda e^{-\lambda(1-b)}\left(T_o + \sum_{n=1}^{\infty} w_n(x) T_n\right)\right]\delta(x-1) H(x=1,\tau) = -\ddot{T}_o - \sum_{n=1}^{\infty} w_n(x)\ddot{T}_n \tag{18}$$

By employing Galerkin's procedure, i.e., by requiring the error in the governing Eq. (18) to be orthogonal to each mode shape including the rigid translation within the definition region of the equation, the following system of ordinary differential equations is obtained:

$$\mathbf{M}\ddot{\mathbf{T}} + \mathbf{K}\mathbf{T} = \mathbf{F} \tag{19}$$

where the dots denote the differentiation with respect to the non-dimensional time τ and

$$\mathbf{T}(\tau) = [T_o\ T_1\ T_2\ T_3\ \ldots]^T \quad diag\ \ \mathbf{M} = [2\ m_1\ m_2\ m_3\ \ldots]$$
$$\mathbf{K}(\tau) = [k_{nm}] \quad \mathbf{F}(\tau) = [f_o\ f_1\ f_2\ f_3\ \ldots]^T$$

$$m_n = 2\int_0^1 w_n(x)w_n(x)dx$$

$$k_{11}(\tau) = k\int_0^1 H(x,\tau)dx + 2g\lambda e^{-\lambda(1-b)}H(x=1,\tau)$$

$$k_{1n}(\tau) = k\int_0^1 H(x,\tau)w_n(x)dx - g\int_0^1 H(x,\tau)w_n''(x)dx + 2gH(x=1,\tau)w_n'(x=1) +$$

$$+2g\lambda H(x=1,\tau)w_n(x=b)e^{-\lambda(1-b)}$$

$$k_{n1}(\tau) = k\int_0^1 H(x,\tau)w_n(x)dx + 2g\lambda H(x=1,\tau)w_n(x=1)e^{-\lambda(1-b)}$$

$$k_{nm}(\tau) = \delta_{nm}\lambda_m^4\int_0^1 w_n(x)w_m(x)dx + k\int_0^1 H(x,\tau)w_n(x)w_m(x)dx - g\int_0^1 H(x,\tau)w_n(x)w_m''(x)dx +$$

$$+2gw_n(x=1)w_m'(x=1)H(x=1,\tau) + 2g\lambda H(x=1,\tau)w_n(x=1)w_m(x=b)e^{-\lambda(1-b)}$$

$$f_o(\tau) = p(\tau) + 2q(\tau) \qquad f_n(\tau) = p(\tau)w_n(x=0) \tag{20}$$

where δ_{nm} denotes Kronecker's delta. Inspection of the equations above justifies once more the use of the Galerkin's approximation, since it is very difficult, if not impossible to find a close solution for the displacement functions of the beam which satisfy all these equations.

The vertical force equilibrium of the beam can be written as follows

$$P(t) + 2AQ(t) = R_K(t) + R_G(t) + R_C(t) + R_I(t) \tag{21}$$

where

$$R_K(t) = 2\int_0^A KW_b(X,t)H(X,t)dX = 2\int_0^B KW_b(X,t)dX$$

$$R_G(t) = -2\int_0^A G\frac{\partial^2 W_b(X,t)}{\partial X^2}H(X,t)dX = -2\int_0^B G\frac{\partial^2 W_b(X,t)}{\partial X^2}dX$$

$$R_C(t) = 2G\left[\frac{\partial W_b(X,t)}{\partial X} - \frac{\partial W_s(X,t)}{\partial X}\right]H(R=A,t)$$

$$R_I(t) = 2\int_0^A M\frac{\partial^2 W_b(X,t)}{\partial t^2}dX \tag{22}$$

$P(t)$ and $Q(t)$ are the external loads, and $R_K(t)$ and $R_G(t)$ correspond to the resultant of the spring stiffness and the membrane stiffness reactions exerted by the foundation proportional to the two stiffness parameters of the foundation K and G, respectively. $R_C(t)$ denotes the foundation edge reaction which is proportional to the difference of the slopes of the foundation surfaces at the two edges of the beam. These two reaction forces develop, when there is a complete contact between the beam and the foundation. As a result of the tensionless character of the two-parameter foundation model, the reaction exerted by the foundation due to the spring and membrane stiffness has to be non-negative, i.e., compression. It means that

$$P_f(X,t) = P_k(X,t) + P_g(X,t) = KW_b(X,t) - G\frac{\partial^2 W_b(X,t)}{\partial X^2} \geq 0 \tag{23}$$

or in terms of the non dimensional parameters

$$p_f(x,\tau) = AP_f(X,t)/(EI) = p_k(x,\tau) + p_g(x,\tau) = kw_b(x,\tau) - gw_b''(x,\tau) \geq 0 \tag{24}$$

Finally, $R_I(t)$ in Eq. (21) corresponds to the resultant of the inertia forces of the beam. By using the assumption for the vertical displacement function $W_b(X,t)$, the force equilibrium in Eq. (21) can be expressed in the following non-dimensional form as follows

$$p(\tau) + 2q(\tau) = r_k(\tau) + r_g(\tau) + r_c(\tau) + r_i(\tau) \tag{25}$$

where

$$r_k(\tau) = \frac{AR_K}{EI} = 2k\int_0^1 \left[T_o + \sum_{n=1}^{\infty} T_n w_n(x)\right] H(x,\tau)dx = 2kbT_o + 2k\int_0^b \sum_{n=1}^{\infty} T_n w_n(x)dx$$

$$r_g(\tau) = \frac{AR_G}{EI} = -2g\int_0^1 \left[\sum_{n=1}^{\infty} T_n w_n''(x)\right] H(x,\tau)dx = -2g\int_0^b \left[\sum_{n=1}^{\infty} T_n w_n''(x)\right] dx$$

$$r_c(\tau) = \frac{AR_C}{EI} = 2g\left[\sum_{n=1}^{\infty} T_n w_n'(x=1) + \lambda e^{-\lambda(1-b)}\left(T_o + \sum_{n=1}^{\infty} T_n w_n(x=b)\right)\right] H(x=1,\tau)$$

$$= 2g\left[\sum_{n=1}^{\infty} T_n w_n'(x=1) + \lambda\left(T_o + \sum_{n=1}^{\infty} T_n w_n(x=1)\right)\right] H(x=1,\tau)$$

$$r_i(\tau) = \frac{AR_I}{EI} = 2\ddot{T}_o \tag{26}$$

and $b(\tau)$ denotes half of the non-dimensional contact length. In the present formulation it is assumed that the foundation can not support tensile reactions and the interaction between the foundation and the beam is only possible, when the reaction under the beam is compressive. In general, a separation takes place to avoid the tensile reactions.

As it can be seen, in addition to the differential equation which governs the problem within the definition region, the continuity of the displacement of the foundation surface at the separation point and at the edge of the beam for the complete contact case (20) are already included into Eq. (26). The edge reactions which develop for the complete contact case are represented in the governing equation of the problem (18).

When a separation takes place in the Winkler foundation model, the foundation displacement is continuous at the point that separates the contact and lift-off regions, whereas no continuity for the slope of the displacement is demanded. Since the vertical foundation reaction is controlled by the displacement solely, the displacement of the lift-off point is zero. On the other hand, a discontinuity in displacements appears at the free end of the beam, when a complete contact develops in case of the Winkler model.

Generally the three anticipated conditions can be stated at the point of separation of a partial contact, they are the continuity of the displacement of foundation, its slope and the zero foundation reaction, as it is the case for the elastic continuum. However, Kerr (1991) pointed out that because of the reduced order of the governing differential equation of the two-parameter foundation model, only two of the three anticipated conditions are to be satisfied as variational analysis yields. For the problem under consideration, continuity of the displacement and vanishing of the foundation reaction are required. An excellent discussion about the boundary conditions involving the two-parameter foundation is given by Kerr (1976). Since a partial contact is in question, no edge reaction exists. Due to the property of the contact function $H(x, \tau)$, the corresponding terms in the governing Eq. (18) and in the reactions (25) will vanish. The contact function also provides that the integrations are evaluated only along the contact region. On the other hand Dirac's delta function in the governing Eq. (18) provides that the concentrated loads $p(\tau)$ at the middle of the beam and at the edges are treated properly in the application of Galerkin's method.

It is worth to note that at least in the present symmetric case a negative value of the beam displacement, i.e., upward displacement guaranties a separation between the beam and the foundation in the two-parameter foundation, as Eq. (1) clearly shows. On the other hand contrary to the Winkler model, a positive value of the beam displacement i.e., a displacement into the foundation, does not always indicate that there is contact at that point between the foundation and the beam, as shown in Fig. 1(a).

Since the mode shapes of the completely free beam is used in the expansion of the beam displacement function, the stiffness matrix **K** will be diagonal, when a conventional Winkler model is assumed. Winkler foundation model is a special case of the two-parameter model for $g = 0$, however the analysis as well as the numerical treatment of the problem can not yield the corresponding result straightforward, due to the definition of $\lambda = k/g$. However, the numerical result having acceptable degree of approximation can be obtained for the Winkler model for $g \to 0$.

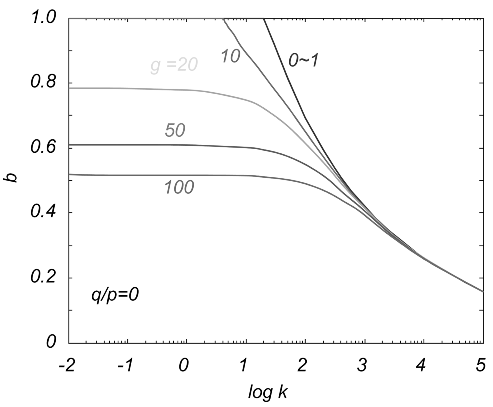
The evaluation of the governing Eq. (19) of the problem involves numerical integration; they are a system of the linear algebraic equations and a system of the linear differential equations, in case of static loads and dynamic loads, respectively, when the complete contact is maintained. However, the coefficients of the governing equations depend on the contact length b between the beam and the foundation in case of a partial contact. In this case the governing equation of the problem (19) becomes highly non-linear due to the tensionless character of the foundation and it requires several iterative procedures for the evaluation of the numerical results, although small displacements for the beam and the foundation are assumed. In this case an iterative solution procedure for b is adopted in the static case. On the other hand, in the dynamic case an iterative solution as in the static case is required to establish as the initial condition of the problem, then the governing equation of the problem is solved in the time domain by updating the contact length continuously.

3. Numerical results and discussion

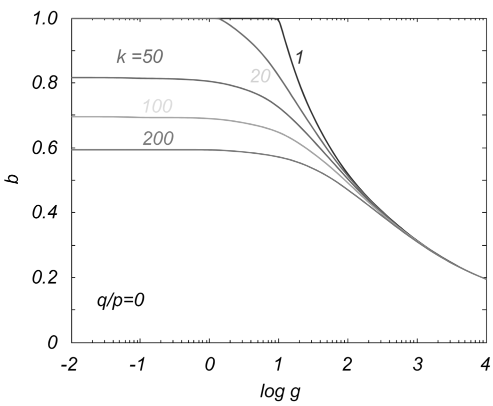
The governing Eq. (19) can be employed for the conventional foundation model by assuming that the full contact is maintained, i.e., $H(x, \tau) = 1$. For a partial contact, an initial value for the contact length is estimated and the elements of the coefficients of the stiffness matrix are evaluated. Eq. (19) can be solved, the displacement functions are obtained. The contact length is evaluated and checked by using Eq. (24). Iterative process is continued until an acceptable approximation is attained. In the dynamic case the initial configuration of the beam evaluated and the governing equation is solved by checking the contact condition (25). The global force equilibrium (24) in the beam is checked in the static case by excluding the inertia force and in the dynamic case at each time step by including the inertia force. The numerical results are obtained and presented in figures by assuming that the beam is subjected to the concentrated load $p(\tau)$ only, i.e., $q = 0$, to keep number of the figures limited.

Figs. 2(a) and 2(b) show the half of the contact length b depending on the spring stiffness k and the membrane stiffness g, respectively. As it is well known, the contact length b, i.e., the position of the lift-off point depends on the foundation stiffnesses k and g, whereas it is independent of the level of the loading. However it depends on the loading ratio q/p, when two types of loadings are involved, as it is the case in the present problem. It means that the contact length does not change, as the loads p and q increase proportionally, whereas the vertical equilibrium is maintained due to increase in vertical displacements proportionally. The beam subjected to a middle load on the tensionless Winkler and Pasternak foundation has been investigated by Celep *et al.* (1989) and Coşkun (2003), respectively. The results presented in Fig. 2 agree very well for $g = 0$ and for $k = 100$ and $g = 1$ with those given in these studies, respectively. Fig. 2(a) shows that the complete contact ($b = 1$) develops for rather low values of the foundation stiffnesses k and g. As the stiffnesses increase, the beam starts to lift off from the foundation and the contact region decreases, consequently the foundation reactions concentrate in a smaller region and get larger. These two figures clearly show that the effect of the membrane stiffness g becomes less pronounced, when the spring stiffness k increases.

Figs. 3(a) and 3(b) show the variations of the resultant of the foundation spring reaction r_k and that of the membrane reaction r_g, as a function of the spring stiffness k for various values of the membrane stiffness g for the beam subjected to a middle load only ($q = 0$). Figs. 3(a) and 3(b) illustrate that the share of the spring reaction and that of the membrane increase as the corresponding stiffnesses get larger. Similarly Fig. 3(c) shows the edge reaction which vanishes in case of partial contact, i.e., for $b < 1$ and comes into being, when the complete contact develops. In these figures, the curves for $g = 10$ seem to display discontinuities, when the beam goes from the complete contact to a partial contact, as the spring stiffness increases. The same is valid other way around, i.e., when the beam goes from a partial contact to the complete contact. In fact these discontinuities are very steep variations, but not an absolute discontinuity in the mathematical sense. For larger values of the membrane stiffness g, the discontinuity gets larger, since the edge force is



(a)



(b)

Fig. 2 Variation of the contact length b for various (a) spring stiffness k and (b) membrane stiffness g for the beam subjected to the load p at the middle

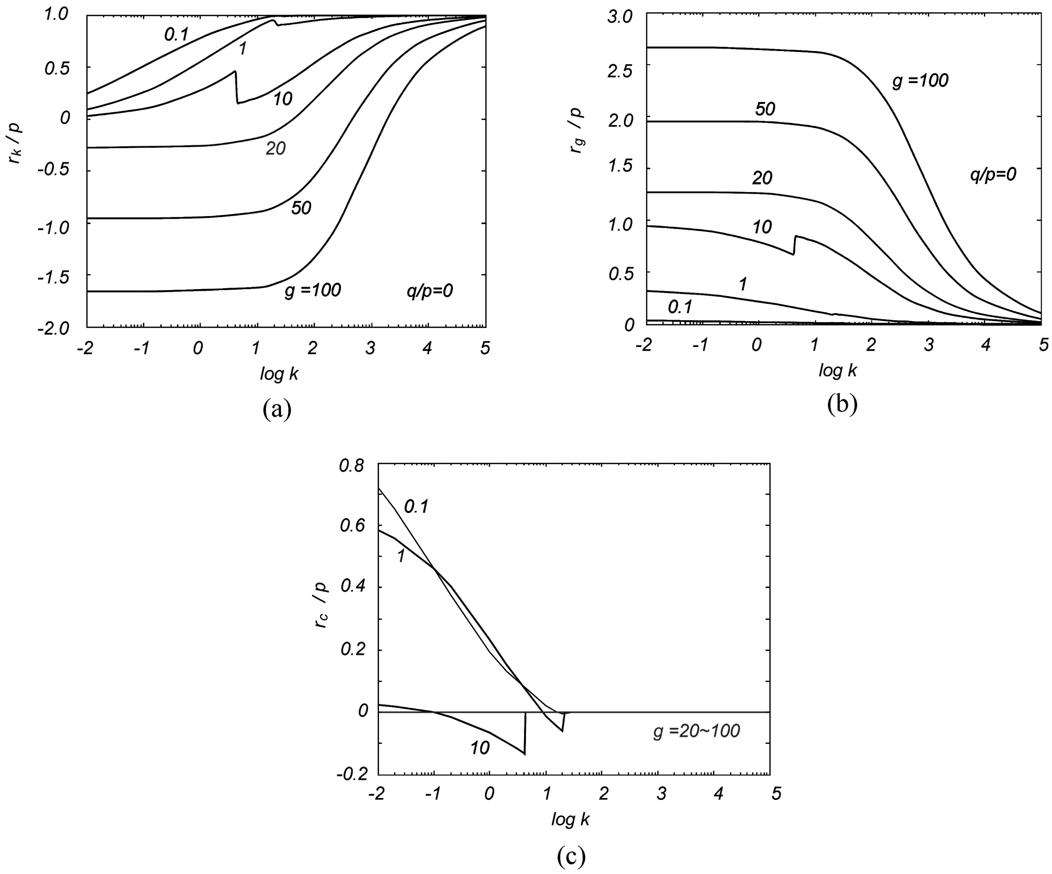


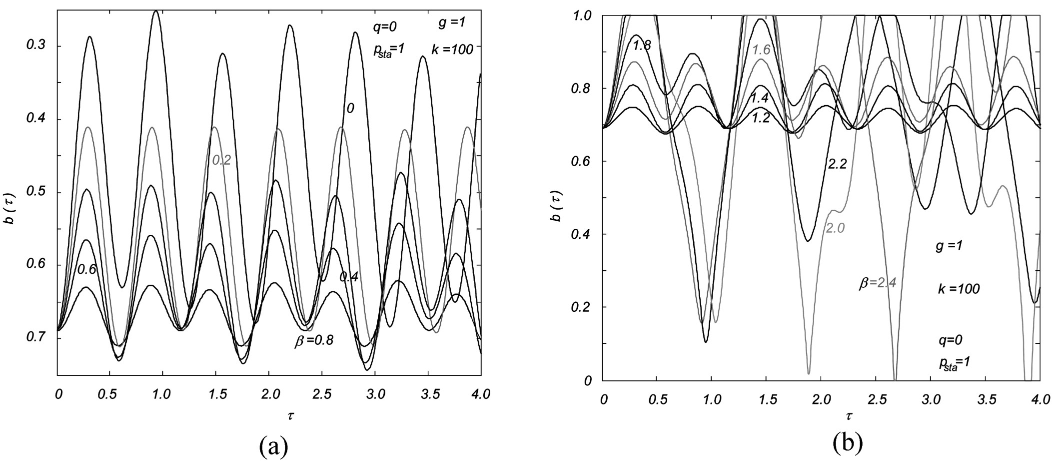
Fig. 3 Variation of (a) the total spring stiffness reaction r_k, (b) the membrane stiffness reaction r_g and (c) the total edge reactions r_c for the beam subjected to the load p at the middle

directly affected by the membrane stiffness. However, no steep variation takes place, when only one type of the contact (partial or complete) appears along a curve in Figs. 3(a) and 3(b). Fig. 3(c) shows the edge membrane reaction r_c which is generated, when the complete contact develops. The edge reaction is proportional to the difference of the slopes at the two sides of the beam edge and to the foundation parameter g. The edge reaction r_c decreases and becomes zero, when the complete contact develops. Comparison of Figs. 2(a) and 3(c) shows that there is no edge reaction exists, when a partial contact comes into being, as expected. Since the sum of the all reactions has to be equal to the external load, an increase in one of the foundation reactions results in a decrease in the other foundation reactions, as Eq. (26) expresses.

When the external force is time depended, oscillations of the beam will take place and the contact region will depend on time. Although the present formulation does not have any restriction concerning the time variation of the loads, in the present study numerical solution of the governing Eq. (19) is carried out for the forced vibrations by assuming that the beam is in static equilibrium under the loading $p_{sta} = 1$ and $q = 0$. Oscillations of the beam starts by changing the loading level to βp_{sta} instantly by using the dynamic load factor β. The time variation of the loading can be written as $p_{sta}+(\beta-1)p_{sta}H(\tau)$, where $H(\tau)$ denotes Heaviside step function. The governing

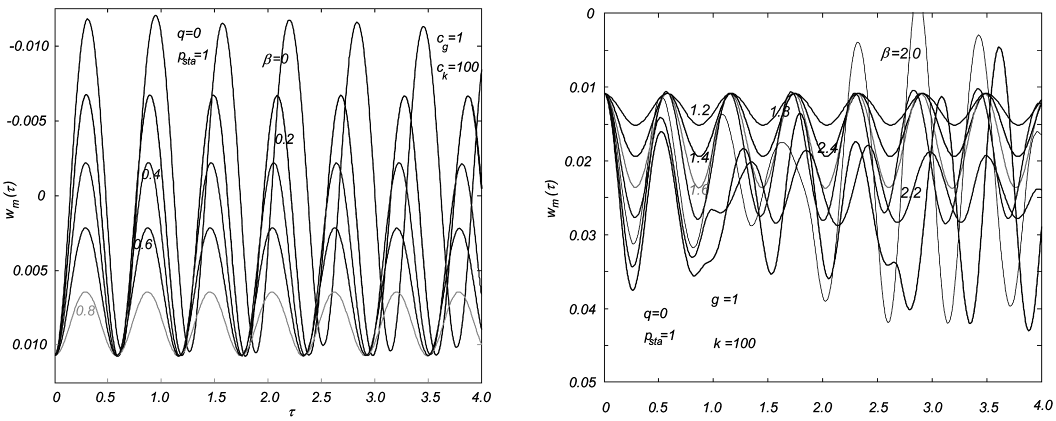
Eq. (26) is a system of non-linear ordinary differential equations, because the coefficients of the stiffness matrix **K** have time dependent terms. When the beam is partially uplifted, the coefficients depend continuously on the vertical displacements of the beam on the contact area. When a partial contact develops, the solution of the static case or the initial configuration of the beam in the dynamic case is found by using an iterative procedure. The governing differential Eq. (26) is solved along the time domain by employing a step-wise numerical integration in the dynamic case. At each time step the contact function $H(x, \tau)$, the contact length $b(\tau)$ and the parameters of the problem including the coefficients of the stiffness matrix are evaluated numerically and updated by considering the displacement configuration of the beam at the previous time step. For recognition of the static and dynamic response of the beam, numerous results are evaluated for selected parameters and they are presented in figures.

Figs. 4(a) and 4(b) show oscillations of the half of the contact length $b(\tau)$ for $g = 1$ and $k = 100$ by assuming $p_{sta} = 1$ and $q = 0$, i.e., only a concentrated load is present for various unloading $\beta < 1$ and $\beta > 1$ loading cases, respectively. As mentioned above, the contact length does not depend on the loading in the static case; however its time variation depends on the level of the loading, since the inertia forces are involved in the dynamic cases. As Fig. 4(a) shows the contact length $b(\tau)$ experiences oscillation which resembles to harmonic variations, since the complete contact does not develop in these unloading cases. However, when the complete contact develops, the time variation of $b(\tau)$ represents highly nonlinear nature, as Fig. 4(b) illustrates. Similar variations can be seen in Figs. 5 and 6, where the time variations of the middle displacement $w_m(\tau) = w(x = 0, \tau)$ and the edge displacement $w_e(\tau) = w(x = \pm 1, \tau)$ are shown. Nonlinear variations appear in all these figures, when the beam goes from a partial contact to the complete contact in course of oscillations and vice versa, which appears in the loading for the numerical values of the parameters used. Figs. 7(a), 7(b), 7(c) and 7(d) represent time variations of the resultant of the spring stiffness force $r_k(\tau)$, the membrane stiffness force $r_g(\tau)$, the edge force $r_c(\tau)$, the inertia force $r_i(\tau)$ and the total force $r_t(\tau)$ of the beam subjected to a load $p_{sta} = 1$ at the middle for $\beta = 0.2, 1.4, 2.0$ and 2.4 assuming $k = 100$, respectively. As Figs. 7(a) and 7(b) show, time variations of these parameters resemble to harmonic oscillations for $\beta = 0.2$ and 1.4, respectively, where the complete contact does not develops and no



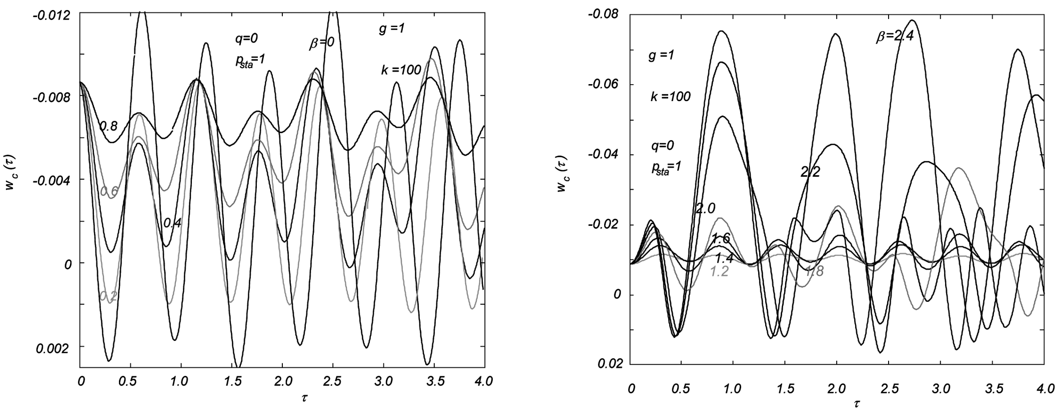
(a) (b)

Fig. 4 Time variation of the contact length $b(\tau)$ of the beam subjected to the load $p_{sta} = 1$ at the middle for (a) unloading $\beta < 1$ and (b) loading $\beta > 1$ cases for $k = 100$ and $g = 1$



(a) (b)

Fig. 5 Time variation of the middle displacement $w_m(\tau) = w(x=0, \tau)$ of the beam subjected to the load $p_{sta} = 1$ at the middle for (a) unloading $\beta < 1$ and (b) loading $\beta > 1$ cases for $k = 100$ and $g = 1$



(a) (b)

Fig. 6 Time variation of the edge displacement $w_c(\tau) = w(x=\pm 1, \tau)$ of the beam subjected to the load $p_{sta} = 1$ at the middle for (a) unloading $\beta < 1$ and (b) loading $\beta > 1$ cases for $k = 100$ and $g = 1$

edge force comes into being, i.e., $r_c = 0$ for $b(\tau) < 1$ and $r_c \neq 0$ for $b(\tau) = 1$. However, the nonlinear character of the problem appears for $\beta = 2.0$ and 2.4, where partial and complete contact develop after one another, as the beam oscillates. The time variation of the total force $r_t(\tau)$ satisfies the global force equilibrium $r_t(\tau) = r_k(\tau) + r_g(\tau) + r_c(\tau) + r_i(\tau) = \beta p_{sta}$ where the resultant of the inertia force included as well. However, it is interesting to note that the edge force $r_c(\tau)$ illustrates very step variation in a very small time interval, as if it is a discontinuity. Consequently the time variation of the total force $r_t(\tau)$ displays a very sudden small change, however it is corrected in a very short time interval, as Figs. 7(c) and 7(d) show.

Fig. 8(a) shows oscillations of the half of the contact length $b(\tau)$ for $g = 1$ and $k = 100$ by assuming the beam being at rest for $\tau = 0$ is subjected to a harmonically varying concentrated load $p(\tau) = p_o \sin\overline{\omega}\tau$, $q(\tau) = 0$ and $p_o = 1$ for various values of the nondimensional circular frequency

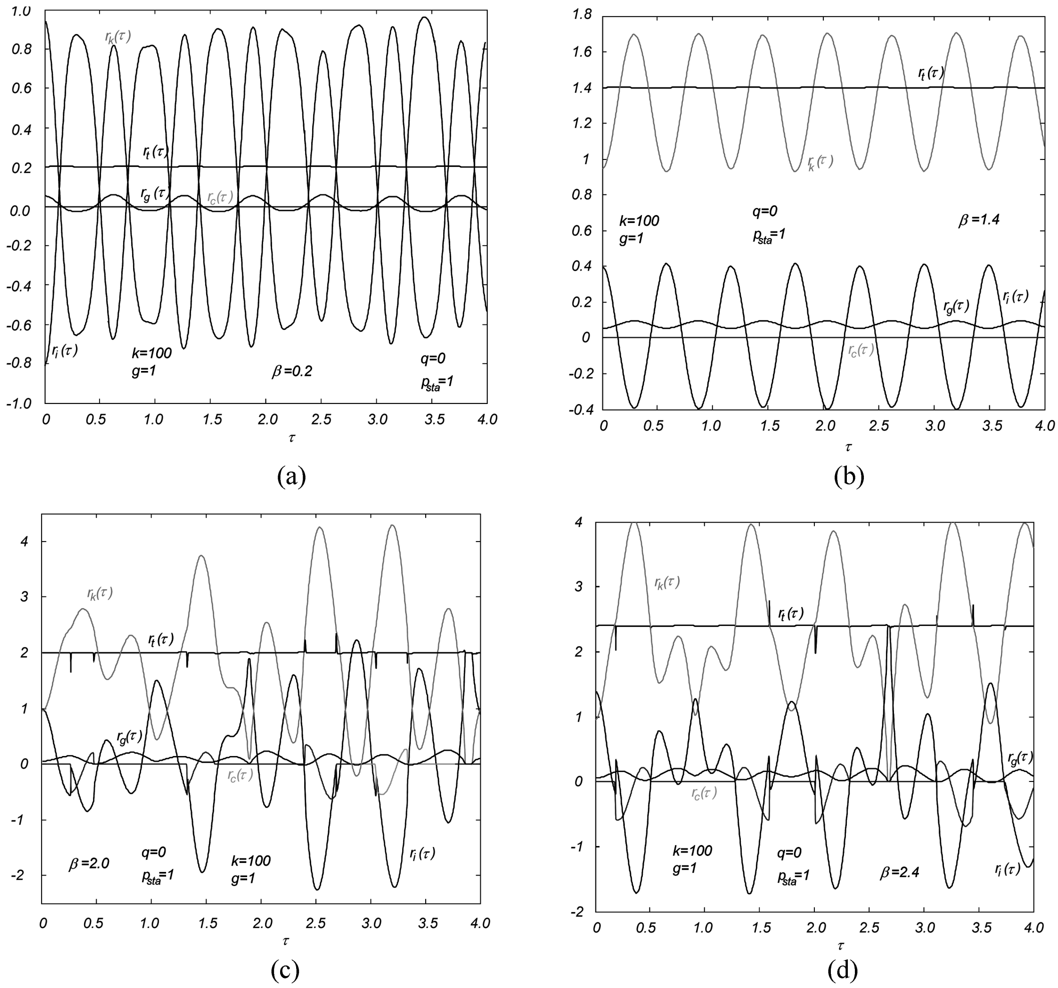
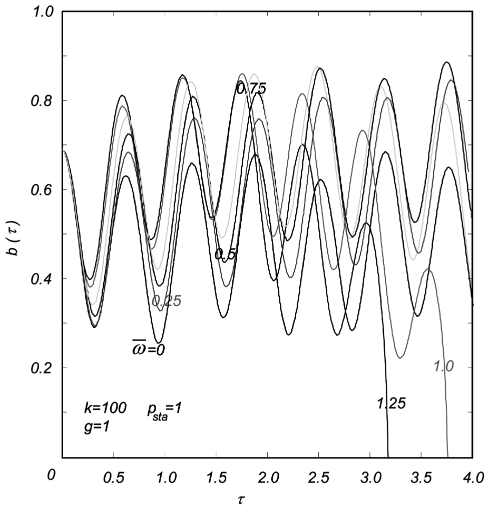
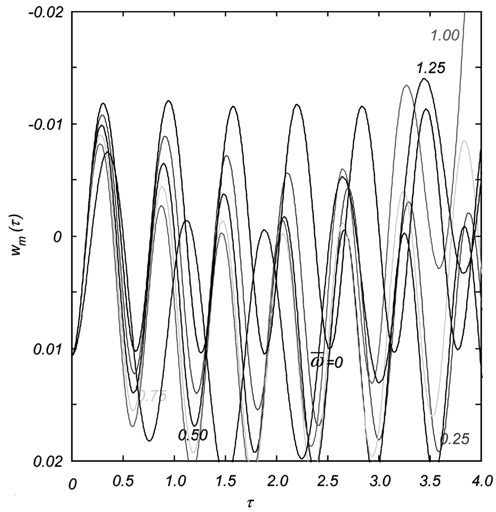


Fig. 7 Time variations of the resultant of the spring stiffness force $r_k(\tau)$, the membrane stiffness force $r_g(\tau)$, the edge force $r_c(\tau)$, the inertia force $r_i(\tau)$ and the total force $r_t(\tau)$ of the beam subjected to the load $p_{sta} = 1$ at the middle for (a) unloading $\beta = 0.2$, (b) loading $\beta = 1.4$, (c) loading $\beta = 2.0$ and (d) loading $\beta = 2.4$ for $k = 100$ and $g = 1$

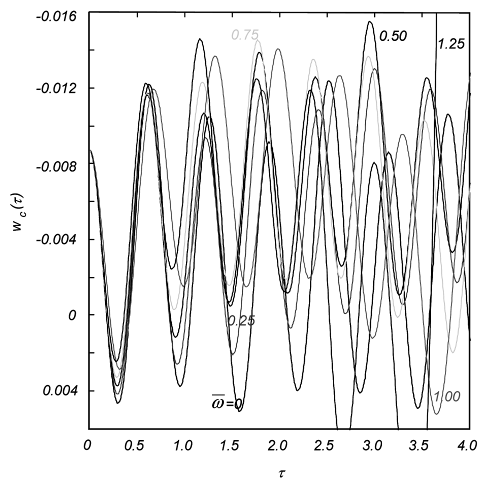
$\overline{\omega}$. The time variations of the middle displacement $w_m(\tau) = w(x = 0, \tau)$ and the edge displacement $w_c(\tau) = w(x = \pm 1, \tau)$ are illustrated in Figs. 8(b) and 8(c) as well. The curves in the figures present very sophisticated variations due to the nonlinearity of the problem. However, two types of nonlinear oscillation can be seen. The first one is controlled by the harmonic variation of the external load having a period of $2\pi/\overline{\omega}$ and the other one is an oscillation which develops due to the elastic response of the beam and the foundation which has an approximate period of 0.6 for the present numerical values. Since these oscillations are heavily nonlinear, periods can not be defined as it is defined in the linear harmonic analysis. Very similar analysis is carried out by assuming that the contact length b is constant and does not depend on time by Coşkun (2003), although the beam is subjected to a harmonically varying concentrated load, where the analysis is focused on the dependency of the contact length on the frequency of the external load mainly.



(a)



(b)



(c)

Fig. 8 Time variation of (a) the contact length $b(\tau)$, (b) the middle displacement $w_m(\tau) = w(x=0, \tau)$ and (c) the edge displacement $w_c(\tau) = w(x=\pm 1, \tau)$ of the beam subjected to the load $q(\tau) = 0$ and $p(\tau) = p_o \sin\overline{\omega}\tau$ where $p_o = 1$ for $k = 100$ and $g = 1$

It is worth to note that the present analytical treatment is valid, when there is only one contact zone in the middle of the beam where the foundation pressure is positive

$$p_f(x, \tau) = kw_b(x, \tau) - gw_b''(x, \tau) \geq 0 \quad \text{for} \quad 0 \leq b(\tau) \leq 1 \tag{27}$$

4. Conclusions

The study presents analysis of the lift-off problem of an elastic beam resting on a two-parameter foundation and subjected to a concentrated force at the middle and a uniformly distributed load. Special attention is paid to the non-dimensionlization of the formulation as well as on the boundary conditions of the beam and the foundation. In order to cover a large spectrum of values of the parameters, the analysis and the numerical results are presented by introducing non-dimensional parameters. Although the displacements of the beam and the foundation are assumed to be small, the governing equation of the problem is non-linear, due to tensionless character of the foundation. The tensile reaction between the beam and the foundation is avoided, as the beam lifts off the foundation. Numerical solution of the problem is carried out by applying Galerkin's method and the numerical results are presented comparatively for various values of the parameters of the problem. The global force equilibrium of the beam in the vertical direction is formulated by including the foundation reactions, the edge reactions and the inertia forces and it is checked numerically in all static and dynamic loading cases. From the numerical analysis presented, the following conclusion can be drawn:

a. In the present analysis for the separation condition between the beam and the foundation is determined by requiring the total foundation reaction which constitutes due to the spring stiffness and to the membrane stiffness of the foundation to be vanished. Due to the shortcoming of the model, the slope of the foundation surface displays a discontinuity contrary to the intuitive approach. However, due to the requirement at the separation point, the continuity of the foundation reaction is guarantied.

b. As it is obvious, separation conditions are not used, when the complete contact is established. However, in this case an edge reaction develops as a result of discontinuity of the slope of the displacement function at that point. In the present formulation and in its numerical evaluation the edge reaction is included into the governing equation of the problem; it is not treated as a boundary condition.

c. The uplift of the beam is influenced mainly by the fundamental mode and the higher modes have lesser effect on the behavior.

d. It is well known that the contact length does not depend on the loading in the static case when only one type of loading is considered. When two types of loading is involved, as it is the case in the present study, the contact length depends on the ratio of the loading. However, when the beam is subjected to dynamic loads, then the time variation of the contact length depends on the level of the loading even for a single loading, because the inertia forces are involved in the dynamic case.

e. Due to the tensionless character of the foundation, oscillations of the beam subjected to dynamic loads are highly nonlinear and no period can be defined, particularly when partial and complete contacts develop subsequently during oscillations.

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