Structural Engineering and Mechanics, Vol. 36, No. 5 (2010) 545-560

DOI: http://dx.doi.org/10.12989/sem.2010.36.5.545

Thermal post-buckling analysis of uniform slender functionally graded material beams

K. Sanjay Anandrao*[1], R.K. Gupta[1a], P. Ramchandran[2a] and G. Venkateswara Rao[3b]

[1]Advanced Systems Laboratory, Kanchanbagh, Hyderabad - 500 058, India
[2]Defense Research and Development Laboratory, Kanchanbagh, Hyderabad - 500 058, India
[3]School of Mechanical Engineering, Sreenidhi Institute of Science and Technology, Hyderabad - 501 301, India

(Received February 22, 2010, Accepted July 16, 2010)

Abstract. Two or more distinct materials are combined into a single functionally graded material (FGM) where the microstructural composition and properties change gradually. Thermal post-buckling behavior of uniform slender FGM beams is investigated independently using the classical Rayleigh-Ritz (RR) formulation and the versatile Finite Element Analysis (FEA) formulation developed in this paper. The von-Karman strain-displacement relations are used to account for moderately large deflections of FGM beams. Bending-extension coupling arising due to heterogeneity of material through the thickness is included. Simply supported and clamped beams with axially immovable ends are considered in the present study. Post-buckling load versus deflection curves and buckled mode shapes obtained from both the RR and FEA formulations for different volume fraction exponents show an excellent agreement with the available literature results for simply supported ends. Response of the FGM beam with clamped ends is studied for the first time and the results from both the RR and FEA formulations show a very good agreement. Though the response of the FGM beam could have been studied more accurately by FEA formulation alone, the authors aim to apply the RR formulation is to find an approximate closed form post-buckling solutions for the FGM beams. Further, the use of the RR formulation clearly demonstrates the effect of bending-extension coupling on the post-buckling response of the FGM beams.

Keywords: Rayleigh-Ritz; finite element analysis; post-buckling; functionally graded materials; load-deflection curves; geometric non-linearity; closed form solution.

1. Introduction

In FGM, two or more distinct materials are combined. The microstructural composition and properties of the constituent materials change gradually through the thickness. The continuous change in microstructural composition of FGM's is determined by a particular distribution of the distinct materials. The choice of constituent materials in FGMs is governed by the functional requirements at the two surfaces of the structure.

*Corresponding author, Scientist, E-mail: sanjayanandkhalane@gmail.com
[a]Scientist
[b]INAE Distinguished Professor and Director

The metal-ceramic FGM components can be widely used in aerospace, nuclear and other specific engineering applications where the structures are subjected to severe thermal loads. Structural members such as uniform beams are the basic components of these structures. It is well known that a linear bifurcation buckling analysis gives a critical value of the thermal loading for a particular isotropic homogenous beam with the prescribed boundary conditions. However, beams (or other structural members) are capable of carrying considerable additional load before the collapse load is reached. Theoretical analysis of post-buckling behavior of beams is non-linear (geometric) and is due to the additional axial strains and stresses caused by the transverse displacements. It is essential to have an in-depth understanding of the post-buckling behavior of the structural members to effectively utilize the additional load carrying capability if the large deflections do not interfere with the functional requirement of the structure.

Buckling aspects of the structures such as isotropic homogenous beams, plates and shells are exhaustively reported by Timoshenko and Gere (1970). Julien *et al.* (2008) carried out the thermal post-buckling analysis of the FGM beams through an analytical model to determine the deflection of a simply supported beam with axially immovable ends for temperatures ranging from pre-buckling to post-buckling temperatures. Han *et al.* (2008) carried out the buckling analysis of the FGM plates and shells using a four noded quasi-conforming shell finite element. Thermal buckling and nonlinear flutter behavior of the FGM panels has been reported by Ibrahim *et al.* (2007). An elasticity solution for the FGM beams is proposed by Sankar (2001), wherein the Young's modulus of the beam is assumed to vary exponentially through the thickness, using a simple Euler-Bernoulli type beam theory. Zhong and Yu (2007) derived an analytical solution of a cantilever FGM beam by presenting a general solution in terms of Airy's stress function. Li (2008) presented a new unified approach for analyzing static and dynamic behavior of the FGM beams considering the rotary inertia and shear deformation and the material properties as arbitrary functions along the thickness. Deschilder *et al.* (2006) carried out non-linear static analysis of a FGM beam. Kitipornchai *et al*. (2009) studied nonlinear vibration of edge cracked functionally graded Timoshenko beams. They used polynomial admissible functions and employed Ritz method to derive the governing eigenvalue equation which is solved by direct iterative method. Lee and Kim (2007) carried out study on thermal stability boundary of FG panel under aerodynamic load. Prakash *et al*. (2006) investigated axisymmetric free flexural vibrations and thermal stability behaviors of functionally graded caps using a three noded axisymmetric curve shell element based on field consistency approach. Jabbari *et al*. (2008) presented an analytical method to obtain the transient thermal and mechanical stresses in a functionally graded hollow cylinder subjected to two dimensional asymmetric loads. They solved the Navier's equations using a direct method of series expansion.

Thermal post-buckling analysis of slender homogenous isotropic columns (beams) using Galerkin finite element formulation is reported by Rao *et al.* (1997). Rao and Raju (1984, 2002) also studied thermal post-buckling of homogenous columns using the Rayleigh-Ritz method. In these studies, the linear critical thermal load and the corresponding non-linear thermal load parameters have been reported for simply supported, clamped, and simply supported-clamped columns. Rao *et al.* (2002, 2003) also presented a simple intuitive method to study thermal post-buckling behavior of uniform columns. Raju *et al*. (2005) used multi-term polynomial admissible functions to study the large amplitude free vibrations of clamped-clamped and pinned-clamped beams using the conservation of total energy principle.

Thermal post-buckling of the FGM beams has been studied earlier using the governing differential equations, wherein, the results for the simply supported end conditions are only reported. It is to be noted here that the effect of the bending-extension coupling for the different boundary conditions of the FGM beams is also not available in the literature. Finite element analysis, although more versatile and accurate, being a numerical method, does not give an elegant closed form solution for the post-buckling behavior of the FGM beams with different boundary conditions and the beam parameters. In this paper, in addition to the versatile finite element analysis, the authors have applied the classical Rayleigh-Ritz method not only to obtain approximate closed form expressions to predict the thermal post-buckling behavior of the FGM beams with axially immovable ends but also to study the effect of the bending - extension coupling. The aim of the present study is to obtain approximate solutions to the thermal post-buckling behavior of the FGM beams, hither to not attempted, using the classical Rayleigh-Ritz method with one-term approximations for axial and transverse deflections that are exact for the isotropic and homogenous beams. Based on the present results, the effect of the bending - extension coupling for both the boundary conditions considered is also clearly demonstrated.

The variation of material properties of the FGM beams is heterogeneous through the thickness. This asymmetry with respect to the beam axis results in the coupling of bending and extensional deformation modes and is included in this study through a coupling matrix. The inclusion of the coupling matrix distinguishes the stability analysis of FGM beams from the conventional homogenous material beams which do not have bending-extension coupling. In the present study the Euler-Bernoulli beam theory is considered for both the RR and FEA formulations. In the finite element formulation, a beam element with two nodes having three degrees of freedom at each node is considered. The governing algebraic non-linear equations are obtained using the principle of virtual work. The Newton-Raphson iterative procedure is used to solve these algebraic non-linear equations. The results obtained from both these formulations are compared with those available in the literature for the simply supported beam. The results for the clamped beam are not readily available in the literature and are reported for the first time using both the formulations developed in this paper. The strain-displacement relations based on von-Karman type non-linearity are used to account for the large deflections. Simple one-term admissible functions for the axial and transverse deflections are assumed that satisfy the kinematic boundary conditions at the two ends of the homogenous beam (Timoshenko and Gere 1970, Rao and Raju 2002, 2003). The Young's modulus and the coefficient of thermal expansion are assumed to vary according to a power law distribution (Ibrahim 2007, Prakash and Ganapathi 2006, Lee and Kim 2007) across the thickness of the beam. The simply supported and clamped beams with axially immovable ends are analyzed and the numerical results are provided to show the effect of the volume fraction exponent on the post-buckling behavior and buckled mode shapes.

2. Functionally Graded Material beam

A FGM beam with ceramic on top face and metal on bottom face is considered in this study as shown in Fig. 1. The variation of properties E and α is governed by a power law distribution, as given in Eq. (1), with the co-ordinate z varying between $-h/2$ to $h/2$. The volume fraction exponent n can take any value between 0 to ∞, with $n = 0$ and $n = \infty$ corresponding to the two extremes of completely homogenous ceramic and aluminum beams respectively.

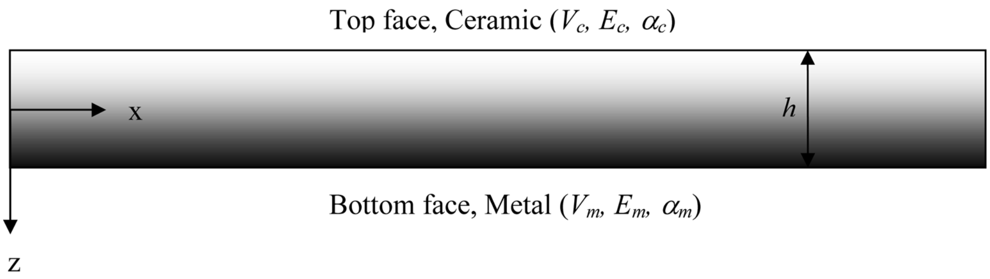

Fig. 1 FGM beam with ceramic and metal as constituents

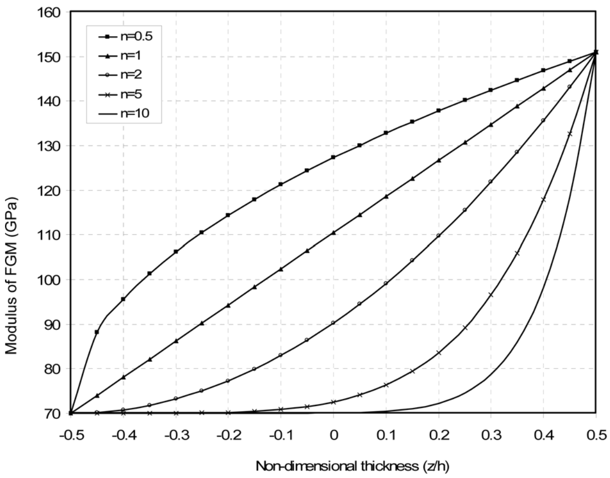

Fig. 2 Variation of Young's modulus across thickness of FGM beam

$$
\begin{aligned}
E(z) &= E_c V_c + E_m(1 - V_c) \\
\alpha(z) &= \alpha_c V_c + \alpha_m(1 - V_c) \\
V_c &= \left(0.5 + \frac{z}{h}\right)^n
\end{aligned}
\tag{1}
$$

Fig. 2 shows the typical variation of Young's Modulus E for the FGM beam across the thickness for different values of n.

3. Formulation

In this section, equations for the extensional, bending and bending-extension coupling stiffness for the FGM beam are established. The expression for equivalent mechanical load developed in the beam due to temperature rise t from the initial stress free temperature is derived. In the present formulation, a slender FGM beam is considered so that the effect of Poisson ratio and transverse shear on the deformation of the beam can be neglected.

3.1 Non-linear strain-displacement relation

The strain-displacement relations considering von Kármán type geometric non-linearity are given by

$$\varepsilon_{xx}^{0} = \frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2$$

$$\kappa_{xx} = -\frac{d^2 w_0}{dx^2}$$

$$\varepsilon_{xx} = \varepsilon_{xx}^{0} + z\kappa_{xx} \tag{2}$$

3.2 Stress-strain equation

$$\sigma_{xx} = E(z)\varepsilon_{xx} \tag{3}$$

3.3 Stress and moment resultant-displacement relations

The stress and moment resultants in the axial direction 'x' can be expressed as

$$N_{xx} = A_{xx}\varepsilon_{xx}^{0} + B_{xx}\kappa_{xx}$$

$$M_{xx} = B_{xx}\varepsilon_{xx}^{0} + D_{xx}\kappa_{xx} \tag{4}$$

where A_{xx}, B_{xx} and D_{xx} are extensional stiffness, bending-extension coupling stiffness and bending stiffness respectively given by

$$A_{xx} = \int_{-h/2}^{h/2} E(z)dz$$

$$B_{xx} = \int_{-h/2}^{h/2} E(z)zdz$$

$$D_{xx} = \int_{-h/2}^{h/2} E(z)z^2 dz \tag{5}$$

The homogenization integration of material properties in z-direction to establish stiffness is the standard procedure well reported in the literature for FGM beams and is similar to the classical lamination theory. The equivalent mechanical load P developed in the beam due to temperature rise t from initial stress free temperature, for unit width of beam is given by

$$P = \int_{-h/2}^{h/2} E(z)\alpha(z)t\,dz \tag{6}$$

where both E and α are assumed to be independent of temperature.

3.4 Rayleigh-Ritz technique

In the present study, accurate kinematically admissible one term trigonometric functions for the displacement variables, based on the available literature (Timoshenko 1970, Rao *et al.* 2002, 2003) for the homogenous beams are used, to get an approximate solution for the FGM beams. Trigonometric functions are often convenient to use as the orthogonality properties of these series simplify the solution process. In the case of FGM beams, the values of volume fraction exponent $n = 0$ and n tends to infinity represent homogenous beam and the one term trigonometric functions used in the present work are exact mode shapes of buckling for the homogeneous beams.

The total potential energy PE of a beam while in equilibrium in a displaced buckling mode is given by Turvey and Lih (1995)

$$PE = V_S + V_B + V_{BS} + V_L \tag{7}$$

where

$$V_S = \frac{1}{2}\int_0^L A_{xx}\left(\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right)^2 dx$$

$$V_{BS} = -\frac{1}{2}\int_0^L 2B_{xx}\left(\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right)\left(\frac{\partial^2 w_0}{\partial x^2}\right) dx$$

$$V_B = \frac{1}{2}\int_0^L D_{xx}\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 dx$$

$$V_L = -\frac{1}{2}\int_0^L P\left(\frac{\partial w_0}{\partial x}\right)^2 dx \tag{8}$$

In the Rayleigh-Ritz method, the total potential energy PE is minimized with respect to the undetermined coefficients of the assumed admissible functions for w and u.

3.4.1 Simply supported beam

For a simply supported homogenous beam with axially immovable ends, exact transverse deflection w satisfying the kinematic and natural boundary conditions, namely

$$w = 0 \text{ at } x = 0, L \text{ and } M = 0 \text{ at } x = 0, L$$

is given by

$$w = a\sin\left(\frac{\pi x}{L}\right) \tag{9}$$

where a is the undetermined coefficient and also represents the central transverse deflection. The admissible function for the axial displacement u of the homogenous beam, satisfying the conditions, namely

$$u = 0 \text{ at } x = 0, \frac{1}{2}, L \text{ is}$$

$$u = b\sin\left(\frac{2\pi x}{L}\right) \tag{10}$$

where b is another undetermined coefficient.

The total potential energy PE from Eq. (7) can be expressed as

$$PE = \frac{A_{xx}}{2}\int_0^L\left(\frac{2\pi b}{L}\cos\left(\frac{2\pi x}{L}\right)+\frac{1}{2}\left(\frac{a\pi}{L}\cos\left(\frac{\pi x}{L}\right)\right)^2\right)^2 dx$$

$$+B_{xx}\int_0^L\left(\frac{2\pi b}{L}\cos\left(\frac{2\pi x}{L}\right)+\frac{1}{2}\left(\frac{a\pi}{L}\cos\left(\frac{\pi x}{L}\right)\right)^2\right)\left(\frac{-a\pi}{L^2}\sin\left(\frac{\pi x}{L}\right)\right)dx$$

$$+\frac{D_{xx}}{2}\int_0^L\left(\frac{-a\pi^2}{L^2}\sin\left(\frac{\pi x}{L}\right)\right)^2 dx-\frac{P}{2}\int_0^L\left(\frac{a\pi}{L}\cos\left(\frac{\pi x}{L}\right)\right)^2 dx \tag{11}$$

Minimizing the total potential energy PE with respect to the undetermined coefficients a and b of the assumed admissible functions, the coefficient a of the transverse deflection of the simply supported beam is obtained as

$$a = \left(\frac{-1}{3\pi^2 A_{xx}}\right)(-18\pi B_{xx}-2(97\pi^2 B_{xx}^2+9\pi^2 PL^2 A_{xx}-9\pi^4 D_{xx}A_{xx})^{1/2}) \tag{12}$$

The above equation gives the relation between temperature rise t (governing the equivalent mechanical load P) and central lateral deflection a of beam. Eq. (12) can also be expressed in a more convenient form as

$$P_{NL} = \frac{a^2 A_{xx}\pi^2}{4L^2}-\frac{3B_{xx}\pi a}{L^2}+\frac{D_{xx}\pi^2}{L^2} \tag{13}$$

For a beam of homogenous material through the thickness such as completely metal or ceramic, the bending - extension coupling B_{xx} will vanish. A_{xx} and D_{xx} can be expressed as EA and EI respectively and the thermal post-buckling load can be expressed as

$$P_{NL} = \frac{a^2 EA\pi^2}{4L^2}+\frac{EI\pi^2}{L^2} \tag{14}$$

Further, substituting $a = 0$ in Eq. (14) gives critical Euler buckling load $P_L = \pi^2 EI/L^2$ for simply supported homogenous and isotropic beam. The non-linear thermal post-buckling load P_{NL} from Eq. (14) can be normalized with respect to the Euler buckling load P_L and this ratio is

$$\frac{P_{NL}}{P_L} = 1+0.25\frac{a^2}{r^2} \tag{15}$$

This is a well known expression for the thermal post-buckling of the simply supported homogenous beam (Rao and Raju 2002, 2003). Comparing Eq. (13) and Eq. (14) for the thermal post-buckling of the FGM beam and homogenous beam, it can be concluded that, the bending-extension coupling 'B_{xx}' will influence the response of the FGM beam from that of the homogenous beam.

3.4.2 Clamped beam

For a beam with the clamped ends, the transverse deflection *w,* satisfying the boundary conditions, namely

$$w = 0 \text{ at } x = 0, L$$
$$\frac{\partial w}{\partial x} = 0 \text{ at } x = 0, L$$

is

$$w = \frac{a}{2}\left[1 - \cos\left(\frac{2\pi x}{L}\right)\right] \tag{16}$$

where a is the undetermined coefficient and also represents the central transverse deflection. The admissible function for the axial displacement u satisfying the conditions, namely

$$u = 0 \text{ at } x = 0, \frac{L}{4}, \frac{L}{2}, \frac{3L}{4}, L$$

is

$$u = b\sin\left(\frac{4\pi x}{L}\right) \tag{17}$$

The total potential energy PE for the clamped beam, from Eq. (7) can be expressed as

$$PE = \frac{A_{xx}}{2}\int_0^L\left(\frac{4\pi b}{L}\cos\left(\frac{4\pi x}{L}\right) + \frac{1}{2}\left(\frac{\pi a}{L}\sin\left(\frac{2\pi x}{L}\right)\right)^2\right)^2 dx$$
$$+ B_{xx}\int_0^L\left(\frac{4\pi b}{L}\cos\left(\frac{4\pi x}{L}\right) + \frac{1}{2}\left(\frac{\pi a}{L}\sin\left(\frac{2\pi x}{L}\right)\right)^2\right)\left(\frac{2a\pi^2}{L^2}\cos\left(\frac{2\pi x}{L}\right)\right)dx$$
$$+ \frac{D_{xx}}{2}\int_0^L\left(\frac{2a\pi^2}{L^2}\cos\left(\frac{2\pi x}{L}\right)\right)^2 dx - \frac{P}{2}\int_0^L\left(\frac{a\pi}{L}\sin\left(\frac{2\pi x}{L}\right)\right)^2 dx \tag{18}$$

Minimizing the total potential energy PE with respect to a and b and equating it to zero gives

$$a = \frac{2}{\pi A_{xx}}(A_{xx}(-4\pi^2 D_{xx} + PL^2))^{1/2} \tag{19}$$

Comparing Eq. (19) with that of the simply supported FGM beam (Eq. (12)) reveals that for the clamped FGM beam, there is no effect of bending-extension coupling (B_{xx}) on the lateral deflection of the beam. Eq. (19) can also be expressed in a more convenient form as

$$P_{NL} = \frac{4D_{xx}\pi^2}{L^2} + \frac{a^2 A_{xx}\pi^2}{4L^2} \tag{20}$$

Eq. (20) gives the thermal post buckling load of the FGM beam and is the same as that of the homogenous beam (completely metal or ceramic beam) as given below.

$$P_{NL} = \frac{a^2 EA\pi^2}{4L^2} + \frac{4EI\pi^2}{L^2} \tag{21}$$

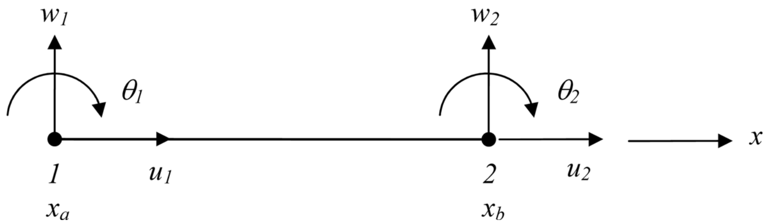

Fig. 3 Finite element model of an Euler Bernoulli beam

Further, substituting $a = 0$ in Eq. (21) gives Euler buckling load $P_L = 4\pi^2 EI/L^2$ for the clamped homogenous beam. The thermal post-buckling load P_{NL} from Eq. (21) can be normalized by P_L and this ratio can be expressed as Rao and Raju (1984, 2002)

$$\frac{P_{NL}}{P_L} = 1 + 0.0625\frac{a^2}{r^2} \tag{22}$$

3.5 Finite element formulation

Fig. 3 shows the finite element model of an Euler-Bernoulli beam.

3.5.1 Displacement/nodal-displacement relation

The degrees of freedom vector for the beam are written as

$$u(x) = N_1 u_1 + N_2 u_2$$

$$w(x) = H_1 w_1 + H_2 \frac{dw_1}{dx} + H_3 w_2 + H_4 \frac{dw_2}{dx} \tag{23}$$

3.5.2 Governing differential equations

The differential equations governing the bending of straight beams are

$$\frac{dN_{xx}}{dx} + f(x) = 0$$

$$\frac{dV}{dx} + q(x) = 0$$

$$\frac{dM_{xx}}{dx} - V + N_{xx}\frac{dw_0}{dx} = 0 \tag{24}$$

3.5.3 Weak form

By using the principle of virtual work, a system of equations is obtained from the finite element method to study thermal postbuckling behavior of FGM beam. The weak forms of governing differential equations are obtained as

$$\int_{x_a}^{x_b} \frac{d(\delta u_0)}{dx}\left\{\left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2\right]A_{xx} - \frac{d^2 w_0}{dx^2}B_{xx} - N^T\right\}dx = \int_{x_a}^{x_b} (\delta u_0) f dx + \delta u_0(x_a)Q_1 + \delta u_0(x_b)Q_4$$

$$\int_{x_a}^{x_b}\left[\begin{array}{l}\left\{\dfrac{d(\delta w_0)}{dx}\left(\left[\dfrac{du_0}{dx}+\dfrac{1}{2}\left(\dfrac{dw_0}{dx}\right)^2\right]A_{xx}-\dfrac{d^2w_0}{dx^2}B_{xx}-N^T\right)\dfrac{dw_0}{dx}\right\}+\\ \left\{\dfrac{d^2(\delta w_0)}{dx^2}\left(-\left[\dfrac{du_0}{dx}+\dfrac{1}{2}\left(\dfrac{dw_0}{dx}\right)^2\right]B_{xx}+\dfrac{d^2w_0}{dx^2}D_{xx}+M^T\right)\right\}\end{array}\right]dx =$$

$$\int_{x_a}^{x_b}(\delta w_0)q\,dx+\delta w_0(x_a)Q_2+\delta w_0(x_b)Q_5+\delta\theta(x_a)Q_3+\delta\theta(x_b)Q_6 \tag{25}$$

Substituting the expressions for u and w in the weak form and rearranging, the finite element system of equations can be expressed for an element as

$$\sum_{j=1}^{2}K_{ij}^{11}u_j+\sum_{j=1}^{4}K_{ij}^{12}d_j=F_i^1 \quad \text{for} \quad i = 1, 2$$

$$\sum_{j=1}^{2}K_{ij}^{21}u_j+\sum_{j=1}^{4}K_{ij}^{22}d_j=F_i^2 \quad \text{for} \quad i = 1, 2, 3, 4 \tag{26}$$

where $u_j = [u_1\ u_2]^T$ and $d_j = [w_1\ \theta_1\ w_2\ \theta_2]^T$ and $U = [u_1\ u_2\ d_1\ d_2\ d_3\ d_4]^T$.

3.5.4 Newton-raphson solution

The assembled non-linear finite element system of equations is of the form

$$K(U)U = F \tag{27}$$

The residual is

$$R = K(U)U - F \tag{28}$$

The load step is given in terms of the temperature increment t from the initial stress free temperature. Using the Newton-Raphson algorithm, for the rth iteration

$$U^{r+1} = U^r - (T^r)^{-1}R^r \tag{29}$$

where the tangent stiffness matrix is given by

$$T^r = \frac{\partial R^r}{\partial U} \tag{30}$$

The iterative procedure is terminated when the Euclidean norm of the residual vector (L_2) is lower than the specified tolerance ε.

$$\sqrt{\sum_{i=1}^{n}R_i^2} \le \varepsilon \tag{31}$$

4. Numerical results

In the present study, a FGM beam having ceramic on the top face and metal on the bottom face is considered with the materials properties as shown in Table 1.

4.1 Simply supported beam

Fig. 4 and Fig. 5 shows the post-buckling load-deflection curves for homogenous as well as FGM beams with different volume fraction exponent *n*. An excellent agreement could be found between the results of present study (Rayleigh Ritz and FEA) with published literature (Julien *et al.* 2008). Completely ceramic beam is simulated in the finite element formulation by reducing the value of the volume fraction exponent near to zero (0.001). Completely aluminum beam is simulated by choosing a higher value (100) of the volume fraction exponent. For the homogenous beams such as the pure metallic or the pure ceramic beams, the slope of the load-deflection diagram has a discontinuity when the buckling temperature is reached. From the Rayleigh-Ritz formulation, for the case of the FGM beams, the exact point corresponding to the buckling temperature can not be predicted as the bending-extension coupling terms in Eq. (12) are non-zero. The beam will start deforming laterally for even a small temperature increment *t*. The central transverse deflection *a* from Eq. (12) is a real number only above certain temperature increment *t*. Below this temperature increment, the term $(97\pi^2 B_{xx}^2 + 9\pi^2 PL^2 A_{xx} - 9\pi^4 D_{xx} A_{xx})$ from Eq. (12) becomes negative. The

Table 1 Material properties of ceramic and metal

Property	Ceramic	Aluminium
E (GPa)	151	70
α ($/K$)	10e-06	23e-06

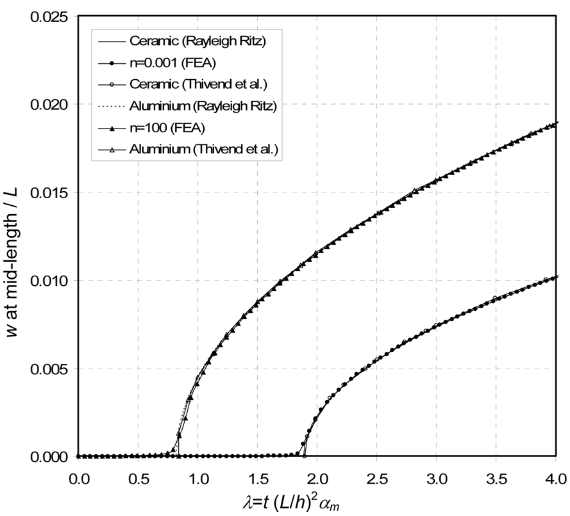

Fig. 4 Post-buckling load vs. deflection curves for homogenous beams, $L/h = 60$ (Simply supported ends)

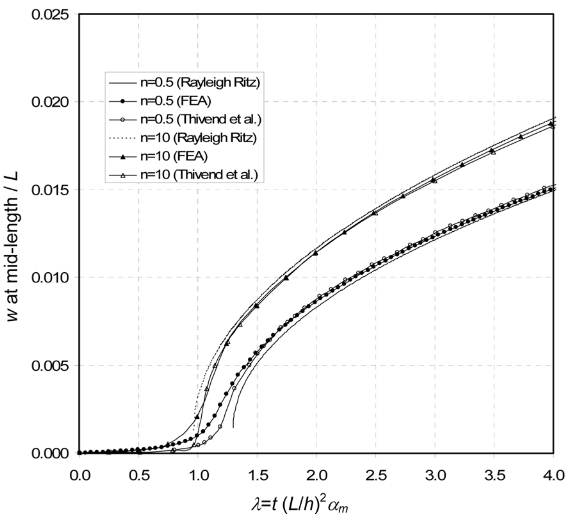

Fig. 5 Post-buckling load vs. deflection curves for FGM beams, $L/h = 60$ (Simply supported ends)

square root of this term makes the central transverse deflection 'a' complex. Equating this term to zero gives

$$(97\pi^2 B_{xx}^2 + 9\pi^2 PL^2 A_{xx} - 9\pi^4 D_{xx} A_{xx}) = 0$$

$$P = \frac{\pi^2}{L^2} D_{xx} - \frac{97 B_{xx}^2}{9 L^2 A_{xx}} \tag{32}$$

The equivalent mechanical load P (corresponding to a temperature increment t) can be obtained only when the right hand side of Eq. (32) is greater than or equal to zero. The post-buckling load deflection curve for the simply supported beam obtained from the Rayleigh-Ritz analysis, thus, does not originate from the x-axis for the FGM beams. However, for the homogenous beam, the above equation reduces to Euler buckling load $P_L = \pi^2 EI/L^2$ with a clear bifurcation point on the x-axis in the load-deflection curve (Fig. 4).

It may be observed from Fig. 5 that, for the FGM beam with simply supported boundary conditions, there is no sudden transverse deflection i.e., buckling phenomenon. This is because any small temperature raise t from the initial stress free temperature results in a transverse deflection of the beam. This makes all simply supported FGM beams lose their distinct buckling temperature. However, for the simply supported homogenous beams (Fig. 4) such as the completely ceramic or the completely aluminum beams, the buckling temperature can be clearly identified from the bifurcation point on the x-axis.

Figs. 6 and 7 show the comparisons of the buckled mode shapes obtained from the Rayleigh-Ritz and the finite element formulation with the available literature (Julien *et al.* 2008) for the simply supported ceramic and the FGM beams at a temperature increment of 45.69°C and 31.54°C respectively. For the ceramic beam, an exact match can be seen between the buckled mode shape obtained from the Rayleigh-Ritz and finite element formulation with the available literature. For the results obtained from the Rayleigh-Ritz formulation of the FGM beam (n = 0.5), however, a 2.7% of difference is observed in the central deflection a as compared to the results obtained from the

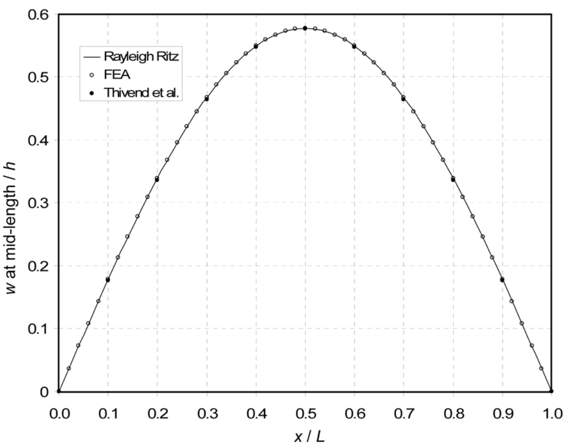

Fig. 6 Buckled mode shape for Ceramic beam, n = 0, L/h = 60 (Simply supported ends)

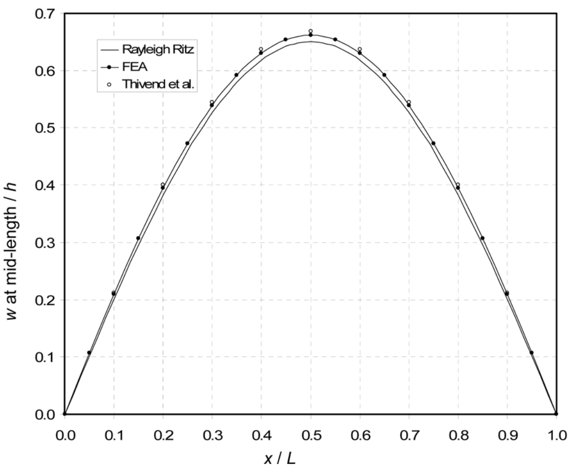

Fig. 7 Buckled mode shape for FGM beam, n = 0.5, L/h = 60 (Simply supported ends)

Table 2 P_{NL}/P_L for homogenous simply supported beam

	$\frac{a}{r}$	0.0	0.2	0.4	0.6	0.8	1.0
$\frac{P_{NL}}{P_L}$	RR (n = 0 and 100)	1.0000	1.0100	1.0400	1.0900	1.1600	1.2500
	Rao and Raju (1984, 2003)	1.0000	1.0100	1.0400	1.0900	1.1600	1.2500

finite element formulation and the available literature. The difference can be reduced by increasing the number of terms in the assumed displacement fields.

The present Rayleigh-Ritz formulation is also validated by comparing the ratio of the non-linear to critical thermal loads (Eq. (15)) for the homogenous cases (n = 0 corresponding to pure ceramic and n = 100 corresponding to pure aluminum). Table 2 shows the ratio of P_{NL}/P_L for different values of a/r for n = 0 and 100 respectively. The values of P_{NL}/P_L for n = 0 and n = 100 match exactly with the available literature values (Rao and Raju 1984, 2003) for the homogeneous beam.

4.2 Clamped beam

Fig. 8 shows the post buckling load-deflection curves of the homogenous as well as the FGM beams for different values of the volume fraction exponent n obtained using Rayleigh-Ritz and finite element formulations. In the case of the clamped beam, Eq. (19) relating the post buckling thermal load and the deflection does not contain the bending-extension coupling term B_{xx}. The term $(-4\pi^2 D_{xx} + PL^2)$ becomes negative only below critical load obtained by equating this term to zero. This critical load is the Euler buckling load $(P_L = 4\pi^2 EI/L^2)$. The absence of B_{xx} term in Eq. (19) makes it possible to capture the exact bifurcation point for the clamped FGM beams for all volume fraction exponents n. The clamped FGM beam shows load-deflection curves with a distinct bifurcation point on the x-axis similar to the homogeneous beam. Fig. 9 shows typical buckled mode shapes of clamped FGM (n = 0.5) beams obtained using the present formulations at λ = 12,

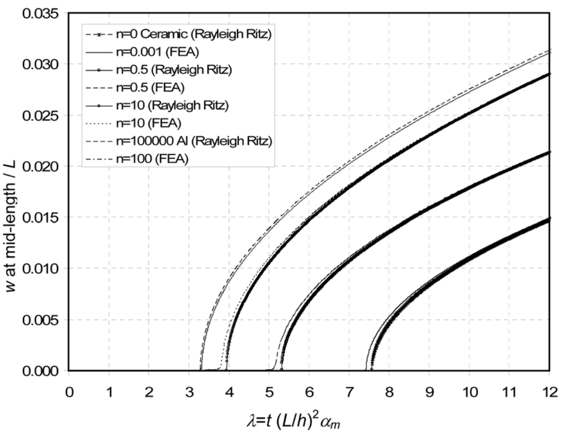

Fig. 8 Post-buckling load vs. deflection curves for FGM beams, L/h = 60 (Clamped ends)

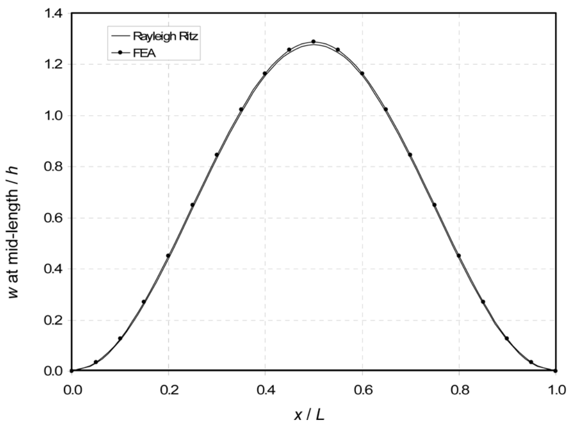

Fig. 9 Buckled mode shape for FGM beam at λ = 12, n = 0.5, L/h = 60 (Clamped ends)

Table 3 P_{NL}/P_L for homogenous clamped beam

	$\frac{a}{r}$	0.0	0.2	0.4	0.6	0.8	1.0
$\frac{P_{NL}}{P_L}$	RR (n = 0 and 100000)	1.0000	1.0025	1.0100	1.0225	1.0400	1.0625
	Rao and Raju (1984, 2003)	1.0000	1.0025	1.0100	1.0225	1.0400	1.0625

where λ is normalized temperature given by $\lambda = t\,(L/h)^2\alpha_m$. An excellent agreement can be found between these two results.

Table 3 shows the ratio of P_{NL}/P_L for different values of a/r for the homogenous cases (n = 0 and n = 100). The values of P_{NL}/P_L for $n = 0$ (pure ceramic) and n=100 (pure aluminium) match exactly with those available in the literature for the homogeneous beam (Rao and Raju 1984, 2003).

5. Conclusions

Thermal post-buckling analysis of uniform slender functionally graded material beams with axially immovable ends is studied using the classical Rayleigh-Ritz and finite element formulations developed separately in this paper. Finite element analysis, although more versatile and accurate, being a numerical method, does not give elegant closed form solution for post-buckling of FGM beams with different boundary conditions and beam parameters. As such, the authors have chosen the Rayleigh-Ritz method to obtain approximate closed form expressions for the thermal post-buckling study of the FGM beams with axially immovable ends and show the effect of bending-extension coupling on structural response. The top face of the FGM beam is taken as ceramic rich and the bottom face is metal rich. The region between these two faces consists of a smooth variation of properties assumed in the form of a power law distribution. Both the simply supported and clamped boundary conditions are considered in the present study. Geometric non-linearity based on von-Karman type large deflection theory is considered in the analysis. Bending-extension coupling arising due to heterogeneity of the material through the thickness is included. The finite element non-linear equations are obtained using the principle of virtual work. Newton-Raphson iterative procedure is used to solve the non-linear equations. Load-deflection curves are obtained from both the Rayleigh-Ritz and the finite element formulations. Numerical results are presented to show the effect of volume fraction exponent on the post-buckling load versus the transverse deflection curves. The load-deflection curves for the simply supported beams obtained from the RR and FEA formulations showed an excellent agreement with the available literature results. The results with clamped ends are not available in the literature and are reported for the first time using both the formulations developed in this paper. It is observed that the simply supported FGM beams do not have a distinct critical buckling temperature. Clamped FGM beams, however, show that the load-deflection curve has a distinct bifurcation point similar to the homogeneous beams.

The usefulness and simplicity of application of the approximate method, such as the Rayleigh-Ritz method, with the well established exact one-term admissible functions for the axial and lateral deflections of the homogenous beams, for approximate closed form thermal post-buckling solution of the FGM beams with simply supported and clamped boundary conditions, is well established in this study. The effect of bending-extension coupling on structural response for both boundary conditions is clearly demonstrated. To this extent, the aim of this study is satisfied as is obvious from the results.

References

Deschilder, M., Eslami, H. and Zhao, Y. (2006), "Non-linear static analysis of a beam made of functionally graded material", *Proceedings of the 47th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference*, Newport, Rhode Island, May.

Han, S.C., Lomboy, G.R. and Kim, K.D. (2008), "Mechanical vibration and buckling analysis of FGM plates and shells using a four node quasi conforming shell element", *Int. J. Struct. Stabil. Dyn.*, **8**(2), 203-229.

Ibrahim, H.H. (2007), "Thermal buckling and nonlinear flutter behavior of functionally graded material panels", *J. Aircraft*, **44**(5), 1610-1618.

Jabbari, M., Vaghari, A.R., Bahtui, A. and Eslami, M.R. (2008), "Exact solution for asymmetric transient thermal and mechanical stresses in FGM hollow cylinders with heat source", *Struct. Eng. Mech.,* **29**(5), 551-565.

Julien, T., Eslami, H. and Zhao, Y. (2008), "Thermal post buckling analysis of FGM beams", *Proceedings of the 47th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference*, Newport, Rhode Island, May..

Kitipornchai, S., Ke, L.L., Yang, J. and Xiang, Y. (2009), "Nonlinear vibration of edge cracked functionally graded Timoshenko beams", *J. Sound Vib.*, **324**, 962-982.

Lee, S.L. and Kim, J.H. (2007), "Thermal stability boundary of FG panel under aerodynamic load", *Int. J. Mech. Syst. Sci. Eng.*, **1**(2), 105-110.

Li, X.F. (2008), "A unified approach for analyzing static and dynamic behaviors of functionally graded Timoshenko and Euler-Bernoulli beams", *J. Sound Vib.*, **318**, 1210-1229.

Prakash, T. and Ganapathi, M. (2006), "Asymmetric flexural vibration and thermoelastic stability of FGM plates using finite element method", *Composites: Part B*, **37**, 642-649.

Prakash, T., Singh, M.K. and Ganapathi, M. (2006), "Vibrations and thermal stability of functionally graded spherical caps", *Struct. Eng. Mech.,* **24**(4), 447-462.

Raju, K.K. and Rao Venkateswara, G. (2005), "Towards improved evaluation of large amplitude free vibration behavior of uniform beams using multi-term admissible functions", *J. Sound Vib.*, **282**, 1238-1246.

Rao Venkateswara, G. (2003), "A simple energy method to predict the thermal post buckling behavior of columns", *J. Aeros. Sci. Technol.*, **55**(2), 141-143.

Rao Venkateswara, G. and Kanaka Raju, K. (1984), "Thermal post buckling of columns", *AIAA J.*, **22**(6), 850-851.

Rao Venkateswara, G. and Kanaka Raju, K. (2002), "Thermal post buckling of uniform Columns: A simple intuitive method", *AIAA J.*, **40**(10), 2138-2140.

Rao Venkateswara, G. and Kanaka Raju, K. (2003), "A simple method to predict the thermal post-buckling behavior of columns on Pasternak foundation", *Ind. J. Eng. Mater. Sci.*, **10**, 177-182.

Rao Venkateswara, G. and Raju, P.C. (1977), "Post-buckling of uniform cantilever columns – Galerkin finite element formulation", *Eng. Fract. Mech.*, **9**, 1-4.

Sankar, B.V. (2001), "An elasticity solution for functionally graded beams", *Compos. Sci. Technol.*, **61**, 689-696.

Timoshenko, S.P. and Gere, J.M. (1970), *Theory of Elastic Stability*, McGraw-Hill.

Turvey, G.J. and Lih, M. (1995), *Buckling and Postbuckling of Composite Plates,* Chapman and Hall, London.

Zhong, Z. and Yu, T. (2007), "Analytical solution of a cantilever functionally graded beam", *Compos. Sci. Technol.*, **67**, 481-488.

Notations

a	: central lateral deflection of beam
A	: cross-sectional area of beam
A_{xx}	: extensional stiffness
b	: coefficient associated with axial deflection
B_{xx}	: extension-bending coupling stiffness
D_{xx}	: bending stiffness
E	: effective modulus of elasticity of FGM beam
E_m, E_c	: modulus of elasticity of metal, ceramic
$f(x)$	: generalized axial load
F_i	: element load vector
h	: height of beam
H_i	: Hermite shape functions
I	: area moment of inertia
K	: element stiffness matrix
L	: length of beam
M_{xx}	: moment resultant
M^T	: thermal bending moment
n	: volume fraction exponent
N_{xx}	: stress resultant
N^T, P	: in-plane thermal load
N_i	: Lagrange shape functions
PE	: total potential energy
$q(x)$	: generalized transverse load
Q_i	: generalized forces
r	: radius of gyration
R	: residual
t	: temperature rise from the initial stress free condition
T	: tangent stiffness matrix
u	: deformation along x-axis
V	: transverse shear
V_m, V_c	: volume fraction of metal, ceramic
V_S	: strain energy due to stretching
V_B	: strain energy due to bending
V_{BS}	: strain energy due to bending-stretching coupling
V_L	: potential due to in-plane thermal load P
w	: deformation along z axis
q	: slope
x, x_a, x_b	: coordinate along length of beam
z	: coordinate along thickness of beam
α	: coefficient of thermal expansion of FGM
α_c, α_m	: coefficient of thermal expansion of ceramic, metal
$\varepsilon^0{}_{xx}$	: mid-plane strain
κ_{xx}	: mid-plane curvature
ν	: Poisson's ratio
δu_0	: virtual displacement along x axis
δw_0	: virtual displacement along z axis
$\delta\theta$	: virtual rotation
σ_{xx}	: axial stress