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An analytical solution of bending thin plates with different moduli in tension and compression

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Abstract. Materials which exhibit different elastic moduli in tension and compression are known as bimodular materials. The bimodular materials model, which is founded on the criterion of positive-negative signs of principal stress, is important for the structural analysis and design. However, due to the inherent complexity of the constitutive relation, it is difficult to obtain an analytical solution of a bimodular bending components except in particular simple problems. Based on the existent simplified model, this paper solves analytically bending thin plates with different moduli in tension and compression. By using the continuity conditions of stress components in unknown neutral layer, we determine the location of the neutral layer, and derive the governing differential equation for deflection, the flexural rigidity, and the internal forces in the thin plate. We also use a circular thin plate with bimodulus to illustrate the application of this solution derived in this paper. The results show that the introduction of different moduli has influences on the flexural stiffness of the bending thin plate.

Keywords: bimodulus; tension and compression; thin plate; bending; continuity.

1. Introduction

Classical elasticity theory assumes that materials have the same elastic properties in tension and compression, but this is only a simplified interpretation, and does not account for material nonlinearities. Many studies have indicated that most materials, including concrete, ceramics, graphite, and some composites, exhibit different tensile and compressive strains given the same stress applied in tension or compression. Thus, materials exhibit different elastic moduli in tension and compression. These materials are known as bimodular materials (Jones 1976, 1977). Overall, there are two basic material models widely used in theoretical analysis within the engineering profession. One of these models is the criterion of positive-negative signs in the longitudinal strain of fibers put forward by Bert (1977). This model is mainly applicable to orthotropic materials, and is therefore widely used for research on laminated composites (Bert 1983, Reddy and Chao 1983, Srinivasan and Ramachandra 1989, Zinno and Greco 2001, Patel *et al*. 2004, Patel *et al*. 2005,

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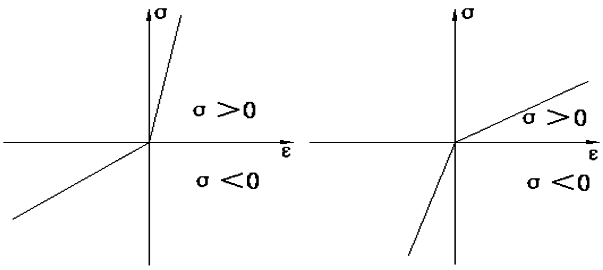
Khan *et al*. 2009). Another model is the criterion of positive-negative signs of principal stress put forward by Ambartsumyan (1986). This model is mainly applicable to isotropic materials. In structural engineering, the stress state in a principal direction is a key point in the analysis of some components like beams, columns, plates and shells. This paper will focus on discussions of the latter model based on principal direction.

The bimodular materials model proposed by Ambartsumyan asserts that Young's modulus of elasticity depends not only on material properties, but also on the stress state of that point. There are only a few applications of the constitutive equation to stress analyses of components because of the inherent complexity in analysis of the bimodular materials, i.e., the elastic constants involved in the governing equations, which depend on the stress state of that point, are not correctly indicated beforehand. In other words, except in particularly simple problems it is not easy to estimate a-priori the stress state in a point in the deformed body. In some complex problems, it is necessary to resort to FEM based on an iterative strategy (Zhang and Wang 1989, Ye *et al*. 2004, Yao *et al*. 2006, Cai and Yu 2009). Because the stress state of the point in question is unknown in advance, we have to begin with a single modulus problem, thus gaining the initial stress state to form a corresponding elasticity matrix for each element. Generally, direct iterative methods based on an incrementally evolving stiffness have been adopted by many researchers. These methods include an improved algorithm by supplementing the shear stress and strain equal to zero in the elastic matrix (Zhang and Wang 1989, Liu and Zhang 2000, He *et al*. 2009), an improved algorithm by keeping Poisson's ratio constant while modifying the elastic matrix (Ye 1997), an initial stress technique (Yang *et al*. 1999), a smoothing function technique (Yang and Zhu 2006, Yang and Wang 2008) and a new analytical-iterative method for statically indeterminate structures (Yao and Ye 2006).

Analytical solutions are available in a few cases, and they only concern beams and columns. By isolating the differential element and then considering its static equilibrium, Yao and Ye (2004a) derived the analytical solution of bending-compression column with different moduli in tension and compression. Aimed at a bimodular beam in lateral-force bending, Yao and Ye (2004b) proposed an assumption that shear stress makes no contribution to position of the neutral axis, and derived the analytical solution of a lateral-force bending beams with different moduli in tension and compression. Yao and Ye (2004c) also derived the analytical solution of the bimodular retaining wall. In order to simplify the derived process, He *et al*. (2007a) proposed the bimodular beams may be turned into the classical beams by the equivalent section method. Using continuity conditions of stresses on the neutral axis, i.e., all the stress components should be continuously variable at the neutral axis, He *et al*. (2007b) obtained the approximate analytical solution of a bimodular deep beam under uniformly-distributed loads. Although there are many analytical solutions concerning beams and columns, little analytical work concerning bimodular plates has been done due to the inherent complexity of the constitutive relation.

When a bending plate problem with bimodular is solved in an analytical way, the judgment criterion and the continuity conditions of unknown neutral layer are of vital importance. Once the unknown neutral layer can be judged, the subarea in tension and compression of a bimodular thin plate will be consequently determined, which opens up possibilities for the simplification of the complex constitutive relation. In small-deflection bending of thin plates, vanishing in-plane stress on middle layer make it possible to judge the existence of a neutral layer. Based on classical Kirchhoff hypotheses, this paper judges the existence of the neutral layer, and uses stresses' continuity conditions in the unknown neutral layer, which is owned by a bimodular bending problem, to determine firstly the location of the unknown neutral layer. Then, we derive the governing differential



(*a*) $E^+ > E^-$ (*b*) $E^+ < E^-$

Fig. 1 Constitutive model of bimodulus materials proposed by Ambartsumyan

equation for deflection, the flexural rigidity, and the internal forces in the thin plate. We also use a circular thin plate with bimodulus to illustrate the application of this solution derived in this paper. The discussions on the solution present the influences introduced by bimodularity of the materials.

2. Bimodular materials model proposed by Ambartsumyan

Ambartsumyan (1986) linearized the nonlinear model, the second material model mentioned above, into two straight lines whose tangents at the origin are discontinuous, as shown in Fig. 1. Based on the bilinear materials model, Ambartsumyan founded the elasticity theory with different moduli in tension and compression. The basic assumptions of this theory are as follows:

(1) The studied body is continuous, homogeneous and isotropic.

(2) Small deformation is assumed.

(3) Young's modulus of elasticity and Poisson's ratio of materials are E^+ and μ^+, respectively while the materials is in tension along certain direction; and they are E^- and μ^-, respectively while the materials is in compression along certain direction.

(4) When the three principal stresses are uniformly positive or uniformly negative, the point in question belongs to the first class; when the signs of the three principal stresses are different, the point in question belongs to the second class. In the first class discussed above, the three basic equations are essentially the same as those of classical theory. However, in the second class discussed above, the differential equations of equilibrium and the geometrical equations are the same as those of classical materials theory, with the exception of the physical equations. We will therefore focus our discussion on this latter case, which a new characteristic concerning bimodulus in tension and compression is inevitably introduced here.

(5) A restrictive assumption $\mu^+/E^+ = \mu^-/E^-$ is introduced and it satisfies symmetry characteristics of the flexibility matrix.

In a spatial problem, let the stress and strain components in general coordinates x, y, z be, respectively

$$\{\sigma\} = (\sigma_x \ \ \sigma_y \ \ \sigma_z \ \ \tau_{yz} \ \ \tau_{zx} \ \ \tau_{xy})^T \tag{1}$$

and

$$\{\varepsilon\} = (\varepsilon_x \ \ \varepsilon_y \ \ \varepsilon_z \ \ \varepsilon_{yz} \ \ \varepsilon_{zx} \ \ \varepsilon_{xy})^T \tag{2}$$

Let the stress and strain components in the principal coordinates α, β, γ be, respectively

$$\{\sigma_I\} = (\sigma_\alpha \ \ \sigma_\beta \ \ \sigma_\gamma)^T \tag{3}$$

and

$$\{\varepsilon_I\} = (\varepsilon_\alpha \ \ \varepsilon_\beta \ \ \varepsilon_\gamma)^T \tag{4}$$

The constitutive model proposed by Ambartsumyan is

$$\begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\gamma \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\gamma \end{Bmatrix} \tag{5}$$

where, a_{ij} $(i,j = 1,2,3)$ denotes the flexibility coefficients determined by the polarity of the signs of the principal stress. For instance, if $\sigma_\alpha > 0, \sigma_\beta < 0, \sigma_\gamma > 0$, the physical equation should be

$$\begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\gamma \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{E^+} & -\dfrac{\mu^-}{E^-} & -\dfrac{\mu^+}{E^+} \\ -\dfrac{\mu^+}{E^+} & \dfrac{1}{E^-} & -\dfrac{\mu^+}{E^+} \\ -\dfrac{\mu^+}{E^+} & -\dfrac{\mu^-}{E^-} & \dfrac{1}{E^+} \end{bmatrix} \begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\gamma \end{Bmatrix} \tag{6}$$

The rest of the physical equations may be deduced analogously.

Because the stress state of the point in question is unknown in advance, we have to begin with a single modulus problem, thus gaining the initial stress state to form a corresponding elasticity matrix for each element. This method is only available for the numerical iterative technology based on FEM. However, once via coordinates conversion, we will find that in the physical equations in general coordinates x, y, z there are some nonlinear items involving the principal stress and its direction cosine. It is very difficult for us to solve analytically such a problem. Therefore, it is necessary to simplify the mechanical model to obtain the approximate analytical solution.

3. Bending analysis of bimodular rectangular plates

Based on the existent work, we may consider such a bimodular thin plate in which the x~y plane coincides with the unknown neutral layer, analogous to the analysis in a bimodular beam. Bounded by the unknown neutral layer, the whole cross sections in x and y directions are divided into the tensile area and the compressive area, respectively, as shown in Fig. 2, in which t is the thickness of the plate, t_1 and t_2 is the thickness of the plate in tension and compression, respectively, and q is the intensity of the uniformly distributed loads.

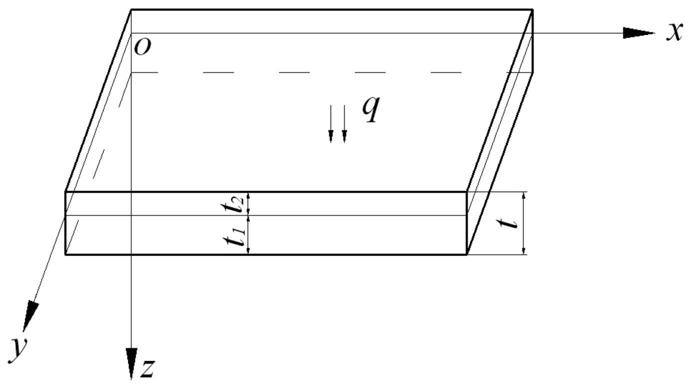


Fig. 2 Bimodular rectangular plate under the uniformly distributed loads

3.1 Stress components

The displacement components at a point, occurring in the *x* and *y* directions, are denoted by *u* and *v*, respectively. Due to lateral loading, deformation takes place; any point at the neutral layer has deflection *w*. According to Kirchhoff hypotheses, the displacement components, *u* and *v*, may be expressed as, respectively

$$u = -\frac{\partial w}{\partial x} z, \quad v = -\frac{\partial w}{\partial y} z \tag{7}$$

Via geometrical equations, the strain components $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ may be expressed as follows

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} = -\frac{\partial^2 w}{\partial x^2} z = \chi_x z \\ \varepsilon_y &= \frac{\partial v}{\partial y} = -\frac{\partial^2 w}{\partial y^2} z = \chi_y z \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -2\frac{\partial^2 w}{\partial x \partial y} z = 2\chi_{xy} z \end{aligned}\right\} \tag{8}$$

where

$$\chi_x = -\frac{\partial^2 w}{\partial x^2}, \quad \chi_y = -\frac{\partial^2 w}{\partial y^2}, \quad \chi_{xy} = -\frac{\partial^2 w}{\partial x \partial y} \tag{9}$$

denote the curvature and the twist with respect to coordinates, respectively. The derivations above are all the same as the classical plate's theory.

Next, we will derive the stress components expressed in terms of *w*. Note that due to the subarea in tension and compression, the physical equation may be determined beforehand. From the corresponding physical equations, the stress components in tensile and compressive areas, $\overset{+}{\sigma}_x, \overset{+}{\sigma}_y, \overset{+}{\tau}_{xy}$ and $\overset{-}{\sigma}_x, \overset{-}{\sigma}_y, \overset{-}{\tau}_{xy}$, may be solved as, respectively

$$\left.\begin{aligned} \sigma_x^+ &= \frac{E^+}{1-(\mu^+)^2}(\varepsilon_x + \mu^+ \varepsilon_y) \\ \sigma_y^+ &= \frac{E^+}{1-(\mu^+)^2}(\varepsilon_y + \mu^+ \varepsilon_x) \\ \tau_{xy}^+ &= \frac{E^+}{2(1+\mu^+)}\gamma_{xy} \end{aligned}\right\} \tag{10a}$$

$$\left.\begin{aligned} \sigma_x^- &= \frac{E^-}{1-(\mu^-)^2}(\varepsilon_x + \mu^- \varepsilon_y) \\ \sigma_y^- &= \frac{E^-}{1-(\mu^-)^2}(\varepsilon_y + \mu^- \varepsilon_x) \\ \tau_{xy}^- &= \frac{E^-}{2(1+\mu^-)}\gamma_{xy} \end{aligned}\right\} \tag{10b}$$

where, E^+, μ^+ denote the tensile Young's modulus of elasticity and Poisson's ratio, respectively, and E^-, μ^- the compressive ones. Due to the deformation consistency, the strain and displacement components in the tensile or compressive parts are the same. Substituting Eq. (8) into Eq. (10a,b), respectively, we have

$$\left.\begin{aligned} \sigma_x^+ &= -\frac{E^+ z}{1-(\mu^+)^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu^+ \frac{\partial^2 w}{\partial y^2}\right) \\ \sigma_y^+ &= -\frac{E^+ z}{1-(\mu^+)^2}\left(\frac{\partial^2 w}{\partial y^2} + \mu^+ \frac{\partial^2 w}{\partial x^2}\right) \\ \tau_{xy}^+ &= -\frac{E^+ z}{1+\mu^+}\frac{\partial^2 w}{\partial x \partial y} \end{aligned}\right\} \quad 0 \le z \le t_1 \tag{11a}$$

$$\left.\begin{aligned} \sigma_x^- &= -\frac{E^- z}{1-(\mu^-)^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu^- \frac{\partial^2 w}{\partial y^2}\right) \\ \sigma_y^- &= -\frac{E^- z}{1-(\mu^-)^2}\left(\frac{\partial^2 w}{\partial y^2} + \mu^- \frac{\partial^2 w}{\partial x^2}\right) \\ \tau_{xy}^- &= -\frac{E^- z}{1+\mu^-}\frac{\partial^2 w}{\partial x \partial y} \end{aligned}\right\} \quad -t_2 \le z \le 0 \tag{11b}$$

w does not vary with z, and the three stress components are direct proportional to z.

Moreover, the stress components in tensile and compressive areas, τ_{zx}^+, τ_{zy}^+ and τ_{zx}^-, τ_{zy}^-, will be expressed in terms of w. Due to no any longitudinal loads, the body force is zero and the first two expressions of the differential equations of equilibrium in the tensile or compressive parts may be written as, respectively

$$\frac{\partial \overset{+}{\tau}_{zx}}{\partial z} = -\frac{\partial \overset{+}{\sigma}_x}{\partial x} - \frac{\partial \overset{+}{\tau}_{yx}}{\partial y}, \quad \frac{\partial \overset{+}{\tau}_{zy}}{\partial z} = -\frac{\partial \overset{+}{\sigma}_y}{\partial y} - \frac{\partial \overset{+}{\tau}_{xy}}{\partial x} \tag{12a}$$

$$\frac{\partial \overset{-}{\tau}_{zx}}{\partial z} = -\frac{\partial \overset{-}{\sigma}_x}{\partial x} - \frac{\partial \overset{-}{\tau}_{yx}}{\partial y}, \quad \frac{\partial \overset{-}{\tau}_{zy}}{\partial z} = -\frac{\partial \overset{-}{\sigma}_y}{\partial y} - \frac{\partial \overset{-}{\tau}_{xy}}{\partial x} \tag{12b}$$

Substituting Eqs. (11a,b) into Eqs. (12a,b) and considering $\overset{+}{\tau}_{yx} = \overset{+}{\tau}_{xy}$ and $\overset{-}{\tau}_{yx} = \overset{-}{\tau}_{xy}$, respectively, we have

$$\left.\begin{aligned} \frac{\partial \overset{+}{\tau}_{zx}}{\partial z} &= \frac{\overset{+}{E} z}{1-(\overset{+}{\mu})^2}\frac{\partial}{\partial x}\nabla^2 w \\ \frac{\partial \overset{+}{\tau}_{zy}}{\partial z} &= \frac{\overset{+}{E} z}{1-(\overset{+}{\mu})^2}\frac{\partial}{\partial y}\nabla^2 w \end{aligned}\right\} \tag{13a}$$

$$\left.\begin{aligned} \frac{\partial \overset{-}{\tau}_{zx}}{\partial z} &= \frac{\overset{-}{E} z}{1-(\overset{-}{\mu})^2}\frac{\partial}{\partial x}\nabla^2 w \\ \frac{\partial \overset{-}{\tau}_{zy}}{\partial z} &= \frac{\overset{-}{E} z}{1-(\overset{-}{\mu})^2}\frac{\partial}{\partial y}\nabla^2 w \end{aligned}\right\} \tag{13b}$$

Considering w does not vary with z and integrating Eqs. (13a,b) with respect to z, respectively, we have

$$\left.\begin{aligned} \overset{+}{\tau}_{zx} &= \frac{\overset{+}{E} z^2}{2[1-(\overset{+}{\mu})^2]}\frac{\partial}{\partial x}\nabla^2 w + \overset{+}{F}_1(x,y) \\ \overset{+}{\tau}_{zy} &= \frac{\overset{+}{E} z^2}{2[1-(\overset{+}{\mu})^2]}\frac{\partial}{\partial y}\nabla^2 w + \overset{+}{F}_2(x,y) \end{aligned}\right\} \tag{14a}$$

$$\left.\begin{aligned} \overset{-}{\tau}_{zx} &= \frac{\overset{-}{E} z^2}{2[1-(\overset{-}{\mu})^2]}\frac{\partial}{\partial x}\nabla^2 w + \overset{-}{F}_1(x,y) \\ \overset{-}{\tau}_{zy} &= \frac{\overset{-}{E} z^2}{2[1-(\overset{-}{\mu})^2]}\frac{\partial}{\partial y}\nabla^2 w + \overset{-}{F}_2(x,y) \end{aligned}\right\} \tag{14b}$$

where, $\overset{+}{F}_1(x,y), \overset{+}{F}_2(x,y)$ and $\overset{-}{F}_1(x,y), \overset{-}{F}_2(x,y)$ are two undetermined functions in tensile and compressive parts, respectively. The boundary conditions at the bottom and top of the thin plate are, respectively

$$(\overset{+}{\tau}_{zx})_{z=t_1} = 0, \quad (\overset{+}{\tau}_{zy})_{z=t_1} = 0 \tag{15a}$$

$$(\overset{-}{\tau}_{zx})_{z=-t_2} = 0, \quad (\overset{-}{\tau}_{zy})_{z=-t_2} = 0 \tag{15b}$$

By Eqs. (15a,b), Eqs. (14a,b) with unknown functions may be determined as, respectively

$$\left.\begin{aligned}\overset{+}{\tau}_{zx} &= \frac{\overset{+}{E}}{2[1-(\overset{+}{\mu})^2]}(z^2-t_1^2)\frac{\partial}{\partial x}\nabla^2 w\\ \overset{+}{\tau}_{zy} &= \frac{\overset{+}{E}}{2[1-(\overset{+}{\mu})^2]}(z^2-t_1^2)\frac{\partial}{\partial y}\nabla^2 w\end{aligned}\right\} \quad 0\le z\le t_1 \tag{16a}$$

$$\left.\begin{aligned}\overset{-}{\tau}_{zx} &= \frac{\overset{-}{E}}{2[1-(\overset{-}{\mu})^2]}(z^2-t_2^2)\frac{\partial}{\partial x}\nabla^2 w\\ \overset{-}{\tau}_{zy} &= \frac{\overset{-}{E}}{2[1-(\overset{-}{\mu})^2]}(z^2-t_2^2)\frac{\partial}{\partial y}\nabla^2 w\end{aligned}\right\} \quad -t_3\le z\le 0 \tag{16b}$$

The stress components in the tensile and compressive area, $\overset{+}{\sigma}_z$ and $\overset{-}{\sigma}_z$, will be expressed in terms of w, respectively. Using the third expression of the differential equations of equilibrium in the tensile and compressive parts, we have

$$\frac{\partial \overset{+}{\sigma}_z}{\partial z} = -\frac{\partial \overset{+}{\tau}_{xz}}{\partial x} - \frac{\partial \overset{+}{\tau}_{yz}}{\partial y} \tag{17a}$$

$$\frac{\partial \overset{-}{\sigma}_z}{\partial z} = -\frac{\partial \overset{-}{\tau}_{xz}}{\partial x} - \frac{\partial \overset{-}{\tau}_{yz}}{\partial y} \tag{17b}$$

Substituting Eqs. (16a,b) into Eqs. (17a,b) and considering $\overset{+}{\tau}_{xz} = \overset{+}{\tau}_{zx}$, $\overset{+}{\tau}_{yz} = \overset{+}{\tau}_{zy}$ and $\overset{-}{\tau}_{xz} = \overset{-}{\tau}_{zx}$, $\overset{-}{\tau}_{yz} = \overset{-}{\tau}_{zy}$ respectively, we have

$$\frac{\partial \overset{+}{\sigma}_z}{\partial z} = \frac{\overset{+}{E}}{2[1-(\overset{+}{\mu})^2]}(t_1^2-z^2)\nabla^4 w \tag{18a}$$

$$\frac{\partial \overset{-}{\sigma}_z}{\partial z} = \frac{\overset{-}{E}}{2[1-(\overset{-}{\mu})^2]}(t_2^2-z^2)\nabla^4 w \tag{18b}$$

Integrating Eqs. (18a,b) with respect to z, respectively, we have

$$\overset{+}{\sigma}_z = \frac{\overset{+}{E}}{2[1-(\overset{+}{\mu})^2]}\left(t_1^2 z-\frac{z^3}{3}\right)\nabla^4 w + \overset{+}{F}_3(x,y) \tag{19a}$$

$$\overset{-}{\sigma}_z = \frac{\overset{-}{E}}{2[1-(\overset{-}{\mu})^2]}\left(t_2^2 z-\frac{z^3}{3}\right)\nabla^4 w + \overset{-}{F}_3(x,y) \tag{19b}$$

where, $\overset{+}{F}_3(x,y)$ and $\overset{-}{F}_3(x,y)$ are two undetermined functions. At the bottom and top of the plate, the boundary conditions are, respectively

$$(\overset{+}{\sigma}_z)_{z=t_1} = 0 \tag{20a}$$

$$(\overset{-}{\sigma}_z)_{z=-t_2} = -q \tag{20b}$$

Applying the boundary conditions above, we obtain

$$\sigma_z^+ = \frac{E^+}{2[1-(\mu^+)^2]}\left(t_1^2 z - \frac{z^3}{3} - \frac{2}{3}t_1^3\right)\nabla^4 w, \qquad 0 \le z \le t_1 \tag{21a}$$

$$\sigma_z^- = \frac{E^-}{2[1-(\mu^-)^2]}\left(t_2^2 z - \frac{z^3}{3} - \frac{2}{3}t_2^3\right)\nabla^4 w, \qquad -t_2 \le z \le 0 \tag{21b}$$

Thus, all the stress components in the tensile and compressive areas have been expressed in terms of w, as shown in Eqs. (11a,b), (16a,b) and (21a,b).

3.2 Continuity conditions

Firstly, all the stress components in the tensile and compressive areas are continuous variables on the neutral layer, so we have

$$(\sigma_x^+)_{z=0} = (\sigma_x^-)_{z=0},\ (\sigma_y^+)_{z=0} = (\sigma_y^-)_{z=0},\ (\tau_{xy}^+)_{z=0} = (\tau_{xy}^-)_{z=0} \tag{22}$$

$$(\tau_{zx}^+)_{z=0} = (\tau_{zx}^-)_{z=0},\ (\tau_{zy}^+)_{z=0} = (\tau_{zy}^-)_{z=0} \tag{23}$$

and

$$(\sigma_z^+)_{z=0} = (\sigma_z^-)_{z=0} \tag{24}$$

From Eqs. (11a,b), we know

$$(\sigma_x^+)_{z=0} = (\sigma_x^-)_{z=0} = 0,\ (\sigma_y^+)_{z=0} = (\sigma_y^-)_{z=0} = 0,\ (\tau_{xy}^+)_{z=0} = (\tau_{xy}^-)_{z=0} = 0 \tag{25}$$

It indicates that Eq. (22) is satisfied naturally. From Eqs. (16a,b) and (23), we have

$$\frac{E^+ t_1^2}{1-(\mu^+)^2} = \frac{E^- t_2^2}{1-(\mu^-)^2} \tag{26}$$

Combining $t_1 + t_2 = t$, we may solve the thicknesses of the plate in tension and compression as follows

$$t_1 = \frac{\sqrt{E^-[1-(\mu^+)^2]}}{\sqrt{E^+[1-(\mu^-)^2]} + \sqrt{E^-[1-(\mu^+)^2]}}t, \quad t_2 = \frac{\sqrt{E^+[1-(\mu^-)^2]}}{\sqrt{E^+[1-(\mu^-)^2]} + \sqrt{E^-[1-(\mu^+)^2]}}t \tag{27}$$

From Eqs. (21a,b) and (24), we obtain

$$\left\{\frac{E^+ t_1^3}{3[1-(\mu^+)^2]} + \frac{E^- t_2^3}{3[1-(\mu^-)^2]}\right\}\nabla^4 w = q \tag{28}$$

Eq. (28) is the governing differential equation of the neutral layer. If we let

$$D^* = \frac{E^+ t_1^3}{3[1-(\mu^+)^2]} + \frac{E^- t_2^3}{3[1-(\mu^-)^2]} \tag{29}$$

where, D^* denotes the flexural rigidity of the bimodular plate, Eq. (28) may be written in a familiar form as follows

$$D^{*}\nabla^{4}w = q \tag{30}$$

When $E^{+} = E^{-} = E$, $\mu^{+} = \mu^{-} = \mu$, Eqs. (27) and (28) are simplified as $t_1 = t_2 = t/2$ and $D\nabla^{4}w = q$, respectively, where $D = Et^{3}/[12(1-\mu^{2})]$. Thus, a bimodular plate problem is reduced to the classical problem.

3.3 Internal forces

Due to the different stress formulas along the thickness direction of the plate, it is necessary to integrate in subsection to obtain the formulas of the internal forces. Using Eqs. (11a,b), we may compute the bending moment M_x and M_y as follows

$$M_x = \int_0^{t_1} \sigma_x^{+} z dz + \int_{-t_2}^{0} \sigma_x^{-} z dz = -\frac{E^{+}t_1^3}{3[1-(\mu^{+})^2]}\left(\frac{\partial^2 w}{\partial x^2} + \mu^{+}\frac{\partial^2 w}{\partial y^2}\right) - \frac{E^{-}t_2^3}{3[1-(\mu^{-})^2]}\left(\frac{\partial^2 w}{\partial x^2} + \mu^{-}\frac{\partial^2 w}{\partial y^2}\right) \tag{31}$$

$$M_y = \int_0^{t_1} \sigma_y^{+} z dz + \int_{-t_2}^{0} \sigma_y^{-} z dz = -\frac{E^{+}t_1^3}{3[1-(\mu^{+})^2]}\left(\frac{\partial^2 w}{\partial y^2} + \mu^{+}\frac{\partial^2 w}{\partial x^2}\right) - \frac{E^{-}t_2^3}{3[1-(\mu^{-})^2]}\left(\frac{\partial^2 w}{\partial y^2} + \mu^{-}\frac{\partial^2 w}{\partial x^2}\right) \tag{32}$$

The twist moment M_{xy} is

$$M_{xy} = \int_0^{t_1} \tau_{xy}^{+} z dz + \int_{-t_2}^{0} \tau_{xy}^{-} z dz = -\frac{1}{3}\left(\frac{E^{+}t_1^3}{1+\mu^{+}} + \frac{E^{-}t_2^3}{1+\mu^{-}}\right)\frac{\partial^2 w}{\partial x \partial y} \tag{33}$$

Similarly, by using Eqs. (16a,b), we may compute the shear forces Q_x and Q_y as follows

$$Q_x = \int_0^{t_1} \tau_{xz}^{+} dz + \int_{-t_2}^{0} \tau_{xy}^{-} dz = \left\{-\frac{E^{+}t_1^3}{3[1-(\mu^{+})^2]} - \frac{E^{-}t_2^3}{3[1-(\mu^{-})^2]}\right\}\frac{\partial}{\partial x}\nabla^2 w \tag{34}$$

$$Q_y = \int_0^{t_1} \tau_{yz}^{+} dz + \int_{-t_2}^{0} \tau_{yz}^{-} dz = \left\{-\frac{E^{+}t_1^3}{3[1-(\mu^{+})^2]} - \frac{E^{-}t_2^3}{3[1-(\mu^{-})^2]}\right\}\frac{\partial}{\partial y}\nabla^2 w \tag{35}$$

If we introduce the following relations

$$D^{+} = \frac{E^{+}t_1^3}{3[1-(\mu^{+})^2]}, \quad D^{-} = \frac{E^{-}t_2^3}{3[1-(\mu^{-})^2]} \tag{36}$$

which satisfies $D^{*} = D^{+} + D^{-}$, Eqs. (31) to (35) may be written as

$$\left.\begin{aligned} M_x &= D^{+}(\chi_x + \mu^{+}\chi_y) + D^{-}(\chi_x + \mu^{-}\chi_y) \\ M_y &= D^{+}(\chi_y + \mu^{+}\chi_x) + D^{-}(\chi_y + \mu^{-}\chi_x) \\ M_{xy} &= D^{+}(1-\mu^{+})\chi_{xy} + D^{-}(1-\mu^{-})\chi_{xy} \\ Q_x &= -D^{*}\frac{\partial}{\partial x}\nabla^2 w, \quad Q_y = -D^{*}\frac{\partial}{\partial y}\nabla^2 w \end{aligned}\right\} \tag{37}$$

Obviously, when $E^{+} = E^{-}$, $\mu^{+} = \mu^{-}$, $t_1 = t_2 = t/2$ holds, and all the formulas may be reduced to the counterparts of classical plate problems.

4. Application in polar coordinate systems

4.1 Non-axisymmetric bending

Now that the solution of a bimodular thin plate with rectangular cross section has been obtained, we may easily obtain the solution of a circular thin plate via coordinate conversion. In polar coordinate systems, the governing differential equation of the neutral layer is

$$\nabla^4 w(r, \theta) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2}\right)^2 w(r, \theta) = \frac{q}{D^*} \tag{38}$$

where, D^* is the flexural rigidity of the bimodular circular plate and is determined by Eq. (29). The bending moments M_r, M_θ, the twist moment $M_{r\theta} = M_{\theta r}$ and the shear forces Q_r, Q_θ are, respectively

$$\left.\begin{aligned}
M_r &= -D^+\left(\frac{\partial^2 w}{\partial r^2} + \frac{\mu^+}{r}\frac{\partial w}{\partial r} + \frac{\mu^+}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) - D^-\left(\frac{\partial^2 w}{\partial r^2} + \frac{\mu^-}{r}\frac{\partial w}{\partial r} + \frac{\mu^-}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\\
M_\theta &= -D^+\left(\mu^+\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) - D^-\left(\mu^-\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\\
M_{r\theta} &= -D^+(1-\mu^+)\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right) - D^-(1-\mu^-)\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right)\\
Q_r &= -D^*\frac{\partial}{\partial r}\nabla^2 w, \quad Q_\theta = -D^*\frac{1}{r}\frac{\partial}{\partial \theta}\nabla^2 w
\end{aligned}\right\} \tag{39}$$

4.2 Axisymmetric bending: an example

If we consider axisymmetric bending of a bimodular circular plate, the governing differential equation of the neutral layer should be simplified as

$$\nabla^2 w(r) = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 w(r) = \frac{q}{D^*} \tag{40}$$

and the internal forces should be

$$\left.\begin{aligned}
M_r &= -D^+\left(\frac{\partial^2 w}{\partial r^2} + \frac{\mu^+}{r}\frac{\partial w}{\partial r}\right) - D^-\left(\frac{\partial^2 w}{\partial r^2} + \frac{\mu^-}{r}\frac{\partial w}{\partial r}\right)\\
M_\theta &= -D^+\left(\mu^+\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) - D^-\left(\mu^-\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)\\
M_{r\theta} &= M_{\theta r} = 0\\
Q_r &= -D^*\frac{d}{dr}\nabla^2 w, \quad Q_\theta = 0
\end{aligned}\right\} \tag{41}$$

Let us consider a bimodular circular plate with its perimeter simply supported, which is under the action of the uniformly distributed loads q at its central potion, as shown in Fig. 3, in which r is the

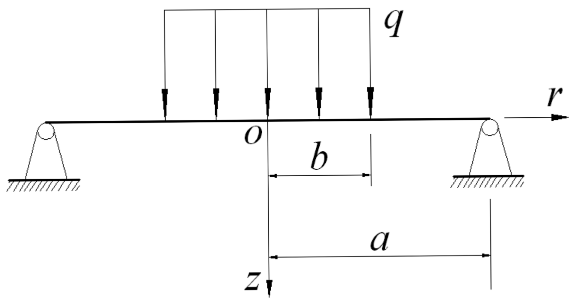


Fig. 3 Bimodular circular plate under the uniformly distributed loads at its central potion

radius of the plate and b is the area radius of the loads applied. Obviously, the bimodular thin plate is under axisymmetric bending and so we can solve this problem according to the basic formulas which have been obtained.

Due to the discontinuous loading mode, the deflection function w should be taken as

$$\left.\begin{aligned} w_1 &= C_1 r^2 + D_1 + \frac{qr^4}{64D^*}, \quad 0 \le r \le b \\ w_2 &= A_2 \ln r + B_2 r^2 \ln r + C_2 r^2 + D_2, \quad b \le r \le a \end{aligned}\right\} \tag{42}$$

where, $C_1, D_1, A_2, B_2, C_2, D_2$ are undetermined constants and D^* is the flexural rigidity of the bimodular circular plate, which has been obtained, as shown in Eq. (29). Next, we will use the boundary conditions at $r = a$ and the continuity conditions at $r = b$ to determine these unknown constants.

The boundary conditions at $r = a$ is

$$(w_2)_{r=a} = 0 \tag{43}$$

and

$$(M_{r2})_{r=a} = \left[-D^+ \left(\frac{d^2 w_2}{dr^2} + \frac{\mu^+}{r} \frac{dw_2}{dr} \right) - D^- \left(\frac{d^2 w_2}{dr^2} + \frac{\mu^-}{r} \frac{dw_2}{dr} \right) \right]_{r=a} = 0 \tag{44}$$

The continuity conditions at $r = b$ is

$$(w_2)_{r=b} = (w_1)_{r=b} \tag{45}$$

$$\left(\frac{dw_2}{dr} \right)_{r=b} = \left(\frac{dw_1}{dr} \right)_{r=b} \tag{46}$$

$$(M_{r2})_{r=b} = (M_{r1})_{r=b}, \text{ i.e.}$$

$$\begin{gathered} \left[-D^+ \left(\frac{d^2 w_2}{dr^2} + \frac{\mu^+}{r} \frac{dw_2}{dr} \right) - D^- \left(\frac{d^2 w_2}{dr^2} + \frac{\mu^-}{r} \frac{dw_2}{dr} \right) \right]_{r=b} \\ \left[-D^+ \left(\frac{d^2 w_1}{dr^2} + \frac{\mu^+}{r} \frac{dw_1}{dr} \right) - D^- \left(\frac{d^2 w_1}{dr^2} + \frac{\mu^-}{r} \frac{dw_1}{dr} \right) \right]_{r=b} \end{gathered} \tag{47}$$

and

$$(Q_{r2})_{r=b} = (Q_{r1})_{r=b}, \text{ i.e., } \left(\frac{d}{dr}\nabla^2 w_2\right)_{r=b} = \left(\frac{d}{dr}\nabla^2 w_1\right)_{r=b} \tag{48}$$

Note that Eqs. (47) and (48) may be transformed into the second and the third derivative of w_1, w_2, respectively

$$\left(\frac{d^2 w_2}{dr^2}\right)_{r=b} = \left(\frac{d^2 w_1}{dr^2}\right)_{r=b} \tag{49}$$

and

$$\left(\frac{d^3 w_2}{dr^3}\right)_{r=b} = \left(\frac{d^3 w_1}{dr^3}\right)_{r=b} \tag{50}$$

Thus, the six undetermined constants may be solved via Eqs. (43)-(46), (49) and (50).

Alternatively, it is more convenient to use the static equilibrium at $r = a$ to solve firstly B_2. From $\Sigma F_z = 0$, we have

$$2\pi a(Q_{r2})_{r=a} + \pi b^2 q = 0 \tag{51}$$

Substituting $(Q_{r2})_{r=a} = -4D^* B_2/a$ into Eq. (51), we obtain

$$B_2 = \frac{qb^2}{8D^*} \tag{52}$$

From Eqs. (50), (44), (43), (49) and (45) in turn, we obtain

$$A_2 = \frac{qb^4}{16D^*} \tag{53}$$

$$\begin{aligned} C_2 = \frac{qb^2}{32D^* K}\{&D^+[(1-\mu^+)b^2/a^2 - 4(1+\mu^+)\ln a - 2(3+\mu^+)] \\ &+ D^-[(1-\mu^-)b^2/a^2 - 4(1+\mu^-)\ln a - 2(3+\mu^-)]\} \end{aligned} \tag{54}$$

$$\begin{aligned} D_2 = \frac{qb^2}{32D^* K}\{&D^+[(\mu^+ - 1)b^2 + 2(3+\mu^+)a^2 - 2(1+\mu^+)b^2 \ln a] \\ &+ D^-[(\mu^- - 1)b^2 + 2(3+\mu^-)a^2 - 2(1+\mu^-)b^2 \ln a]\} \end{aligned} \tag{55}$$

$$\begin{aligned} C_1 = \frac{qb^2}{32D^* K}\{&D^+[(1-\mu^+)b^2/a^2 + 2(1+\mu^+)(1+2\ln(b/a)) - 2(3+\mu^+)] \\ &+ D^-[(1-\mu^-)b^2/a^2 + 2(1+\mu^-)(1+2\ln(b/a)) - 2(3+\mu^-)]\} \end{aligned} \tag{56}$$

and

$$\begin{aligned} D_1 = \frac{qb^2}{32D^* K}\{&D^+[(\mu^+ - 1)b^2 + 2(1+\mu^+)b^2(\ln(b/a) - 5/4) + 2(3+\mu^+)a^2] \\ &+ D^-[(\mu^- - 1)b^2 + 2(1+\mu^-)b^2(\ln(b/a) - 5/4) + 2(3+\mu^-)a^2]\} \end{aligned} \tag{57}$$

where

$$K = D^{+}(1+\mu^{+}) + D^{-}(1+\mu^{-}) \tag{58}$$

which depends lastly on the plate thickness t and the elastic coefficients of this bimodular material, i.e., $E^{+}, E^{-}, \mu^{+}, \mu^{-}$. Substituting these determined constants into Eq. (42), we obtain the deflection formulas as follows

$$\begin{aligned} w_1 = {} & \frac{qr^4}{64D^*} + \frac{qb^2}{32D^*K}\{D^{+}[(1-\mu^{+})b^2(r^2/a^2-1) - 2(3+\mu^{+})(r^2-a^2) \\ & + 2(1+\mu^{+})(r^2+2r^2\ln(b/a)+b^2\ln(b/a)-b^2 5/4)] + D^{-}[(1-\mu^{-})b^2(r^2/a^2-1) \\ & - 2(3+\mu^{-})(r^2-a^2) + 2(1+\mu^{-})(r^2+2r^2\ln(b/a)+b^2\ln(b/a)-b^2 5/4)]\} \end{aligned}$$

when $0 \le r \le b$ (59)

$$\begin{aligned} w_2 = {} & \frac{qb^4}{16D^*}\ln r^2 + \frac{qb^2}{8D^*}r^2\ln r \\ & + \frac{qb^2}{32D^*K}\{D^{+}[(1-\mu^{+})b^2(r^2/a^2-1) - 2(3+\mu^{+})(r^2-a^2) - 2(1+\mu^{+})(2r^2+b^2)\ln a] \\ & + D^{-}[(1-\mu^{-})b^2(r^2/a^2-1) - 2(3+\mu^{-})(r^2-a^2) - 2(1+\mu^{-})(2r^2+b^2)\ln a]\} \end{aligned}$$

when $b \le r \le a$ (60)

The maximum deflection takes place at the center of the plate, i.e., $w_{\max} = (w_1)_{r=0} = D_1$.

Note that the bimodular circular plate we solve is a relatively-variable mechanical model, in which the area of the applied loads is variable, i.e., $0 \le b/a \le 1$. Although the deflection formulas becomes more complicated due to the continuity conditions at $r = b$, based on the solution above, we can easily obtain the solutions of another two problems. When $b/a \to 1$, i.e., $b = a$, we may obtain the solution of the same bimodular circular plate under the uniformly distributed loads acted on the whole plate surface. Taking $b = a$ in Eq. (59), we obtain

$$w = \frac{qa^4}{64D^*K}\left\{\frac{r^4}{a^4}K - 2\frac{r^2}{a^2}[D^{+}(3+\mu^{+}) + D^{-}(3+\mu^{-})] + [D^{+}(5+\mu^{+}) + D^{-}(5+\mu^{-})]\right\} \tag{61}$$

And, when letting $b/a \to 0$, i.e., $b \to 0$ and keeping the total applied loads $P = \pi b^2 q$ constant, we also obtain the solution of another important problem which a bimodular circular plate is under the centrally concentrated force. Substituting P/π for πb^2, and letting $b \to 0$ in Eq. (60), we obtain

$$\begin{aligned} w = {} & \frac{P}{8\pi D^*K}\{Kr^2\ln r + D^{+}[(3+\mu^{+})(a^2-r^2)/2 - (1+\mu^{+})r^2\ln a] \\ & + D^{-}[(3+\mu^{-})(a^2-r^2)/2 - (1+\mu^{-})r^2\ln a]\} \end{aligned} \tag{62}$$

When $E^{+} = E^{-} = E$, $\mu^{+} = \mu^{-} = \mu$, we have $D^{+} = D^{-} = D/2, K = D(1+\mu)$, where D is the flexural stiffness of singular modulus problem, all the solutions we obtain may be reduced to the counterparts of the classical problems.

5. Results and discussions

From Eqs. (27) and (29), it may be seen that the flexural stiffness of the bimodular plate depends on the elastic coefficients of this bimodular material and the plate thickness in tension and compression. However, it is hard to judge that the introduction of bimodularity increases or decreases the flexural stiffness of the plate because it depends lastly on how the magnitudes of different moduli and singular modulus are taken. For the convenience of comparisons, under the conditions $\mu^+/E^+ = \mu^-/E^-$, let us introduce the following relations (Li 1990)

$$E = \frac{E^+ + E^-}{2}, \quad \beta = \frac{E^+ - E^-}{E^+ + E^-}, \quad E^+ = (1+\beta)E, \quad E^- = (1-\beta)E$$
$$\mu = \frac{\mu^+ + \mu^-}{2}, \quad \beta = \frac{\mu^+ - \mu^-}{\mu^+ + \mu^-}, \quad \mu^+ = (1+\beta)\mu, \quad \mu^- = (1-\beta)\mu \tag{63}$$

The dimensionless expressions of Eqs. (27) and (29) are, respectively

$$t_1' = \frac{t_1}{t} = \left[1 + \sqrt{\frac{(1+\beta)}{(1-\beta)}\frac{1-(1-\beta)^2\mu^2}{1-(1+\beta)^2\mu^2}}\right]^{-1} \tag{64}$$

and

$$D^{*\prime} = \frac{D^*}{Et^3} = \frac{(1+\beta)(t_1')^3}{3[1-(1+\beta)^2\mu^2]} + \frac{(1-\beta)(1-t_1')^3}{3[1-(1-\beta)^2\mu^2]} \tag{65}$$

where, t_1' and $D^{*\prime}$ are the dimensionless quantities of t_1 and D^*, respectively. For the common bimodular materials, e.g., organic glass, graphite and some alloys, the ratio of tensile modulus to compressive one (or compressive modulus to tensile one) is no more than 5, therefore the coefficient β mainly ranges from −2/3 to 2/3. We plot the relation between t_1/t and β as well as the relation between D^*/Et^3 and β, as shown in Figs. 4 and 5, respectively.

From Fig. 4, it is observed that

(1) The relation between t_1/t and β is approximately linear although Eq. (64) indicates that the relation is essentially nonlinear.

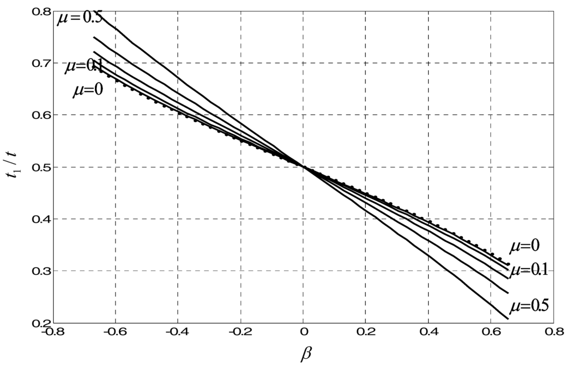


Fig. 4 The relation between t_1/t and β (The dotted line and five curves correspond to $\mu = 0$, 0.1, 0.2, 0.3, 0.4, 0.5, respectively)

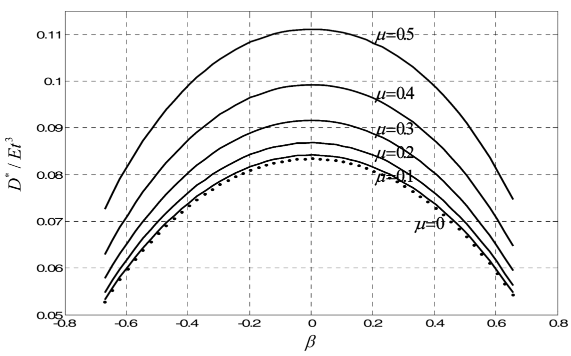


Fig. 5 The relation between D^*/Et^3 and β (The dotted line and five curves correspond to $\mu = 0$, 0.1, 0.2, 0.3, 0.4, 0.5, respectively)

(2) When $\beta = 0$, i.e., $E^+ = E^-$, the tensile thickness of the plate is exactly a half of the whole thickness ($t_1/t = 0.5$); when $\beta > 0$, i.e., $E^+ > E^-$, the tensile part of the plate becomes thinner as Poisson's ratio increases; When $\beta < 0$, i.e., $E^+ < E^-$, the tensile part of the plate becomes thicker as Poisson's ratio increases.

(3) If we take $\mu = 0$, Eq. (27) is simplified as

$$t_1 = \frac{\sqrt{E^-}}{\sqrt{E^+} + \sqrt{E^-}} t, \quad t_2 = \frac{\sqrt{E^+}}{\sqrt{E^+} + \sqrt{E^-}} t \tag{66}$$

which is exactly the formulas of tensile-compressive section height of a bimodular beam and has been obtained in different way (Ambartsumyan 1986, Yao and Ye 2004a, He *et al*. 2007a), therefore, the case of $\mu = 0$ corresponds to the case of bimodular beams. When $\beta > 0$, i.e., $E^+ > E^-$, the tensile height of the beam is the upper limit among all cases of $\mu = 0$, 0.1, 0.2, 0.3, 0.4, 0.5; when $\beta < 0$, i.e., $E^+ < E^-$, however, it becomes the lower limit among all cases.

From Fig. 5, it is observed that

(1) All the curves are axisymmetrical with respect to $\beta = 0$ and are convex with respect to the axis of D^*/Et^3. Obviously, $\beta = 0$ i.e., $E^+ = E^-$, corresponds to the highest points of all curves, which indicates that the introduction of bimodularity always decrease the flexural stiffness of the plate.

(2) The flexural stiffness of the plate will decrease as Poisson's ratio decreases and the case $\mu = 0$ is the lower limit among all cases of $\mu = 0$, 0.1, 0.2, 0.3, 0.4, 0.5.

(3) Similarly, if we take $\mu = 0$, Eq. (29) is simplified as

$$D^* = \frac{E^+ t_1^3 + E^- t_2^3}{3} \tag{67}$$

which is exactly the flexural stiffness of a bimodular beam (the width of beam cross section is taken as unit 1) and has been obtained in different way (Ambartsumyan 1986, Yao and Ye 2004a, He *et al*. 2007a), therefore, the case of $\mu = 0$ corresponds to the bimodular beams.

6. Conclusions

Based on the continuity conditions of the neutral layer, we analyze the bending problem of a bimodular thin plate and derive the governing differential equation for deflection, the flexural rigidity, the thicknesses in tension and compression and the internal forces in the plate. The bimodularity of materials has great influences on the flexural rigidity of the plate and consequently, the great changes may occur in stress, strain and displacement in the plate.

The work in this paper makes it possible to obtain the analytical solution without much effort otherwise we must resort to the numerical method based on iterative strategy. However, it should be noted here that the results obtained in this paper is based on classical Kirchhoff hypotheses, which aims to the relatively thin plates in small deflection. The results are not applicable to the analysis of bimodular thick plates. In thick plates, the shearing stresses are important. Such bimodular thick plates are treated by means of a more general theory owing to the fact that some assumptions from Kirchhoff hypotheses are no longer appropriate. Similarly, the results are not applicable to the analysis of bimodular thin plates in large deflection in that Kirchhoff hypotheses is only valid for the bending problem in small deflection.

This work will be helpful for predicting the mechanical behaviors of bimodular materials. In particular, these results may be useful to analyze concrete-like materials and fiber-reinforced composite materials that contain cracks and undergoing contact, whose macroscopic constitutive behavior depends on the direction of the macroscopic strain, similarly to the case of the bimodular materials (Shin *et al*. 2008, Greco 2009).

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