Structural Engineering and Mechanics, Vol. 35, No. 4 (2010) 387-407
DOI: http://dx.doi.org/10.12989/sem.2010.35.4.387

Nonlinear response of a resonant viscoelastic microbeam under an electrical actuation

M. Zamanian[1], S.E. Khadem[1]* and S.N. Mahmoodi[2]

[1]*Mechanical & Aerospace Engineering Department, Tarbiat Modares University, Tehran, Iran*
[2]*Center for Vehicle Systems & Safety, Department of Mechanical Engineering, Virginia Tech, Blacksburg, VA 24061, USA*

(Received November 7, 2007, Accepted February 1, 2010)

Abstract. In this paper, using perturbation and Galerkin method, the response of a resonant viscoelastic microbeam to an electric actuation is obtained. The microbeam is under axial load and electrical load. It is assumed that midplane is stretched, when the beam is deflected. The equation of motion is derived using the Newton's second law. The viscoelastic model is taken to be the Kelvin-Voigt model. In the first section, the static deflection is obtained using the Galerkin method. Exact linear symmetric mode shape of a straight beam and its deflection function under constant transverse load are used as admissible functions. So, an analytical expression that describes the static deflection at all points is obtained. Comparing the result with previous research show that using deflection function as admissible function decreases the computation errors and previous calculations volume. In the second section, the response of a microbeam resonator system under primary and secondary resonance excitation has been obtained by analytical multiple scale perturbation method combined with the Galerkin method. It is shown, that a small amount of viscoelastic damping has an important effect and causes to decrease the maximum amplitude of response, and to shift the resonance frequency. Also, it shown, that an increase of the DC voltage, ratio of the air gap to the microbeam thickness, tensile axial load, would increase the effect of viscoelastic damping, and an increase of the compressive axial load would decrease the effect of viscoelastic damping.

Keywords: viscoelastic damping; static deflection; mode shape; resonance excitation; perturbation methods; Galerkin method.

1. Introduction

Low weight, small size, low consumption energy and high durability of micro electro mechanical systems (MEMS) have increased the use of these systems. In many circumstances, these systems are used as an actuator or sensor. So, a driving or sensing electrode lay at opposite configuration to the microbeam and an electrical force is applied to the microbeam. Depending on the nature of device, the electrical load is composed of DC and AC part polarization. Often, the DC part is used to apply a constant deflection to the beam and AC part is used to excite the harmonic modes of beam. Increasing of the applied voltage will cause the beam stick to the electrode. This failure voltage, which is recognized as a failure mode of the system is called pull in voltage. Many

*Corresponding author, Professor, E-mail: khadem@modares.ac.ir

researches have been performed to analyze behavior the MEMS system. Ijntema and Tilmans (1992) obtained the natural frequency of vibration for the beam by considering the static deflection of the beam. They used the Rayleigh method and did not consider the midplane stretching. Tilmans and Legtenberg (1994) used the Rayleigh-Ritz method to obtain the natural frequency and static deflection of microbeam. Midplane stretching has been neglected in this paper. They compared their theoretical results with experimental results. It was shown a good agreement between theoretical and experimental for small deflection. If the amplitude of deflection be large, then it causes the system be hardened. Chio and Lovell (1997) obtained the static deflection by numerical shooting method by considering this hardening (midplane stretching). Abdel-Rahman *et al.* (2002) obtained the static deflection and natural frequency of deflected beam by numerical shooting method. In that research, microbeam has been assumed under axial load and an electrical force. Also, they assumed that midplane is stretched when the beam is deflected. This research showed that natural frequency is altered depending on the midplane stretching. Younis *et al.* (2003) accomplished a research using the reduced order method and the Galerkin method. They considered the equation in two parts, i by expanding electrostatic term to the fifth term about undeflected position and ii by multiplying the equation by the denominator of electrostatic force term. They solved the static deflection equation by the Galerkin method. They used five symmetric mode shapes of linear straight beam as admissible function. These admissible functions have been obtained by numerical shooting method. Younis and Nayfeh (2003) investigated the response of a resonant microbeam to a primary resonance electric actuation. They considered midplane stretching and axial load. They used perturbation method and obtained the modulation of the amplitude and phase of the response. The boundary value equation resulted from the solution process have been solved by numerical shooting method. A similar investigation has been performed on a subharmonic and superharmonic electric actuation by Abdel-Rahman and Nayfeh (2003). Zhang and Meng (2005) used a lamp mass model and described the dynamic behavior of micro cantilever beam by Mathieu equation. The harmonic balance method was applied to simulate the resonant amplitude frequency response under the combined parametric and forcing excitation. Najar *et al.* (2005) analyzed the deflection and motion of a shaped microbeam. Their model was included nonlinearities resulted from midplane stretching and electrostatic excitation. They used DQM method to discrete the microbeam partial deferential equation. Nayfeh and Younis (2005) analyzed the dynamics of electrically actuated microbeams under subharmonic and superharmonic resonance excitations. They used the Galerkin method to discrete equations into a finite degrees of freedom system. The linear undamped mode shape of straight microbeam has been used as the fundamental function in the Galerkin method. Zamanian *et al.* (2008, 2009, 2010) studied the static deflection, natural frequency and dynamic behavior of microbeam under DC (DC) electric (piezoelectric), DC (AC) electric (piezoelectric) and AC-DC (DC) electric (piezoelectric) actuation, respectively. Recently using the viscoelastic material in MEMS Devices has been increased. Dufour *et al.* (2007) considered the quality factor of viscoelastic damping in a microbeam that a viscoelastic polymer layer has been deposited on it. This configuration may be assumed a chemical sensor where the shift of resonance frequency is used as criteria of adsorption of molecule. Uncuer *et al.* (2007) studied the linear effect of structural viscoelastic damping on the behavior of clamped-free and clamped-clamped microbeams under shock and electrostatic actuation using the Kelvin-Voigt model. Fu and Zhang (2009) studied nonlinear static and dynamic response of viscoelastic microcantilever beam under combination of AC and DC electric actuation. They used standard viscoelastic model and neglected from the midplane stretching and axial load. They discretized the motion equation using single mode in

Galerkin method and solve them by numerical method In another research they studied the nonlinear dynamic stability for a clamped-guided viscoalestic microbeam under both a periodic axial force and a symmetric electrostatic force, Fu *et al.* (2009). Wenzel *et al.* (2009) studied the sensitivity and the response duration of a viscoelastic microcantilever beam that a stressed layer has been deposited on it.

In this paper, the response of a resonant viscoelastic microbeam to an electric actuation is investigated using the multiple scale perturbation methods. Also, in this paper the static deflection of microbeam is obtained using the comparison function proposed in previous researches and a new comparison function proposed in this paper. In contrast to previous research works.

2. Modeling and formulation

The system model is a clamped-clamped microbeam at distance d from an electrode plate. An electrical voltage is applied between the microbeam and electrode plate which is composed of a DC voltage, v_p, and a AC voltage, $v(t) = v_{ac}\cos(\hat{\Omega}t)$, in which $\hat{\Omega}$ is the frequency of excitation and t is the time (Fig. 1(a)). It must be noted that the microbeam of Fig. 1(a) is not free to deflect in axial direction, but its right end is induced to remain in a constant position which is obtained by applying the axial load. This modeling of axial load is used by Nayfeh and Pai (2004) to model the axial load in the microbeam without viscoelastic damping. It is assumed that the microbeam is initially displaced by U and W with respect to the fixed coordinate system. The fixed coordinate is shown

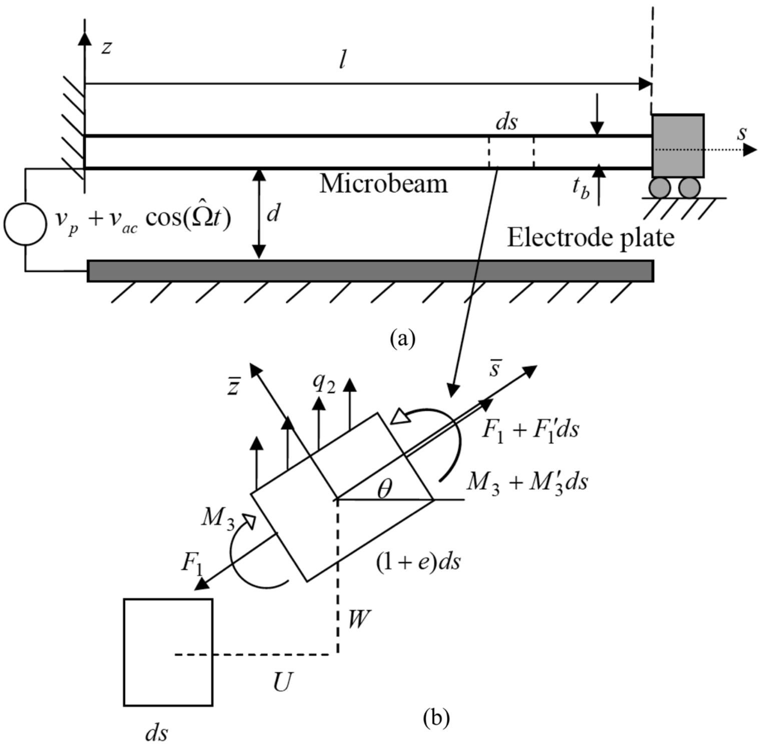

Fig. 1 (a) Model of MEMS resonator, and (b) free body diagrams of a deflected element of microbeam with s and z in the longitudinal and transverse directions, respectively, where its origin lays at the left end of microbeam on the geometric center of microbeam cross section (Fig. 1(b)).

It is assumed that the constant displacement $\bar{\varepsilon}$ is applied at the right end of microbeam. So, the motion equations for a beam subject to the transverse load per unit length q_2 and its boundary conditions may be written as following equations (Nayfeh and Pai 2004)

$$(F_1\cos(\theta))' + \left(\frac{M_3'\sin(\theta)}{1+\bar{e}}\right)' = m(s)\ddot{U}$$

$$(F_1\sin(\theta))' - \left(\frac{M_3'\cos(\theta)}{1+\bar{e}}\right)' + q_2 = m(s)\ddot{W}$$

$$U(0,t) = 0, \quad U(l,t) = \bar{\varepsilon}$$

$$W(0,t) = 0, \quad W(l,t) = 0, \quad \frac{\partial W(0,t)}{\partial s} = 0, \quad \frac{\partial W(l,t)}{\partial s} = 0 \tag{1}$$

Where M_3 is bending moment, and F_1 is the equivalent axial force in the cross section of beam as shown in Fig. 1(b). Also, dot and prime signs show the differentiation with respect to s and t, respectively, and, $m(s)$ is the mass per unit length of the microbeam, so

$$m(s) = w_b\rho_b t_b \tag{2}$$

Where, w_b and t_b are the width and thickness of microbeam cross section, respectively, and, ρ_b is the density of microbeam. Here, q_2 is the sum of load per unit length of microbeam due to the electrical field and air damping. So, it may be written as follow, Zamanian *et al.* (2009)

$$q_2 = -\frac{1}{2}\varepsilon_0 w_b \frac{(v_p + v_{ac}\cos(\hat{\Omega}t))^2}{(d+W)^2} - \hat{c}\dot{W} \tag{3}$$

Where, $\hat{c}$ is the viscous damping per unit length of microbeam, and ε_0 is the dielectric constant of the gap medium between the microbeam and electrode plate. The Kelvin-Voigt model describes the stress/strain, σ_b/ε_b, relation of viscoelastic material as

$$\sigma_b = E\varepsilon_b + \hat{C}\dot{\varepsilon}_b \tag{4}$$

Where, $\hat{C}$ and E are the viscoelastic damping factor and the elasticity modulus of microbeam, respectively. To describe the strain in the cross section of microbeam, the local coordinate $\bar{s}\,\bar{z}$ is used, where its origin lays at the geometric center of deflected microbeam cross section (Fig. 1(b)). The strain in the cross section of microbeam may be written as follow (Nayfeh and Pai 2004)

$$\varepsilon_b = e - \kappa\bar{z} \tag{5}$$

Where, e is the stretching strain of the neutral axis of cross section, and κ is the curvature about the local longitudinal axis $\bar{s}$.

$$F_1 = \int\sigma_b dA = \int_{-t_b/2}^{t_b/2}(E\varepsilon_b + \hat{C}\dot{\varepsilon}_b)w_b d\bar{z}$$

$$M_3 = -\int\sigma_b\bar{z}dA = \int_{-t_b/2}^{t_b/2}\bar{z}(E\varepsilon_b + \hat{C}\dot{\varepsilon}_b)w_b d\bar{z} \tag{6}$$

By substituting Eq. (5) in to Eq. (4), and combining it with Eq. (6)

$$F_1 = EAe + \hat{C}A\frac{\partial e}{\partial t}$$
$$M_3 = EI\kappa + \hat{C}I\frac{\partial \kappa}{\partial t} \tag{7}$$

Where, A and I are area moment of inertia and area of the microbeam cross section, respectively. The Taylor series of midplane stretching, e, curvature, κ, and other geometric relations of microbeam deflection, may be constructed as follow

$$e = \sqrt{(1+U')^2 + W'^2} - 1 = U' + \frac{1}{2}W'^2 - \frac{1}{2}U'W'^2 + \dots$$
$$\kappa = \theta' = \left(W' - U'W' + U'^2W' - \frac{1}{3}W'^3\right)'$$
$$\cos\theta = \frac{1+U'}{1+e} = 1 - \frac{1}{2}W'^2 + U'W'^2 + \dots$$
$$\sin\theta = \frac{W'}{1+e} = W' - U'W' + U'^2W' - \frac{1}{2}W'^3 + \dots$$
$$\frac{1}{1+e} = 1 - U' - \frac{1}{2}W'^2 + \frac{3}{2}U'W'^2 + U'^2\dots \tag{8}$$

By substituting Eqs. (7) and (3) into Eq. (1), and then, substituting Eqs. (8)-(11) into the resulted equation, and keeping the terms up to the third order, one obtains

$$m(s)\ddot{U} - (EAU')' - (\hat{C}A\dot{U}')' = \left(EA\left(\frac{1}{2}W'^2 - U'W'^2\right)\right)' +$$
$$[W'(EI(W' - U'W')')' - 2U'W'(EIW''')']' + (\hat{C}A(-\dot{U}'W'^2 - U'W'\dot{W}' + W'\dot{W}'))' +$$
$$[W'(\hat{C}I(\dot{W}' - \dot{U}'W' - \dot{U}'\dot{W}')')' - 2U'W'(\hat{C}I\dot{W}''')']' \tag{9}$$

and

$$(EIW''')'' + (\hat{C}I\dot{W}''')'' + m(s)\ddot{W} + \hat{c}\dot{W} = \left(EA\left(U'W' - U'^2W' + \frac{1}{2}W'^3\right)\right)' +$$
$$(\hat{C}A^b(\dot{U}'W'U' + W'^2\dot{W}' + \dot{U}'W'))' + [(EI(U'W')')'(1-U') +$$
$$(EIW''')'(U' - U'^2 + W'^2) - \left(EI\left(U'^2W' - \frac{1}{3}W'^3\right)'\right)'\Big]' +$$
$$[(\hat{C}I(\dot{U}'W' + U'\dot{W}')')'(1-U') + (\hat{C}I\dot{W}''')'(U' - U'^2 + W'^2) -$$
$$(\hat{C}I(U'^2\dot{W}' + 2U'\dot{U}'W' - W'^2\dot{W}')')']' - \frac{1}{2}\varepsilon_0 w_b \frac{(v_p + v_{ac}\cos(\hat{\Omega}t))^2}{(d+W)^2} \tag{10}$$

For slender beam the longitudinal inertia may be negligible (Nayfeh and Pai 2004). Also, since the magnitude of $\sqrt{I/A}$ in the microbeam is $O(10^{-12})$, so, the nonlinear terms with coefficient EI may be neglected comparing to the nonlinear terms with coefficient EA. In this case, it follows from Eq. (9) that

$$-EAU'' - \hat{C}A\frac{\partial}{\partial t}U'' = \frac{1}{2}EA(W'^2)' + \frac{1}{2}\hat{C}A\frac{\partial}{\partial t}(W'^2)' \tag{11}$$

It results that $U = O(W^2)$, and

$$U' = -\frac{1}{2}W'^2 + C_1(t) \tag{12}$$

By integrating Eq. (16) from s = 0 to s = l, where l is the microbeam length, one obtains

$$U(l) - U(0) = \frac{-1}{2}\int_0^l W'^2 ds + C_1(t)l \tag{13}$$

Where $C_1(t)$ is the constant of integrating. By substituting the boundary conditions from Eq. (1) into Eq. (17)

$$C_1(t) = \frac{\bar{\varepsilon}}{l} + \frac{1}{2l}\int_0^l W'^2 ds \tag{14}$$

One variable equation may be obtained by substituting U' from Eq. (16) into Eq. (14) and using the assumptions of Eq. (15), considering that $U = O(W^2)$ and keeping the terms up to cubic terms

$$EI\,W'''' + m(s)\ddot{W} + \hat{c}\dot{W} + \hat{C}I\dot{W}'''' = EAC_1(t)W'' + \hat{C}A\dot{C}_1(t)W'' - \frac{1}{2}\varepsilon_0 w_b\frac{(v_p + v_{ac}\cos(\hat{\Omega}t))^2}{(d+W)^2} \tag{15}$$

For obtaining the equation in dimensionless form, the following variables are used.

$$w = \frac{-W}{d},\quad x = \frac{s}{l},\quad \tau = \frac{t}{T},\quad T = \sqrt{\frac{\rho_b w_b t_b l^4}{EI}} \tag{16}$$

So, the dimensionless form of Eq. (20) will be

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial \tau^2} + C\frac{\partial^5 w}{\partial x^4 \partial \tau} + c\frac{\partial w}{\partial \tau} = [\alpha_1\Gamma(w,w) + N]\frac{\partial^2 w}{\partial x^2} + 2\alpha_1 C\Gamma(\frac{\partial w}{\partial \tau}, w)\frac{\partial^2 w}{\partial x^2} + \frac{\alpha_2(v_p + v_{ac}\cos(\Omega\tau))^2}{(1-w)^2}$$

$$w\big|_{x=0} = 0,\quad \frac{\partial w}{\partial x}\Big|_{x=0} = 0,\quad w\big|_{x=1} = 0,\quad \frac{\partial w}{\partial x}\Big|_{x=1} = 0 \tag{17}$$

in which

$$\Gamma(f_1, f_2) = \int_0^l \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial x}dx$$

$$N = \frac{\bar{\varepsilon}Al}{I},\quad c = \frac{\hat{c}l^4}{EIT},\quad C = \frac{\hat{C}}{ET},\quad \alpha_1 = 6(\frac{d}{t_b})^2,\quad \alpha_2 = \frac{6\varepsilon_0 l^4}{Et_b^3 d^3},\quad \Omega = \hat{\Omega}T \tag{18}$$

Comparing the motion equations obtained for the viscoelastic microbeam with motion equations obtained for elastic microbeam by Abdel-Rahman *et al.* (2002) shows that one linear term with coefficient C and one nonlinear term with coefficient $2\alpha_1 C$ have been added to the elastic equation. By assuming that differentiation of time and AC voltage will be equal to zero in Eq. (22), the differential equation of static deflection will be

$$\frac{d^4 w_s}{dx^4} = [\alpha_1 \Gamma(w_s, w_s) + N]\frac{d^2 w_s}{dx^2} + \frac{\alpha_2 v_p^2}{(1-w_s)^2}$$

$$w_s\big|_{x=0} = 0, \quad \frac{dw_s}{dx}\Big|_{x=0} = 0, \quad w_s\big|_{x=1} = 0, \quad \frac{dw_s}{dx}\Big|_{x=1} = 0 \tag{19}$$

The displacement of system is the sum of static deflection w_s, and dynamic deflection $u(x,t)$, so

$$w(x,t) = u(x,\tau) + w_s \tag{20}$$

By substituting Eq. (20) into Eq. (17), and expanding the electrical force about the static position and by using Eq. (19), the terms that represent equilibrium position are eliminated, and one obtains:

By assuming that $\phi(x)$ is the linear mode shapes and ω is its natural frequency of vibration of deflected microbeam about static position, then

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial \tau^2} + c\frac{\partial u}{\partial \tau} + C\frac{\partial^5 u}{\partial x^4 \partial \tau} = [\alpha_1 \Gamma(w_s, w_s) + N]\frac{\partial^2 u}{\partial x^2} + 2\alpha_1 \Gamma(w_s, u)\frac{d^2 w_s}{dx^2} + \alpha_1 \Gamma(u,u)\frac{d^2 w_s}{dx^2}$$

$$+2\alpha_1 \Gamma(w_s, u)\frac{\partial^2 u}{\partial x^2} + \alpha_1 \Gamma(u,u)\frac{\partial^2 u}{\partial x^2} + 2\alpha_1 C\Gamma(\frac{\partial u}{\partial \tau}, w_s)\frac{d^2 w_s}{dx^2} + 2\alpha_1 C\Gamma(\frac{\partial u}{\partial \tau}, u)\frac{d^2 w_s}{dx^2}$$

$$+2\alpha_1 C\Gamma(\frac{\partial u}{\partial \tau}, w_s)\frac{\partial^2 u}{\partial x^2} + 2\alpha_1 C\Gamma(\frac{\partial u}{\partial \tau}, u)\frac{\partial^2 u}{\partial x^2} + \frac{2\alpha_2 v_p^2}{(1-w_s)^3}u + \frac{3\alpha_2 v_p^2}{(1-w_s)^4}u^2 + \frac{4\alpha_2 v_p^2}{(1-w_s)^5}u^3$$

$$+\frac{2\alpha_2 v_p v_{ac}\cos(\Omega\tau)}{(1-w_s)^2} + \frac{4\alpha_2 v_p v_{ac}\cos(\Omega\tau)}{(1-w_s)^3}u + \frac{\alpha_2}{(1-w_s)^2}(v_{ac}\cos(\Omega\tau))^2 + \ldots$$

$$u\big|_{x=0} = u\big|_{x=1} = 0, \qquad \frac{\partial u}{\partial x}\Big|_{x=0} = \frac{\partial u}{\partial x}\Big|_{x=1} = 0 \tag{21}$$

$$u = \phi(x)e^{i\omega\tau} \tag{22}$$

By substituting this equation into Eq. (21), the differential equation of modal system will be

$$\frac{d^4\phi}{dx^4} - [\alpha_1 \Gamma(w_s, w_s) + N]\frac{d^2\phi}{dx^2} - 2\alpha_1 \Gamma(w_s, \phi)\frac{d^2 w_s}{dx^2} - (\frac{2\alpha_2 v_p^2}{(1-w_s)^3} + \omega^2)\phi = 0$$

$$\phi\big|_{x=0} = 0, \quad \frac{d\phi}{dx}\Big|_{x=0} = 0, \quad \phi\big|_{x=1} = 0, \quad \frac{d\phi}{dx}\Big|_{x=1} = 0 \tag{23}$$

3. Static deflection of microbeam

In this part, the solution of Eq. (25) is obtained using the Galerkin method. Right selection of comparison function is a very important step in this method. Here, the symmetric mode shapes of straight microbeam proposed by Younis *et al.* (2003), and a new function proposed in this paper are used as comparison functions. The proposed function in this paper is to be chosen deflection of microbeam by assuming that microbeam is under constant transverse load equal to electrical force applied to a straight microbeam. By using this assumption, and considering axial load and neglecting midplane stretching, the differential equation of static deflection $w_{s[0]}$ will be

$$\frac{d^4 w_{s[0]}}{dx^4} = N \frac{d^2 w_{s[0]}}{dx^2} + \alpha_2 {v_p}^2$$

$$w_{s[0]}\big|_{x=0} = 0, \quad \frac{dw_{s[0]}}{dx}\Big|_{x=0} = 0, \quad w_{s[0]}\big|_{x=1} = 0, \quad \frac{dw_{s[0]}}{dx}\Big|_{x=1} = 0 \tag{24}$$

This equation is a linear differential equation with constant coefficients. The solution of equation is composed of homogenous solution and particular solution, which is dependent on the sign of axial load. So

If ($N > 0$)

$$w_{s[0]} = c_1 \cosh(\sqrt{N}x) + c_2 \sinh(\sqrt{N}x) + c_3 x + c_4 - \frac{\alpha_2 v_p^2}{2N} x^2 \tag{25}$$

If ($N < 0$)

$$w_{s[0]} = c_1 \cos(\sqrt{-N}x) + c_2 \sin(\sqrt{-N}x) + c_3 x + c_4 - \frac{\alpha_2 v_p^2}{2N} x^2 \tag{26}$$

If ($N = 0$)

$$w_{s[0]} = c_1 x^3 + c_2 x^2 + c_3 x + c_4 + \frac{\alpha_2 v_p^2}{24} x^4 \tag{27}$$

The coefficients c_1, c_2, c_3 and c_4 are unknown coefficients that are obtained by substituting the boundary conditions of Eq. (24) into Eqs. (25), (26) and (27). Considering Eq. (23), the differential equation that is governed to the mode shapes of straight microbeam without an electrical force may be written as following form

$$\frac{d^4 \phi_{1[j]}}{dx^4} - N \frac{d^2 \phi_{1[j]}}{dx^2} - (\omega_{1[j]})^2 \phi_{1[j]} = 0, \qquad j = 1, 2, 3, \ldots$$

$$\phi_{1[j]}\big|_{x=0} = 0, \quad \frac{d\phi_{1[j]}}{dx}\Big|_{x=0} = 0, \quad \phi_{1[j]}\big|_{x=1} = 0, \quad \frac{d\phi_{1[j]}}{dx}\Big|_{x=1} = 0 \tag{28}$$

Where $\phi_{1[j]}$ is jth mode shape and $\omega_{1[j]}$ is jth natural frequency of the straight microbeam without an electrical force. This equation is a linear differential equation with constant coefficients. So, its characteristic equation is

$$r^4 - Nr^2 - (\omega_{1[j]})^2 = 0 \tag{29}$$

The roots of this equation are

$$r_{1,2} = \pm \sqrt{\frac{N + \sqrt{N^2 + 4(\omega_{1[j]})^2}}{2}}$$

$$r_{3,4} = \pm \sqrt{\frac{\sqrt{N^2 + 4(\omega_{1[j]})^2} - N}{2}}\, i \tag{30}$$

The positive sign is for r_1 and r_3 and the negative sign is for r_2 and r_4, also, i shows imaginary part. So, the homogeneous solution of Eq. (29) will be

$$\phi_{1[j]} = c_1'\cosh(r_1 x) + c_2'\sinh(r_1 x) + c_3'\sin(r_3 x) + c_4'\cos(r_3 x) \tag{31}$$

The coefficients c_1', c_2', c_3' and c_4' are unknown coefficients of homogeneous solution of Eq. (28). An algebraic system is obtained by applying boundary conditions of Eq. (28) into (31). The resulted equation has nonzero solution if determinant of coefficients of algebraic system be zero. So, by equating determinant of coefficients to zero, one can obtain the natural frequency and mode shapes of vibration of a straight microbeam. By denoting the jth symmetric mode shape of the straight microbeam by $w_{s[j]}$, one has

$$w_{s[j]} = \phi_{1[2j-1]}, \quad j = 1, 2, 3, \ldots \tag{32}$$

It is assumed that solution of Eq. (19) is

$$w_s = a_{[0]} w_{s[0]} + \sum_{j=1}^{n} a_{[j]} w_{s[j]} = \sum_{j=0}^{n} a_{[j]} w_{s[j]} \tag{33}$$

Where, $a_{[j]}$ are the constants that must be obtained, when the Glaerkin method is applied to Eq. (25). By multiplying $(1-w_s)^2$ by Eq. (19), and substituting Eq. (33) into Eq. (19), and multiplying the resulting equation by $w_{s[n]}$, $n = 0, 1, 2, 3, \ldots, M$ and integrating the outcome from $x = 0$ to $x = 1$, one obtains ($M+1$) algebraic equations as following

$$\begin{aligned}
&\sum_{j=0}^{M} a_{[j]} \int_0^1 w_{s[n]} \frac{d^4 w_{s[j]}}{dx^4} dx + \int_0^1 \sum_{j,k,m=0}^{M} a_{[j]} a_{[k]} a_{[m]} \frac{d^4 w_{s[j]}}{dx^4} w_{s[k]} w_{s[m]} w_{s[n]} dx - \\
&2\int_0^1 \sum_{j,k=0}^{M} a_{[j]} a_{[k]} \frac{d^4 w_{s[j]}}{dx^4} w_{s[k]} w_{s[n]} dx - \alpha_1 \sum_{j,k,m=0}^{M} a_{[j]} a_{[k]} a_{[m]} \Gamma(w_{s[j]}, w_{s[k]}) \int_0^1 \frac{d^2 w_{s[m]}}{dx^2} w_{s[n]} dx \\
&-\alpha_1 \sum_{j,k,m,p,q=0}^{M} a_{[j]} a_{[k]} a_{[m]} a_{[p]} a_{[q]} \Gamma(w_{s[j]}, w_{s[k]}) \int_0^1 \frac{d^2 w_{s[m]}}{dx^2} w_{s[p]} w_{s[q]} w_{s[n]} dx + \\
&2\alpha_1 \sum_{j,k,m,p=0}^{M} a_{[j]} a_{[k]} a_{[m]} a_{[p]} \Gamma(w_{s[j]}, w_{s[k]}) \int_0^1 \frac{d^2 w_{s[m]}}{dx^2} w_{s[p]} w_{s[n]} dx - \\
&N \sum_{j=0}^{M} \int_0^1 a_{[j]} \frac{d^2 w_{s[j]}}{dx^2} w_{s[n]} dx - N \int_0^1 \sum_{j,k,m=0}^{M} a_{[j]} a_{[k]} a_{[m]} \frac{d^2 w_{s[j]}}{dx^2} w_{s[k]} w_{s[m]} w_{s[n]} dx + \\
&2N \int_0^1 \sum_{j,k=0}^{M} a_{[j]} a_{[k]} \frac{d^2 w_{s[j]}}{dx^2} w_{s[k]} w_{s[n]} dx - \alpha_2 v_p^2 \int_0^1 w_{s[n]} dx = 0
\end{aligned} \tag{34}$$

The coefficients $a_{[j]}$ will be obtained by solving this algebraic system, then, one can obtain static deflection by using Eq. (33).

4. Natural frequencies and mode shapes

It is clear that Eq. (23) is similar to the equation governing the mode shapes of a microbeam on an elastic foundation with spring coefficient $2\alpha_2 v_p^2$ and the axial load of $(\alpha_1 \Gamma(w_s, w_s) + N)$. So, the mode shapes of this similar system may be used as comparison functions in the Galerkin method for obtaining the solution of Eq. (23). These comparison functions may be obtained using the similar process that is used for obtaining the solution of Eq. (28). The only difference is that the terms N, $(\omega_{1[j]})^2$ and $\omega_{s[j]}$ have been replaced by $(\alpha_1\Gamma(w_s,w_s)+N),(2\alpha_2v_p^2+(\omega_{a[j]})^2)$ and $\phi_{a[j]}$, respectively. Where, $\omega_{a[j]}$ and $\phi_{a[j]}$ are jth natural frequency and the mode shape of the similar system. The solution of Eq. (23) is assumed as

$$\phi = \sum_{j=1}^{M} b_{[j]}\phi_{a[j]} \tag{35}$$

Where $b_{[j]}$ is the unknown coefficients that must be obtained using the Galerkin method. By substituting Eq. (35) into Eq. (23), and multiplying the result by $\phi_{a[n]}$, $n = 1,2,\ldots,M$, and integrating from $x = 0$ to $x = 1$ an algebraic system is obtained as

$$\int_0^1\sum_{j=1}^{M} b_{[j]}\,(1-w_s)^3\,\frac{d^4\phi_{a[j]}}{dx^4}\,\phi_{a[n]}\,dx - \int_0^1\sum_{j=1}^{M} b_{[j]}(1-w_s)^3(\alpha_1\Gamma(w_s,w_s)+N)\frac{d^2\phi_{a[j]}}{dx^2}\phi_{a[n]}dx$$

$$-2\alpha_1\int_0^1\sum_{i=1}^{M} b_{[j]}\,\Gamma(\phi_{a[j]},w_s)\phi_{a[n]}(1-w_s)^3\,\frac{d^2w_s}{dx^2}dx - \omega^2\int_0^1\sum_{j=1}^{M} b_{[j]}\,(1-w_s)^3\,\phi_{a[j]}\,\phi_{a[n]}\,dx$$

$$-2\alpha_2{v_p}^2\int_0^1\sum_{j=1}^{M} b_{[j]}\phi_{a[j]}\,\phi_{a[n]}\;dx = 0 \tag{36}$$

Eq. (23) has a nonzero solution, if the determinant of coefficient $b_{[j]}$ in Eq. (36) be equal to zero. So, by equating the determinant of coefficients to zero, one can obtain the natural frequency.

5. Dynamic response

5.1 Primary resonance

The solution of Eq. (21) may be obtained using the multiple scale perturbation method. By considering $T_0 = \tau$, $T_2 = \varepsilon^2\tau$, where ε is small non-dimensional book keeping parameter, the solution of Eq. (21) will be as following form (Younis and Nayfeh 2003)

$$u(x,\tau,\varepsilon) = \varepsilon u_1(x,T_0,T_1,T_2) + \varepsilon^2 u_2(x,T_0,T_1,T_2) + \varepsilon^3 u_3(x,T_0,T_1,T_2) + \ldots \tag{37}$$

In order that nonlinearity balances the effects of air damping c, viscoelastic damping C and excitation v_{ac}, they are considered as order $\varepsilon^2, \varepsilon^2$ and ε^3 respectively. The following equations will be obtained by substituting Eq. (37) into Eq. (21), and equaling coefficients of the same power of ε.

order (ε)

$$L(u_1) = \frac{\partial^2 u_1}{\partial {T_0}^2} + \frac{\partial^4 u_1}{\partial x^4} - [\alpha_1\Gamma(w_s,w_s)+N]\frac{\partial^2 u_1}{\partial x^2} - 2\alpha_1\Gamma(w_s,u_1)\frac{d^2 w_s}{dx^2} - \frac{2\alpha_2{v_p}^2}{(1-w_s)^3}u_1 = 0$$

$$u_1\big|_{x=0} = 0,\quad \frac{\partial u_1}{\partial x}\big|_{x=0} = 0,\quad u_1\big|_{x=1} = 0,\quad \frac{\partial u_1}{\partial x}\big|_{x=1} = 0 \tag{38}$$

order (ε^2)

$$L(u_2)=-2\frac{\partial^2 u_1}{\partial T_0 \partial T_1}\alpha_1\Gamma(u_1,u_1)\frac{d^2 w_s}{dx^2}+2\alpha_1\Gamma(w_s,u_1)\frac{\partial^2 u_1}{\partial x^2}+\frac{3\alpha_2 v_p^2}{(1-w_s)^4}u_1{}^2$$

$$u_2\big|_{x=0}=0,\quad \frac{\partial u_2}{\partial x}\Big|_{x=0}=0,\quad u_2\big|_{x=1}=0,\quad \frac{\partial u_2}{\partial x}\Big|_{x=1}=0 \tag{39}$$

order (ε^3)

$$L(u_3)=-2\frac{\partial^2 u_2}{\partial T_0 \partial T_1}-(\frac{\partial^2 u_2}{\partial T_1^2}+2\frac{\partial^2 u_1}{\partial T_0 \partial T_2})-c\frac{\partial u_1}{\partial T_0}-C\frac{\partial^5 u_1}{\partial x^4 \partial T_0}+2\alpha_1\Gamma(u_1,u_2)\frac{d^2 w_s}{dx^2}+$$

$$2\alpha_1\Gamma(w_s,u_2)\frac{\partial^2 u_1}{\partial x^2}+2\alpha_1\Gamma(w_s,u_1)\frac{\partial^2 u_2}{\partial x^2}+\alpha_1\Gamma(u_1,u_1)\frac{\partial^2 u_1}{\partial x^2}+2\alpha_1 C\,\Gamma(\frac{\partial u}{\partial T_0},w_s)\frac{d^2 w_s}{dx^2}+$$

$$\frac{6\alpha_2 v_p^2}{(1-w_s)^4}u_1 u_2+\frac{2\alpha_2 v_p v_{ac}\cos(\Omega T_0)}{(1-w_s)^2}+\frac{4\alpha_2 v_p^2}{(1-w_s)^5}u_1{}^3$$

$$u_3\big|_{x=0}=0,\quad \frac{\partial u_3}{\partial x}\Big|_{x=0}=0,\quad u_3\big|_{x=1}=0,\quad \frac{\partial u_3}{\partial x}\Big|_{x=1}=0 \tag{40}$$

In previous section, static deflections and vibration, i.e., natural frequencies and mode shapes about equilibrium position obtained. Using the normalized mode shape of system ($\int_0^1 \phi^2 dx = 1$), the solution of Eq. (38) is

$$u_1 = A(T_1,T_2)e^{i\omega T_0}\phi(x)+\bar{A}(T_1,T_2)e^{-i\omega T_0}\phi(x) \tag{41}$$

Where, $A(T_1,T_2)$ is a complex function, which is obtained by imposing solvability condition at the third order term. By substituting Eq. (41) into Eq. (39)

$$L(u_2) = (A^2 e^{2i\omega T_0}+2A\bar{A}+\bar{A}^2 e^{-2i\omega T_0})h(x)-2\omega i\Big(\frac{\partial A(T_1,T_2)}{\partial T_1}e^{i\omega T_0}-\frac{\partial \bar{A}(T_1,T_2)}{\partial T_1}e^{-i\omega T_0}\Big) \tag{42}$$

where

$$h(x) = \alpha_1\Gamma(\phi,\phi)\frac{d^2 w_s}{dx^2}+2\alpha_1\Gamma(w_s,\varphi)\frac{d^2\phi}{dx^2}+\frac{3\alpha_2 v_p^2}{(1-w_s)^4}\phi^2 \tag{43}$$

If A is only depend on T_2, then the secular term does not arise in Eq. (42). Using this assumption the particular solution of Eq. (42) would be as follow

$$u_2 = \psi_1(x)A^2 e^{2i\omega T_0}+2\psi_2(x)A\bar{A}+\psi_1(x)\bar{A}^2 e^{-2i\omega T_0} \tag{44}$$

where

$$\frac{d^4\psi_j}{dx^4}-4\omega^2\delta_{1j}\psi_j-(\alpha_1\Gamma(w_s,w_s)+N)\frac{d^2\psi_j}{dx^2}-2\alpha_1\Gamma(w_s,\psi_j)\frac{d^2 w_s}{dx^2}-\frac{2\alpha_2 v_p^2}{(1-w_s)^3}\psi_j=h(x)$$

$$j=1,2,\qquad \psi_j\big|_{x=0}=0,\quad \frac{d\psi_j}{dx}\Big|_{x=0}=0,\quad \psi_j\big|_{x=1}=0,\quad \frac{d\psi_j}{dx}\Big|_{x=1}=0 \tag{45}$$

The solution of Eq. (45) may be obtained by using the linear symmetric mode shapes of vibration of the microbeam about the static position as comparison functions in Galerkin method. So, it is assumed that

$$\psi_j = \sum_{k=1}^{M} c[k]\,\phi[k] \tag{46}$$

Where, $\phi[k]$ show kth linear symmetric mode shape of deflected microbeam about static position, and $c[k]$ are coefficients that must be obtained using the Galerkin method. In fact by solving the algebraic system that is obtained by substituting Eq. (46) into Eq. (45), multiplying the results by $\phi[n]$, $n = 1,2,\ldots,M$ and integrating the results from $x = 0$ to $x = 1$, one has

$$\begin{aligned}&\int_0^1 \sum_{k=1}^{M} c[k]\,\frac{d^4\phi[k]}{dx^4}\phi[n]\,dx - 4\omega^2\delta_{1j}\int_0^1\sum_{k=1}^{M} c[k]\phi[k]\phi[n]\,dx - \\ &\int_0^1\sum_{k=1}^{M} c[k](\alpha_1\Gamma(w_s,w_s)+N)\frac{d^2\phi[k]}{dx^2}\phi[n]dx - 2\alpha_1\int_0^1\sum_{k=1}^{M} c[k]\Gamma(\phi[k],w_s)\phi[n]\frac{d^2w_s}{dx^2}dx \\ &-2\alpha_2 {v_P}^2\int_0^1\sum_{k=1}^{M} c[k]\frac{\phi[k]\phi[n]}{(1-w_s)^3}\,dx - \int_0^1 h(x)\phi[n]dx = 0, \quad j=1,\,2\end{aligned} \tag{47}$$

Now, by substituting u_1 and u_2 from Eqs. (41) and (44) into Eq. (40), introducing fundamental natural frequency by detuning parameter σ as $\Omega = \omega + \varepsilon^2\sigma$, and keeping only the terms that produce secular terms, it results

$$L(u_3) = \left[-2\,\omega i\frac{dA}{dT_2}\varphi(x) - i\,\omega A(c\,\phi(x)+C\chi^v) + \chi(x)A^2\bar{A} + \bar{F}(x)e^{i\sigma T_2}\right]e^{i\omega T_0} + cc + NST \tag{48}$$

NST shows all terms, that is not secular and *cc* denotes the complex conjugate terms, where

$$\bar{F}(x) = \frac{\alpha_2 v_p v_{ac}}{(1-w_s)^2} \tag{49}$$

and

$$\begin{aligned}&\chi(x) = \chi_q^G + \chi_c^G + \chi_q^E + \chi_c^G \\ &\chi_q^G = (2\alpha_1\Gamma(\psi_1,\varphi)+4\alpha_1\Gamma(\psi_2,\varphi))\frac{d^2w_s}{dx^2} + (2\alpha_1\frac{d^2\psi_1}{dx^2}+4\alpha_1\frac{d^2\psi_2}{dx^2})\Gamma(w_s,\varphi) + \\ &\quad (2\alpha_1\Gamma(w_s,\psi_1)+4\alpha_1\Gamma(w_s,\psi_2))\frac{d^2\varphi}{dx^2} \\ &\chi_c^G(x) = 3\alpha_1\Gamma(\varphi,\varphi)\frac{d^2\varphi}{dx^2} \\ &\chi_q^E = \frac{6\alpha_2 v_p^2}{(1-w_s)^4}(2\varphi\psi_2+\varphi\psi_1) \\ &\chi_c^E = \frac{12\alpha_2 v_p^2}{(1-w_s)^5}\varphi^3 \\ &\chi^v = \frac{d^4\varphi}{dx^4} - 2\alpha_1\,\Gamma(\varphi,w_s)\frac{d^2w_s}{dx^2}\end{aligned} \tag{50}$$

The expression χ^v may be written as following form by considering Eq. (23)

$$\chi^v = [\alpha_1\Gamma(w_s, w_s)+N]\frac{d^2\phi}{dx^2}+\left(\frac{2\alpha_2 v_p^2}{(1-w_s)^3}+\omega^2\right)\phi \tag{51}$$

The left hand side of Eq. (48) is self adjoint, so the adjoint solution is like the solution of homogenous Eq. (38). The non-homogenous Eq. (48) has a solution only if the right hand side of Eq. (48) is orthogonal to every solution of the self adjoint homogenous equation i.e., $\phi(x)e^{i\omega T_0}$. So, by multiplying the right hand side of Eq. (48) to $\phi(x)e^{i\omega T_0}$ and integrating the outcome from $x = 0$ to $x = 1$, one can obtain the solvability condition as

$$2i\omega\left(\frac{dA}{dT_2}+\frac{\mu_1 A}{2}+\frac{\mu_2 CA}{2}\right)+8SA^2\bar{A}-Fe^{i\sigma T_2} = 0 \tag{52}$$

where

$$\mu_1 = \int_0^1 c\phi^2 dx = c$$

$$\mu_2 = \int_0^1 \chi^v \phi dx$$

$$S_q^G = -\frac{1}{8}\int_0^1 \chi_q^G \phi dx$$

$$S_c^G = -\frac{1}{8}\int_0^1 \chi_c^G \phi dx$$

$$S_q^E = -\frac{1}{8}\int_0^1 \chi_q^E \phi dx$$

$$S_c^E = -\frac{1}{8}\int_0^1 \chi_c^E \phi dx$$

$$F = \int_0^1 \bar{F}\phi dx$$

$$S = S_q^G + S_c^G + S_q^E + S_c^E \tag{53}$$

Now, A is denoted in polar form with amplitude a and phase $\sigma T_2 - \gamma$, so

$$A = 1/2ae^{i(\sigma T_2-\gamma)} \tag{54}$$

By substituting Eq. (54) into Eq. (52) and separating the real and imaginary part, it results that

$$\frac{da}{dT_2} = -\left(\frac{\mu_1+\mu_2 C}{2}\right)a+\frac{F}{\omega}\sin\gamma$$

$$a\frac{d\gamma}{dT_2} = a\sigma-\frac{Sa^3}{\omega}+\frac{F}{\omega}\cos\gamma \tag{55}$$

By using Eqs. (37), (41), (44) and substituting $\varepsilon = 1$, the solution of Eq. (21) will be

$$u(x, \tau) = a\cos(\Omega\tau-\gamma)\phi(x)+\frac{1}{2}a^2[\psi_2(x)+\cos2(\Omega\tau-\gamma)\psi_1(x)] \tag{56}$$

By letting da/dT_2 and $d\gamma/dT_2$ being equal to zero, one can obtain the equilibrium point (a_0, γ_0) as

$$a_0^2\left[\left(\frac{\mu_1+\mu_2 C}{2}\right)^2+\left(\sigma-\frac{Sa_0^2}{\omega}\right)^2\right]=\frac{F^2}{\omega^2} \tag{57}$$

This equation shows that the amplitude a_0 is a maximum, when the expression inside parenthesis vanishes. So, it results that $\sigma = Sa_0^2/\omega$ and $a_0 = 2F/(\omega\mu_1+\mu_2 C\omega)$. Also, by considering that $\sigma = \Omega-\varepsilon^2\omega$ and combing it with the obtained results, the nonlinear resonance frequency is obtained as

$$\Omega = \omega+\frac{4SF^2}{\omega^3(\mu_1+C\mu_2)^2} \tag{58}$$

5.2 Secondary resonance

Now, it is assumed that frequency of AC part is $\Omega = 2\omega+\varepsilon^2\sigma$, so the $2\alpha_2 v_p v(t)/(1-w_s)^2$ in Eq. (29) does not produce a secular term, and it can be assumed a forced term. It is clear that if $\Omega = 2\omega+\varepsilon^2\sigma$ and the order of $v(t)$ be (ε^2), then $v(t)u_1$ produces a secular term, so system have a subharmonic resonance. The damping of system is assumed order (ε^2) in order to nonlinearity balances the effect of them. By substituting Eq. (37) into Eq. (21)

order (ε)

$$L(u_1)=\frac{\partial^2 u_1}{\partial T_0{}^2}+\frac{\partial^4 u_1}{\partial x^4}-[\alpha_1\Gamma(w_s,w_s)+N]\frac{\partial^2 u_1}{\partial x^2}-2\alpha_1\Gamma(w_s,u_1)\frac{d^2 w_s}{dx^2}-\frac{2\alpha_2 v_p^2}{(1-w_s)^3}u_1=0$$

$$u_1\big|_{x=0}=0,\quad \frac{\partial u_1}{\partial x}\bigg|_{x=0}=0,\quad u_1\big|_{x=1}=0,\quad \frac{\partial u_1}{\partial x}\bigg|_{x=1}=0 \tag{59}$$

order (ε^2)

$$L(u_2)=-2\frac{\partial^2 u_1}{\partial T_0\partial T_1}+\alpha_1\Gamma(u_1,u_1)\frac{d^2 w_s}{dx^2}+\alpha_1\Gamma(u_1,u_1)\frac{d^2 w_s}{dx^2}+2\alpha_1\Gamma(w_s,u_1)\frac{d^2 u_1}{dx^2}+\frac{3\alpha_2 v_p{}^2}{(1-w_s)^4}u_1{}^2+$$
$$\frac{2\alpha_2}{(1-w_s)^2}v_p v_{ac}\cos(\Omega T_0)=0$$

$$u_2\big|_{x=0}=0,\quad \frac{\partial u_2}{\partial x}\bigg|_{x=0}=0,\quad u_2\big|_{x=1}=0,\quad \frac{\partial u_2}{\partial x}\bigg|_{x=1}=0 \tag{60}$$

order (ε^3)

$$L(u_3)=-2\frac{\partial^2 u_2}{\partial T_0\partial T_1}-\left(\frac{\partial^2 u_2}{\partial T_1^2}+2\frac{\partial^2 u_1}{\partial T_0\partial T_2}\right)-c\frac{\partial u_1}{\partial T_0}-C\frac{\partial^5 u_1}{\partial x^4\partial T_0}+2\alpha_1\Gamma(u_1,u_2)\frac{d^2 w_s}{dx^2}+$$
$$2\alpha_1\Gamma(w_s,u_2)\frac{\partial^2 u_1}{\partial x^2}+2\alpha_1\Gamma(w_s,u_1)\frac{\partial^2 u_2}{\partial x^2}+\alpha_1\Gamma(u_1,u_1)\frac{\partial^2 u_1}{\partial x^2}+2\alpha_1 C\Gamma\left(\frac{\partial u_1}{\partial t},w_s\right)\frac{d^2 w_s}{dx^2}+$$
$$\frac{6\alpha_2 v_p^2}{(1-w_s)^4}u_1 u_2+\frac{4\alpha_2 v_p{}^2}{(1-w_s)^5}u_1{}^3+\frac{4\alpha_2 v_p}{(1-w_s)^3}v_{ac}\cos(\Omega T_0)u_1=0$$

$$u_3\big|_{x=0}=0,\quad \frac{\partial u_3}{\partial x}\bigg|_{x=0}=0,\quad u_3\big|_{x=1}=0,\quad \frac{\partial u_3}{\partial x}\bigg|_{x=1}=0 \tag{61}$$

It shows that Eq. (59) is identical to Eq. (38), so by using Eq. (41) and substituting into (60), it results that

$$L(u_2)=(A^2e^{2i\omega T_0}+2A\overline{A}+\overline{A}^2e^{-2i\omega T_0})h(x)+\frac{2\alpha_2 v_p}{(1-w_s)^2}v_{ac}\cos(\Omega T_0) \tag{62}$$

By considering Eq. (62), the solution of Eq. (60) is

$$u_2=\psi_1(x)A^2e^{2i\omega T_0}+2\psi_2(x)A\overline{A}+\psi_1(x)\overline{A}^2e^{-2i\omega T_0}+2\alpha_2 v_p v_{ac}\cos(\Omega T_0)\psi_3(x)+ \tag{63}$$

Where, $h(x)$, $\psi_1(x)$, $\psi_2(x)$ are identical with previous section and $\psi_3(x)$ is solution of below equation

$$\frac{\partial^4\psi_3}{\partial x^4}-(\Omega^2+\frac{2\alpha_2 v_p{}^2}{(1-w_s)^3})\psi_3-[\alpha_1\Gamma(w_s,w_s)+N]\frac{\partial^2\psi_3}{\partial x^2}-2\alpha_1\Gamma(w_s,\psi_3)\frac{d^2w_s}{dx^2}=f(x)$$

$$f(x)=\frac{1}{(1-w_s)^2}\qquad \psi_1\big|_{x=0}=0,\quad \frac{\partial\psi_1}{dx}\Big|_{x=0}=0,\quad \psi_1\big|_{x=1}=0,\quad \frac{\partial\psi_1}{dx}\Big|_{x=1}=0 \tag{64}$$

The solution of Eq. (64) may be obtained using the Galerkin method, similar to the process which has been performed for obtaining the solution of Eq. (45). The only difference is that the terms of $\psi_j, h(x)$ and $4m(x)\omega^2\delta_{1j}$ have been replaced by ψ_3, $f(x)$ and Ω^2, respectively. By substituting Eq. (63) into Eq. (61), and considering that $\Omega = 2\omega+\varepsilon^2\sigma$

$$L(u_3)=[-2\omega i\frac{dA}{dT_2}\phi(x)-i\omega A(c\phi(x)+C\chi^v)+\chi(x)A^2\overline{A}+\alpha_2 v_p v_{AC}\overline{A}\zeta(x)e^{i\sigma T_2}]e^{i\omega T_0}+cc+NST \tag{65}$$

Where, $\chi(x)$ and χ^v is identical to Eq. (50) and (51) and

$$\zeta(x)=2\alpha_1\Gamma(\phi,\psi_3)\frac{\partial^2 w_s}{\partial x^2}+2\alpha_1\Gamma(w_s,\psi_3)\frac{\partial^2\phi}{\partial x^2}+2\alpha_1\Gamma(w_s,\phi)\frac{\partial^2\psi_3}{\partial x^2}+\frac{2\phi}{(1-w_s)^3}+\frac{6\alpha_2 v_p^2}{(1-w_s)^4}\phi\psi_3 \tag{66}$$

The left hand side of Eqs. (65) and (48) is identical, so as before solvability condition is obtained by multiplying the right hand side of Eq. (65) with $e^{-i\omega T_0}\varphi(x)$ and integrate the result from $x = 0$ to $x = 1$.

$$2i\omega(\frac{dA}{dT_2}+\frac{\mu_1 A+\mu_2 CA}{2})+8SA^2\overline{A}-\alpha_2 v_p v_{ac}\Lambda\overline{A}e^{i\sigma T_2}=0$$

$$\Lambda=\int_0^1\zeta(x)dx \tag{67}$$

Where, μ_1, μ_2 and S are obtained from Eq. (53).

By substituting $A = (1/2)ae^{i(\gamma+\sigma T_2)/2}$ into Eq. (66) and separating the real and imaginary part and by assuming that, it result that

$$\frac{da}{dT_2}=-\frac{\mu_1+\mu_2 C}{2}a-\frac{\alpha_2 v_p}{2\omega}\Lambda v_{ac}a\sin\gamma$$

$$a\frac{d\gamma}{dT_2}=-a\sigma+\frac{2Sa^3}{\omega}-\frac{\alpha_2 v_p}{\omega}\Lambda v_{ac}a\cos\gamma \tag{68}$$

By substituting Eqs. (41) and (33) into (37), and substituting $\varepsilon = 1$, the solution of Eq. (21) will be

$$u(x,\tau) = a\cos\frac{1}{2}(\Omega\tau+\gamma)\phi(x) + \frac{1}{2}a^2[\psi_2(x) + \cos 2(\Omega\tau+\gamma)\psi_1(x)] + 2\alpha_2 v_p v_{ac}\cos(\Omega\tau)\psi_3(x) + .. \tag{69}$$

By letting da/dT_2 and $d\gamma/dT_2$ equal to zero in Eq. (68), the equilibrium solution (a_0, γ_0) will be obtained

$$a_0^2\left[(\mu_1 + C\mu_2)^2 + \left(\sigma - \frac{Sa_0^2}{\omega}\right)^2\right] = \frac{v_{ac}^2}{\omega^2}\alpha_2^2 v_p^2 \Lambda^2 a_0^2 \tag{70}$$

6. Results and discussion

The following values are used in the simulation: $\alpha_1 = 3.7, N_b = 8.7, \mu_1 = 0$ and $\alpha_2 = 3.9$ in addition to those values which have been mentioned in the caption and the legend of figures. The differential equations of static deflection, mode shapes and nonlinear coefficients ($S_q^G, S_c^G, S_q^E, S_c^E$ have been obtained identical to those of elastic microbeam. So, the analytical results of this paper can be validated by comparing them with numerical results of previous research works. Table 1, compares the first and the second natural frequencies calculated by approximate analytical method (this paper) and numerical shooting method (Younis and Nayfeh 2003). This comparison shows an excellent agreement.

The comparison between nonlinear coefficients obtained in this paper with the nonlinear coefficient obtained by Younis and Nayfeh (2003), is shown in Fig. 2 which shows an excellent agreement. It must be noted that nonlinear coefficient S_c^E in previous work has not been plotted, and so, this coefficient has not been compared to previous work in Fig. 2.

In previous research works the symmetric mode shapes of straight microbeam have been used as comparison functions in using the Galerkin method. These mode shapes have been obtained by numerical shooting method. So, an analytical expression that gives deflection at all point was not available. Here, it is available, since the comparison functions are obtained exactly. In this paper the new function $w_s[0]$ has been proposed as a comparison function. A comparison between the values

Table 1 The first and second natural frequency of system, calculated using approximate analytical method (this paper) and numerical method (Younis and Nayfeh 2003)

Coefficients	Analytical method	Numerical method
$\alpha_1 = 3.7$ $\alpha_2 v_p^2 = 35.3$ $N = 0$	$\omega_1 = 20.344$ $\omega_2 = 61.04$	$\omega_1 = 20.35$ $\omega_2 = 61$
$\alpha_1 = 3.7$ $\alpha_2 v_p^2 = 56.8$ $N = 10$	$\omega_1 = 21.58$ $\omega_2 = 64.24$	$\omega_1 = 21.6$ $\omega_2 = 64.2$

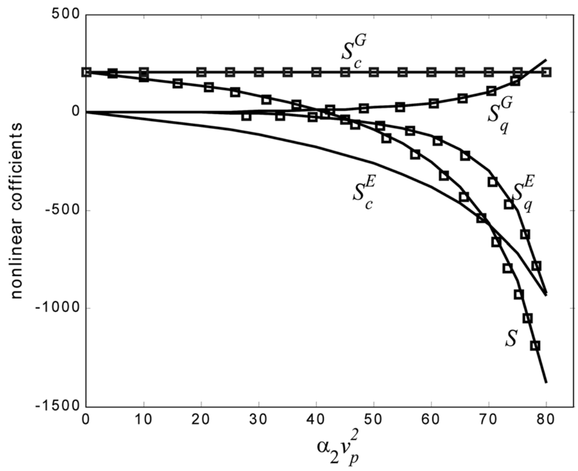

Fig. 2 Variations of nonlinear coefficients with respect to $\alpha_2 v_p^2$ for a microbeam with $N = 8.7$ and $\alpha_1 = 3.7$; the square point belongs to Younis and Nayfeh (2003), the solid line belongs to this paper

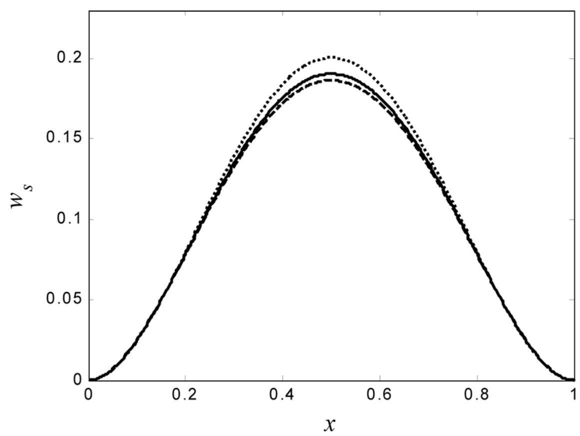

Fig. 3 The variations of w_s with respect to the variations of x for the system with $\alpha_2 v_p^2 = 45$. The dotted line belongs to using the first mode shape of straight microbeam as comparison, dashed line belongs to using the proposed function in this paper as comparison function, and the solid line is the converged solution

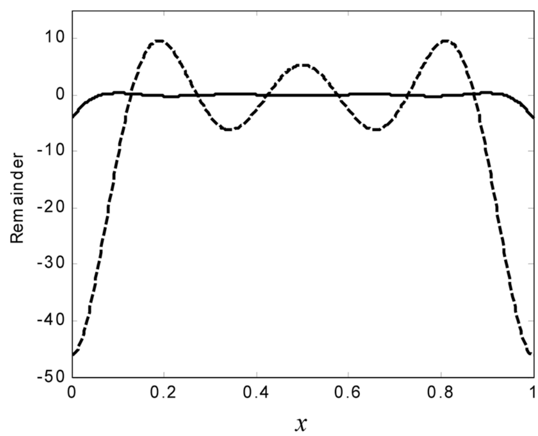

Fig. 4 The remainder of w_s obtained using the Galerkin method for the system with $\alpha_2 v_p^2 = 45$. Solid line belongs to using two symmetric mode shapes of straight microbeam and proposed function in this paper as comparison functions, and dashed line belongs to using three symmetric mode shapes of straight microbeam as comparison function

of static deflection using only the first mode shape of straight microbeam, and using only the new proposed function in this paper as comparison function is shown in Fig. 3. It shows that the using new proposed function as comparison function has less error than using first mode shape as comparison function. This decrease is due to the fact that the effect of the electrical force has also been accounted in the new comparison function. So, one can obtain almost a good approximate analytical solution for Eq. (19) by using only the proposed function in this paper, $w_{s[0]}$, as comparison function. Remainder of the static solution can be obtained by substituting the solution for w_s into differential equation of static deflection. Fig. 4 shows remainder of solution by using both $w_{s[0]}$ and two symmetric mode shapes as comparison function, and using only three symmetric mode shapes as comparison function, respectively. It shows that using both $w_{s[0]}$ and two symmetric mode shapes as comparison function decrease error of solution further.

Eq. (58) denotes that nonlinear resonance frequency for a primary excitation depends on the values $C\mu_2$, S and μ_1. It demonstrates that for $S > 0$, the system has a hardening behavior, i.e., $\Omega/\omega > 0$, and for $S < 0$, system has a softening behavior i.e., $\Omega/\omega < 0$. The variations of nonlinear coefficient S and nonlinear resonance frequency Ω with respect to the variations of system parameters have been studied in previous researches (Younis and Nayfeh 2003), so, here only the effect of viscoelastic damping on the behavior of system is studied. Figs. 5, 6 and Fig. 7 show the variation of nonzero equilibrium point a_0 with variation of σ for primary and secondary resonance,

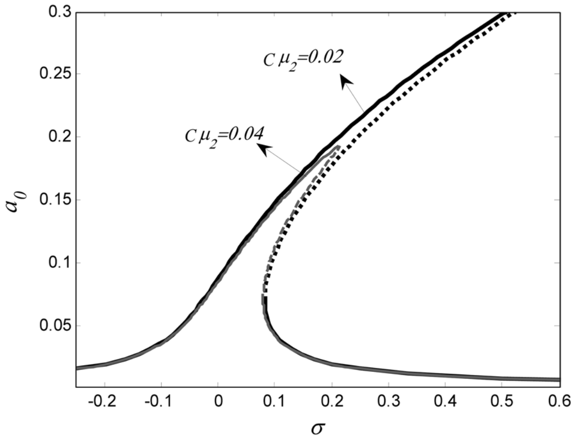

Fig. 5 The variations of a_0 with respect to the variations of σ. The solid lines belong to the stable solution and the dashed lines belong to the unstable solution where $\alpha_2 v_p^2 = 20$, $v_{ac} = 0.02$

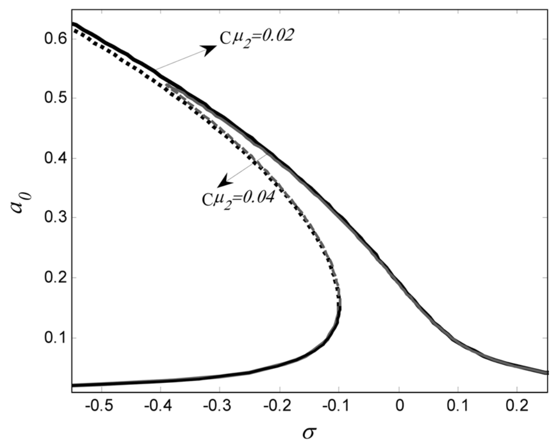

Fig. 6 The variations of a_0 with respect to the variations of σ. The solid lines belong to stable and dashed lines belong to the unstable solution where $\alpha_2 v_p^2 = 45.08$, $v_{ac} = 0.03$

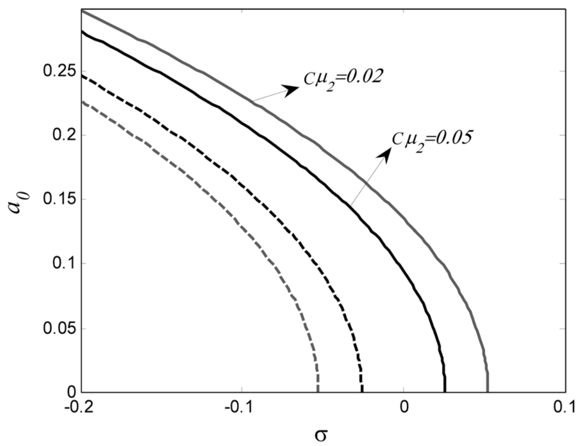

Fig. 7 The variations of a_0 with respect to the variations of σ. The solid lines belong to stable and dashed lines belong to the unstable solution where $\alpha_2 v_p^2 = 45.08$, $v_{ac} = 0.038$

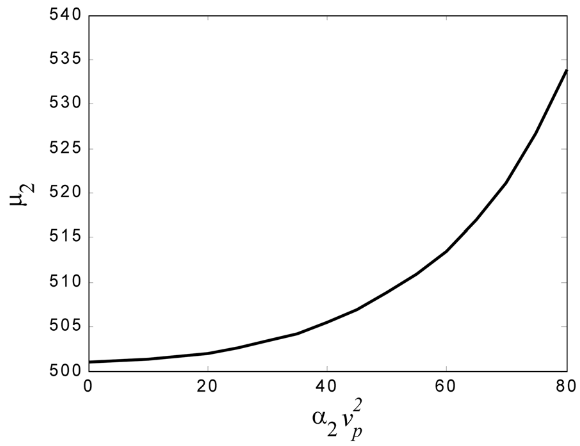

Fig. 8 The variations of viscoelastic coefficient μ_2 with respect to the variations of $\alpha_2 v_p^2$

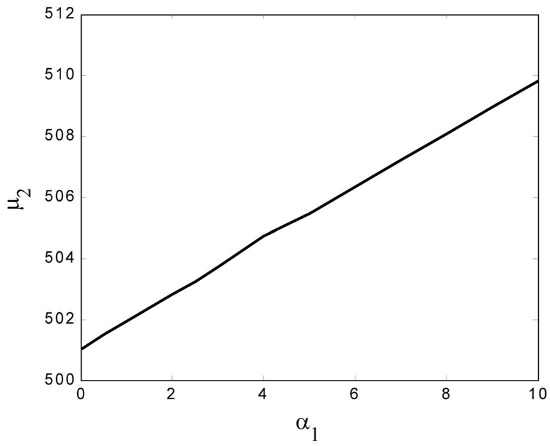

Fig. 9 The variations of viscoelastic coefficient μ_2 with respect to the variations of α_1 where $\alpha_2 v_p^2 = 45.08$

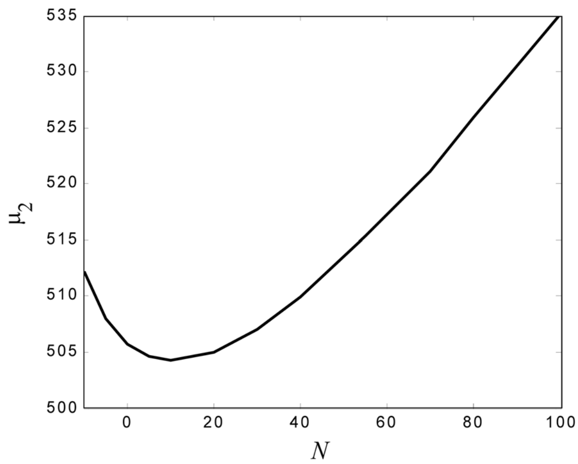

Fig. 10 The variations of viscoelastic coefficient μ_2 with respect to the variations of N for the system with $\alpha_1 = 3.7$, $\alpha_2 v_p^2 = 45.08$

respectively. These figures show that amplitude of a_0 decreased by increasing of viscoelastic factor. Figs. 5 and 6 show that if system has hardening behavior, then, an increase of viscoelastic damping decreases the value of nonlinear resonance frequency, and if system has softening behavior, then, an increase of viscoelastic damping increases the value of nonlinear resonance frequency. It means that when the nonlinear terms is considered, then the viscoelastic damping shift the resonance frequency which is not seen in linear system studied by Uncuer *et al.* (2007).

Figs. 8, 9 and 10 show the variations of μ_2 for different values of system parameters. Figs. 8 and 9 show that the value of μ_2 increases by an increase of $\alpha_2 v_p^2$ and α_1. Also, Fig. 10 shows that by increasing the value of N from the negative values to a special value of N, the value of μ_2 decreases and by more increasing of N, this behavior is inversed. These variations may be verified by considering Eq. (53). Eq. (53) demonstrates that $\mu_2 = \int_0^1 \chi^v \phi dx$, where χ^v is obtained from Eq. (51). This equation denotes that the variations of χ^v and so the variations of μ_2 depend to the variations of $\alpha_1\Gamma(w_s, w_s)+N$, $2\alpha_2 v_p^2/(1-w_s)^3$ and ω^2. It has been shown by Abdel-Rahman *et al.* (2002) that by increasing the value of $\alpha_2 v_p^2$ or decreasing α_1 or decreasing the value of N, the value of w_s increases, and the value of ω decreases. It means that by increasing $\alpha_2 v_p^2$ or decreasing α_1 or decreasing N the value of $\Gamma(w_s, w_s)$ and $2/(1-w_s)^3$ increases and the value of ω^2 decreases. When the value of $\alpha_2 v_p^2$ increases, the increasing of χ^v due to increasing the value of $\alpha_1\Gamma(w_s, w_s)$ and $2\alpha_2 v_p^2/(1-w_s)^3$ is more than the decreasing of χ^v due to decreasing the value of ω^2 and so, by increasing $\alpha_2 v_p^2$, the value of μ_2 increases. When the value of α_1 increases, the increasing of χ^v due to increasing the value of ω^2 and $\alpha_1\Gamma(w_s, w_s)$ is more than the decreasing of χ^v due to decreasing the value of $2\alpha_2 v_p^2/(1-w_s)^3$, and so, by increasing α_1, the value of μ_2 increases. When N increases from negative values, the decreasing of χ^v due to decreasing the value of $\alpha_1\Gamma(w_s, w_s)$ and $2\alpha_2 v_p^2/(1-w_s)^3$ is more than the increasing of χ^v due to increasing the value of N and ω^2, and so, the value of μ_2 decreases by increasing the value of N from a negative value to an especial value. By more increase of N from this special value, this competing is inversed, and so, χ^v and μ_2 increases.

Eqs. (58) and (70) demonstrates that the effect of $C\mu_2$ is similar to the effect of μ_1. So, it may be assumed that the vibrating system with viscoelastic damping is equivalent to an elastic system vibrating in the fluid with viscous damping per unit length equal to $C\mu_2$.

7. Conclusions

The nonlinear equations of motion for the viscoelastic microbeam under axial and electric loads have been derived using the Newton' s second law. The Voigt-Kelvin viscoelastic model has been used, and it has been assumed that the midplane of microbeam is stretched, when it is deflected. A new function, $w_{s[0]}$, has been proposed as comparison function for obtaining the static deflection of microbeam. It has been shown that a good approximate analytical solution may be obtained using only this function as comparison function. It has been shown that in contrast to the previous research works, using this function as a comparison function decreases the computational errors.

It is shown that small amount of viscoelastic damping has an important effect and causes the maximum amplitude of response to be decreased, and resonance frequency be displaced. Also, it is shown that under a primary excitation if nonlinear coefficient, $S < 0$, then the value of nonlinear resonance frequency increases by an increase of the value of $C\mu_2$, and if nonlinear coefficient, $S > 0$, then the value of nonlinear resonance frequency decreases by an increase of the value of $C\mu_2$. It has been shown that the value of μ_2 increases by increasing the DC voltage, ratio of the air gap to the microbeam thickness. Also, it has been shown that by increasing the value of N from negative value to a special value, μ_2 increases and by more increasing of N from this special value, μ_2 decreases. It has been shown that the vibrating system with viscoelastic damping is equivalent to an elastic system vibrating in the fluid with viscous damping per unit length equal to $C\mu_2$.

References

Abdel-Rahman, E.M., Younis, M.I. and Nayfeh, A.H. (2002), "Characterization of the mechanical behavior of an

electrically actuated microbeam", *J. Micromech. Microeng*, **12**, 759-766.
Abdel-Rahman, E.M., Younis, M.I. and Nayfeh, A.H. (2003), "Secondary resonances of electrically actuated resonant microsensors", *J. Micromech. Microeng*, **13**, 491-501.
Choi, B. and Lovell, E.G. (1997), "Improved analysis of microbeams under mechanical and electrostatic loads", *J. Micromech. Microeng,* **7**, 24-29.
Dufour, I., Lochon, F., Heinrich, S.M. and Josse, F. (2007), "Effect of coating viscoelasticity on quality factor and limit of detection of microcantilever chemical sensors", *IEEE Sens. J.*, **7**, 230-236.
Fu, Y.M. and Zhang, J. (2009), "Nonlinear static and dynamic responses of an electrically actuated viscoelastic microbeam", *Acta Mech. Sinica,* **25**, 211-218.
Fu, Y.M. and Zhang, J. (2009), "Active control of the nonlinear static and dynamic responses for piezoelectric viscoelastic microplates", *Smart Mater. Struct.*, **18**, 095037.
Fu, Y.M., Zhang, J. and Bi, R.G. (2009), "Analysis of the nonlinear dynamic stability for an electrically actuated viscoelastic microbeam", *Microsyst. Technol.,* **15**, 763-769.
Ijntema, D.J. and Tilmans, H.A. (1992), "Static and dynamic aspects of an air-gap capacitor", *Sensor. Actuat. A-Phys.,* **35**, 121-128.
Mahmoodi, S.N., Khadem, S.E. and Jalili, N. (2006), "Theoretical development and closed-form solution of nonlinear vibrations of a directly excited nanotube-reinforced composite cantilevered beam", *Arch. Appl. Mech*., **75**, 153-163.
Najar, F., Choura, S., Borgi, S.E., Abdel-Rahman, E.M. and Nayfeh, A.H. (2005), "Modeling and design of variable-geometryelectrostatic microactuators", *J. Micromech. Microeng,* **15**, 419-429.
Nayfeh, A.H. and Pai, P.F. (2004), *Linear and Nonlinear Structural Mechanics*, John Wiley & Sons, New York.
Nayfeh, A.H. and Younis, M.I. (2005), "Dynamics of MEMS resonators under superharmonic and subharmonic excitations", *J. Micromech. Microeng*, **15**, 1840-1847.
Rochus, V., Rixen, D.J. and Golinval, J.C. (2005), "Electrostatic coupling of MEMS structures: transient simulations and dynamic pull-in", *Nonlinear Anal.*, **63**, e1619-e1633.
Tilmans, H.A. and Legtenberg, R. (1994), "Electrostatically driven vacuum-encapsulated polysilicon resonators Part II: Theory and performance", *Sensor. Actuat. A-Phys.,* **45**, 67-84.
Uncuer, M., Marinkovic, B. and Koser, H. (2007), "Simulation of clamped-free and clamped- clamped microbeams dynamics for nonlinear mechanical switch applications", *Proceedings of the COMSOL Conference,* Boston.
Wenzel, M.J., Josse, F. and Heinrich, S.M. (2009), "Deflection of a viscoelastic cantilever under a uniform surface stress: Applications to static-mode microcantilever sensors undergoing adsorption", *J. Appl. Phys.,* **105**, 064903.
Younis, M.I., Abdel-Rahman, E.M. and Nayfeh, A.H. (2003), "A reduced-order model for electrically actuated microbeam-based MEMS", *J. Microelectomech. S.,* **12**, 672-680.
Younis, M.I. and Nayfeh, A.H. (2003), "A Study of the nonlinear response of a resonant microbeam to an electric actuation", *Nonlinear Dyn.,* **31**, 91-117.
Zamanian, M., Khadem, S.E. and Mahmoodi, S.N. (2008), "The effect of a piezoelectric layer on the mechanical behavior of an electrostatic actuated microbeam", *Smart Mater. Struct.*, **17**, 065024.
Zamanian, M., Khadem, S.E. and Mahmoodi, S.N. (2009), "Analysis of non-linear vibrations of a microresonator under piezoelectric and electrostatic actuations", *Proc. IMechE Part C: J. Mech. Eng. Sci.,* **223**, 329-344.
Zamanian, M. and Khadem, S.E. (2010), "Nonlinear vibration of an electrically actuated microresonator tuned by combined DC piezoelectric and electric actuations", *Smart Mater. Struct.,* **19**, 015012.
Zhang, W. and Meng, G. (2005), "Nonlinear dynamical system of micro-cantilever under combined parametric and forcing excitations in MEMS", *Sensor. Actuat. A-Phys.,* **119**, 291-299.