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Technical Note

Closed-form solutions on bending of cantilever twisted Timoshenko beams under various bending loads

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1. Introduction

The static equilibrium equations of a cantilever rectangular blade subject to either a concentrated or uniformly distributed transverse load were established by Carnegie (1957) based on Euler-Bernoulli beam theory. The exact solution for static bending of a cantilever blade under a distributed transverse force including the shear deformation was obtained by Subrahmanyam *et al.* (1980). However, the solution is not accurate as a result of improper boundary conditions used. A general review of the structural and dynamic aspects of pretwisted beams can be found in review paper by Rosen (1991). The exact analytical solution of in-plane static problems for a curved beam with variable curvature and cross-section was established by Tufekci and Arpaci (2006) by using the initial value method. The differential equations governing the structural behavior for a curved-beam with varying cross-section and under generalized load were derived by Gimena *et al.* (2009).

In this paper the equilibrium equations of static bending of a twisted Timoshenko beam subjected to simultaneous concentrated and distributed bending loads are established. The closed-form solutions for the uniform cantilever blade under either a concentrated or uniformly distributed transverse force or moment are derived. The effects of the twist angle, thickness ratio, length-to-thickness ratio and loading conditions on the static bending behaviors of the cantilever blades are investigated.

2. Formulation

Fig. 1 shows the twisted beam configuration, the inertial coordinate xyz and the twist coordinate XYz. By applying the principle of minimum potential energy (Reddy 1986) to the total potential energy of the beam system, the static equilibrium equations of a twisted Timoshenko beam subjected to distributed transverse forces (f_x, f_y) and bending couples (q_x, q_y), concentrated bending forces (F_x, F_y) and bending moments (Q_x, Q_y) are obtained as

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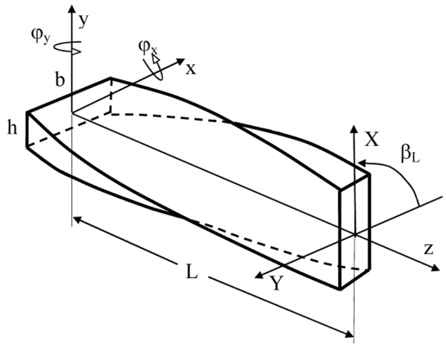

Fig. 1 Beam configuration and coordinate systems

$$[\kappa GA(u_x' - \varphi_y)]' - f_x = 0 \tag{1a}$$

$$[\kappa GA(u_y' - \varphi_x)]' - f_y = 0 \tag{1b}$$

$$(EI_{xx}\varphi_x')' + (EI_{xy}\varphi_y')' + \kappa GA(u_y' - \varphi_x) - q_x = 0 \tag{1c}$$

$$(EI_{yy}\varphi_y')' + (EI_{xy}\varphi_x')' + \kappa GA(u_x' - \varphi_y) + q_y = 0 \tag{1d}$$

and the boundary conditions at $z = 0$, L are

$$\text{either} \quad \kappa GA(u_x' - \varphi_y) + F_x = 0 \quad \text{or} \quad u_x \text{ is prescribed} \tag{2a}$$

$$\text{either} \quad \kappa GA(u_y' - \varphi_x) + F_y = 0 \quad \text{or} \quad u_y \text{ is prescribed} \tag{2b}$$

$$\text{either} \quad EI_{xx}\varphi_x' + EI_{xy}\varphi_y' + Q_x = 0 \quad \text{or} \quad \varphi_x \text{ is prescribed} \tag{2c}$$

$$\text{either} \quad EI_{yy}\varphi_y' + EI_{xy}\varphi_x' + Q_y = 0 \quad \text{or} \quad \varphi_y \text{ is prescribed} \tag{2d}$$

By performing mathematical operations on Eqs. (1) and (2) (Carnegie 1957, Subrahmanyam *et al.* 1980), the closed-form solutions of deflection u_x and u_y for cantilevered twisted beams subjected to a concentrated load F_y at the free end along the negative y-axis can be shown to be

$$u_x = (F_y L^3/8EI_{XX})(1-R)[(\bar{z}-1)\beta_L \sin 2\beta_L \bar{z} + \cos 2\beta_L \bar{z} + 2\bar{z}\beta_L^2 - 1]/\beta_L^3 \tag{3a}$$

$$\begin{aligned} u_y = {} & -F_y L\bar{z}/\kappa GA - (F_y L^3/3EI_{XX})\{(1+R)[(1-\bar{z})^3 + 3\bar{z} - 1]/4 - \\ & -3(1-R)[(1-\bar{z})\beta_L \cos 2\beta_L \bar{z} + \sin 2\beta_L \bar{z} - (1+\bar{z})\beta_L]/8\beta_L^3\} \end{aligned} \tag{3b}$$

where $R = I_{XX}/I_{YY}$, $\bar{z} = z/L$ and β_L = total twist angle.

Similarly, for cantilevered twisted beams subjected to a uniformly distributed lateral load f_y along the negative y-axis, the expressions of deflections u_x and u_y are given as

$$\begin{aligned} u_x = {} & -(f_y L^4/32EI_{XX})(1-R)\{[2(1-\bar{z})^2\beta_L^2 \sin 2\beta_L \bar{z} - 4(1-\bar{z})\beta_L \cos 2\beta_L \bar{z}]/\beta_L^4 - \\ & -[3\sin 2\beta_L \bar{z} + 2\beta_L \bar{z}(1-2\beta_L^2) + 4\beta_L]/\beta_L^4\} \end{aligned} \tag{4a}$$

$$u_y = -f_y L^2(2\bar{z}-\bar{z}^2)/2\kappa GA - (f_y L^4/8EI_{XX})\{(1+R)[(1-\bar{z})^4+4\bar{z}-1]/6+$$
$$+(1-R)[(3-2(1-\bar{z})^2\beta_L^2)\cos 2\beta_L\bar{z} - 4(1-\bar{z})\beta_L\sin 2\beta_L\bar{z} + 2\beta_L^2(2\bar{z}+1)-3]/4\beta_L^3\} \tag{4b}$$

For the case of cantilevered twisted beams under a single moment Q_x at the free end along the positive x-axis, the expressions for displacements u_x and u_y are

$$u_x = -(Q_x L^2/2EI_{XX})\{(1-R)(\sin 2\beta_L\bar{z} - 2\beta_L\bar{z})/4\beta_L^2\} \tag{5a}$$

$$u_y = -(Q_x L^2/2EI_{XX})\{(1+R)\bar{z}^2/2 - (1-R)(\cos 2\beta_L\bar{z} - 1)/4\beta_L^2\} \tag{5b}$$

For the example of cantilevered twisted beams under a uniformly distributed couple q_x along the positive x-axis, the expressions for deflections u_x and u_y are

$$u_x = (q_x L^3/8EI_{XX})(1-R)[(\bar{z}-1)\beta_L\sin 2\beta_L\bar{z} + \cos 2\beta_L\bar{z} + 2\bar{z}\beta_L^2 - 1]/\beta_L^3 \tag{6a}$$

$$u_y = -(q_x L^3/3EI_{XX})\{(1+R)[(1-\bar{z})^3+3\bar{z}-1]/4-$$
$$-3(1-R)[(1-\bar{z})\beta_L\cos 2\beta_L\bar{z} + \sin 2\beta_L\bar{z} - (1+\bar{z})\beta_L]/8\beta_L^3\} \tag{6b}$$

3. Numerical results and remarks

Figs. 2(a) and 2(b) present the effects of thickness ratio and length-to-thickness ratio on bending deflections of cantilever rectangular blades subject to the concentrated force at the free end,

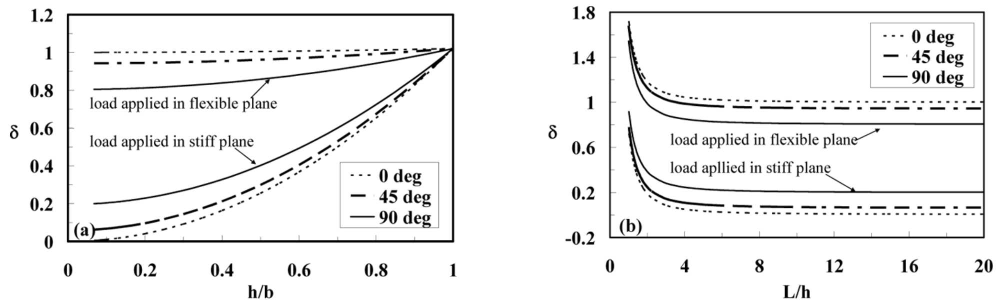

Fig. 2 Relative tip displacements (δ) versus (a) thickness ratio and (b) length-to-thickness ratio for cantilever rectangular blades under tip transverse forces

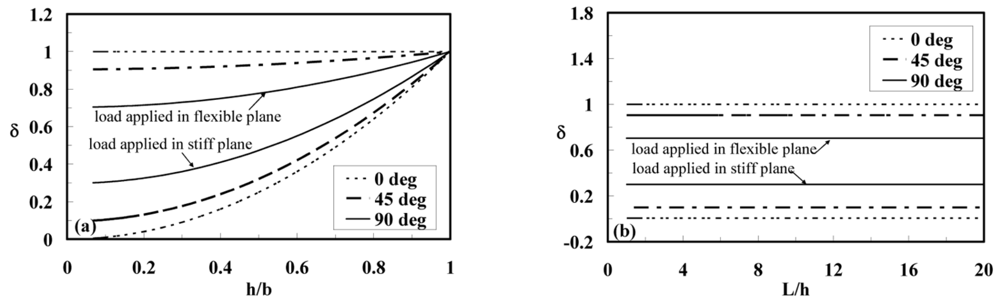

Fig. 3 Relative tip displacements (δ) versus (a) thickness ratio and (b) length-to-thickness ratio for cantilever rectangular blades under tip bending moments

respectively. Similar results can also be obtained for blades under a uniformly distributed load. Figs. 3(a) and 3(b) illustrate the influence of thickness ratio and length-to-thickness ratio on deflections of rectangular blades under a single moment at the free end, respectively. Likewise, the displacements for blades under a uniformly distributed couple have the same tendency. For each example, the blade is subjected to the loading applied in the flexible-plane (thickness) or stiff-plane (width) direction. The maximum tip deflections are nondimensionalized by division by that of the corresponding straight blade obtained by the Euler-Bernoulli beam theory.

Based on the results, several remarks can be summarized as follows. The relative maximum deflection decreases for the blade loaded in flexible-plane direction and increases for the blade loaded in stiff-plane direction with the increasing twist angle for thickness ratio other than one. The relative maximum deflection always increases when the thickness ratio is increased regardless of the twist angle and applied loading. The shear deformation or length-to-thickness ratio will merely affect the bending behaviors of the blades subject to transverse forces. When the length-to-thickness ratio is increased, the relative maximum deflection is decreased for the blades under transverse forces, and remains unchanged for the blades subject to bending moments.

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