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Free vibration analysis of moderately thick rectangular laminated composite plates with arbitrary boundary conditions

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Abstract. A semi-analytical method is presented for accurately prediction of the free vibration behavior of generally laminated composite plates with arbitrary boundary conditions. The method employs the technique of separation of spatial variables within Hamilton's principle to obtain the equations of motion, including two systems of coupled ordinary homogeneous differential equations. Subsequently, by applying the laminate constitutive relations into the resulting equations two sets of coupled ordinary differential equations with constant coefficients, in terms of displacements, are achieved. The obtained differential equations are solved for the natural frequencies and corresponding mode shapes, with the use of the exact state-space approach. The formulation is exploited in the framework of the first-order shear deformation theory to incorporate the effects of transverse shear deformation and rotary inertia. The efficiency and accuracy of the present method are demonstrated by obtaining solutions to a wide range of problems and comparing them with finite element analysis and previously published results.

Keywords: extended Kantorovich method; laminated composite plates; free vibration; arbitrary boundary conditions; shear deformation; rotary inertia.

1. Introduction

Nowadays, the use of laminated composite plates in structural components has been more and more extended and this issue, undoubtedly, may be owed to widely engineering analyses performed on this kind of structures. It doesn't need to be noted that the free vibration analysis generally is of most important steps in structures design process. On these facts, one can find a great number of published articles in which the free vibration of laminated composite plates has been studied. Nevertheless, nearly all of these works are confined to the special laminate stacking sequences (symmetric or antisymmetric angle-ply and cross-ply laminations) with special boundary conditions and don't have actually the ability to analyze laminated plates with arbitrary laminations and boundary conditions.

Many of theories developed in this field have been dedicated to the analysis of simply supported plates with symmetric or antisymmetric angle-ply and cross-ply lay-ups. For example, the reader is

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referred to the works of Srinivas *et al*. (1970), Srinivas and Rao (1970), Noor (1973), Jones (1973), Bert and Chen (1978), Putcha and Reddy (1986), Whitney (1987), Leissa and Narita (1989), Noor and Burton (1990), Marco and Ugo (1995), Salimt *et al*. (1998), Fares and Zenkour (1999), Matsunaga (2000), Kant and Swaminathan (2001a,b), Aydogdu and Timarci (2001), Guo *et al*. (2002), Swaminathan and Patil (2008), Ćetković and Vuksanović (2009), and Zhen *et al*. (2009).

Following laminated plates with simply supported boundary conditions, most of the research works have been devoted to study the free vibration behavior of antisymmetric angle-ply and cross-ply rectangular laminates subjected to simply supported boundary conditions at their opposite two edges and different boundary conditions at the remaining ones (they are usually known as Levy's analytical solution). Khdeir (1988a,b, 1989) developed an exact methodology to analyze the free vibration of symmetric and unsymmetric cross-ply and antisymmetric angle-ply laminated plates, respectively. The procedure was based on a generalized Levy-type solution considered in conjunction with a higher order laminate theory and the state-space concept. Similarly, Khdeir and Librescu (1988) analyzed the free vibration and buckling problems of rectangular cross-ply laminated plates. Reddy and Khdeir (1989) and Bose and Reddy (1998a,b) presented analytical and finite element solutions of the classical, first-order and third-order laminate theories for the free vibration study of cross-ply rectangular composite laminates under Levy's admissible boundary conditions. Hadian and Nayfeh (1993) applied a modified state-space technique to overcome the ill-conditioning problem occurring in the cases of thick composite plates for the same type of problem. Using the first-order shear deformation theory (FSDT) and Levy's solution, Chen and Liu (1990) studied the free vibration and static behavior of laminated composite plates. Also, within the FSDT, Palardy and Palazotto (1990) obtained the buckling loads and fundamental frequencies of cross-ply laminated plates. Khdeir and Reddy (1997) used the second-order theory of laminated composite plates and a generalized Levy-type solution in conjunction with the state-space concept to analyze the free vibration behavior of cross-ply and antisymmetric angle-ply laminated plates.

A number of investigators have been conducted the vibration analysis of some other laminations and boundary conditions. Taking the idea of Green (1944) (calculation derivatives of a function represented by a Fourier series that violates the boundary conditions) for isotropic plates, Whitney (1971) obtained the natural frequencies of flexural vibrations of a thin symmetrically laminated rectangular plate with clamped edges. Chaudhuri and Kabir (1993) and Kabir and Chaudhuri (1994) extended the method of Whitney (1971) for free vibration of clamped general cross-ply and arbitrarily laminated plates, respectively. They applied the boundary-continuous generalized Navier solution technique to solve five highly coupled linear second-order partial differential equations with constant coefficients, and the associated geometric boundary conditions arising from the FSDT. Kabir *et al*. (2001) and Kabir (1999) presented a similar work on thin and moderately thick simply supported plates with arbitrary laminations, respectively. Also, Khalili *et al*. (2005) employed a similar methodology for the analysis of symmetric cross-ply laminated plates with different boundary conditions. They had to fulfill an elaborate mathematical procedure to obtain the unknown due to the every set of boundary conditions on the edges of the plate.

Gorman (1990, 1993) exploited the superposition method to obtain accurate analytical type solutions for the free vibration of fully clamped and completely free especially orthotropic rectangular plates. Gorman (1995) extended his previous works for analysis of cantilever plates with rectangular orthotropy. Yu and Cleghorn (1993) investigated the generic free vibration of especially orthotropic rectangular plates with combinations of clamped and simply supported edges (clamped, one simply supported, and two adjacent simply supported edges) by the superposition technique. Yu *et al*. (1994) used the superposition method in order to study the free vibration behavior of symmetric cross-ply laminated plates having all possible combinations of clamped-simply supported edge support. Gorman and Ding (2003) utilized the method of superposition to obtain accurate analytical solutions for natural frequencies and mode shapes of completely free symmetric cross-ply laminated rectangular plates. Based on the use of Navier-type double Fourier-series solutions, Kshirsagar and Bhaskar (2008) developed a new superposition approach for free vibration and buckling studies of thin orthotropic rectangular plates with any combination of clamped, simply supported, and free edges. In their latter work, Kshirsagar and Bhaskar (2009) used the same technique to analyze free vibration and stability of moderately thick orthotropic rectangular plates with any combination of simply supported and clamped edges.

All of the solution methods reviewed in the last three paragraphs is situated in the category of analytical approaches. On the other hand, many of researchers have utilized the approximate methods, such as finite element, finite strip, Ritz, Galerkin, and Rayleigh-Ritz methods, to analyze the free vibrations of laminated plates for which there exist no Levy-type solutions. Zenkour and Youssif (2000) obtained a generalized mixed variational formulation based on the third-order plate theory to study the vibration behavior of symmetric cross-ply laminated plates subjected to three sets of boundary conditions; all edges simply supported, all edges clamped, and two opposite edges simply supported and the others clamped. On the basis of three-dimensional elasticity, Ye (1997) presented a free vibration analysis of cross-ply laminated rectangular plates with clamped boundaries. Clamped boundary conditions are imposed by suppressing the edge displacements of a number of planes which are parallel to the mid-plane. This is achieved by coupling a number of different vibration modes of the corresponding simply supported plate using a Lagrange multiplier method. Hearmon (1959) applied the Rayleigh-Ritz method to obtain frequency of flexural vibrations of special orthotropic plates, with clamped and simply supported boundary conditions, neglecting transverse shear and rotary inertia deformations. Bert (1969) by the use of Rayleigh-Ritz energy method and the classical laminated plate theory (CLPT) presented an approximate solution for determining the natural frequencies of unsymmetrically laminated rectangular plates with clamped edges. Baharlou and Leissia (1987), based on the CLPT, conducted a vibration analysis of cross-ply and angle-ply laminated plates having arbitrarily edge conditions by using the Ritz method. Qatu (1991) employed a similar method with algebraic polynomial displacement functions to solve the vibration problem for laminated composite plates having two adjacent free edges and the remaining edges simply supported, clamped or free. They presented the results for symmetrically laminated plates. Leissa and Narita (1989) applied the Ritz method with the displacement components assumed as the double series of trigonometric functions for the vibration problem of symmetrically laminated composite rectangular plates with simply boundary conditions at all of their edges. Soldatos and Messina (2001) extended a Ritz-type formulation that had been applied in connection with vibrations of cross-ply laminated structural elements in their previous works (Messina and Soldatos 1999a,b,c) towards the investigation of the free vibration problem of shear deformable composite laminated plates, closed cylindrical shells, and open cylindrical panels having an arbitrary angle-ply lay-up. Using Ritz solution, Aydogdu and Timarci (2003) carried out the vibration analysis of symmetric and antisymmetric cross-ply laminated square plates subjected to different sets of boundary conditions. The analysis was based on a five-degree-of-freedom shear deformable plate theory. Liew (1996) employed a global p-Ritz method for the vibration analysis of thick rectangular symmetrically laminated plates with various combinations of boundary conditions. In the method a set of boundary beam characteristic orthogonal polynomials was used as admissible functions in the Ritz minimization procedures. Chen *et al*. (1997) presented a similar work for symmetrically laminated thick rectangular plates. Their formulation was basically applicable to rectangular laminates with any sets of boundary conditions. Parametric studies on the symmetrical angle-ply laminates were performed. Based on the Ritz method and on the classical laminated plate theory, Nallima and Oller (2008) developed a simple and accurate formulation for the study of the free vibration of arbitrarily laminated composite plates.

Gorman and Ding (1995, 1996) exploited the superposition-Galerkin method to obtain an accurate analysis of the free vibration of symmetric cross-ply, and antisymmetric angle-ply laminated plates with combinations of clamped and simply supported edge conditions based on the FSDT. Shi *et al*. (2004) performed the free vibration analysis of arbitrarily laminated plate with all four edges clamped by using Galerkin method. Liew *et al*. (2003) adopted the FSDT in the moving least squares differential quadrature procedure for predicting the free vibration behavior of moderately thick symmetrically laminated composite plates under Levy's admissible boundary conditions. Lanhe *et al*. (2005) applied the same approximate method to solve the vibration problems of arbitrarily laminated moderately thick plates with different boundary conditions. The results were reported for (0^o, 45^o) and antisymmetric angle-ply clamped and symmetric cross-ply laminates under various boundary conditions. Dawe and Wang (1995) developed a spline finite strip method in the context of the FSDT for predicting the natural frequencies of rectangular laminated plates. Their proposed method had potential of analysis of arbitrary lay-ups and general boundary conditions. Akhrast *et al*. (1995) presented a finite strip method for the vibration and stability analyses of general cross-ply, symmetric and anti-symmetric angle-ply laminated composite plates according to the higher-order shear deformation theory. Because of some limitations inherent in the semi-analytical solution procedures, the analysis could only be carried out efficiently for the plates with two opposite ends simply supported. Akhras and Li (2007) improved their previous methodology (Akhrast *et al*. 1995) to find the critical buckling load and natural frequencies of composite plates with the same lay-ups but under more general boundary conditions. The results were presented for the plates with all edges simply-supported or clamped. Numayr *et al*. (2004) applied the finite difference method to solve differential equations of motion of free vibration of composite plates with different boundary conditions (fixed at four edges, fixed from two adjacent edges and simple at the other two edges, and simply supported at four edges). Solutions were obtained for symmetric and angle-ply laminated plates. Bambill *et al*. (2000) used the Rayleigh-Ritz method and the finite element method (FEM) to analyze the free vibration of orthotropic plates with a free edge and varying thickness in one direction. Employing the finite strip transition matrix technique, Ashour (2001) studied the flexural vibration of orthotropic plates with variable thickness in one direction, under various boundary conditions. Also, Huang *et al*. (2005) developed a discrete method for analyzing the transverse vibration of orthotropic rectangular plates with variable thickness in two directions. They obtained the natural frequencies for plates with two opposite edges having the same supports, but the other edges had arbitrary boundary conditions.

In this paper, the extended Kantorovich method (EKM) is employed to study the free vibration behavior of laminated composite plates with arbitrary lamination and boundary conditions. With the extended Kantorovich approach, it may be assumed that a mode shape is in the form of either a product of two independent functions of problem spatial variables (e.g., $f(x)$ and $g(y)$ for a rectangular plate) or a sum of product of independent functions of problem spatial variables. Taking this assumption along with an energy method, two coupled sets of ordinary differential equations - instead of one set of partial differential equations- are obtained. The coupled differential equations are solved in an iterative manner which starts by guessing a solution for a set of the equations. Then the solution of the other set may be derived analytically or numerically. Subsequently, the obtained solution is used as a beginning point to solve the former set of the equations. This iterative procedure will be continued until the solution is converged. The EKM was used for the problem of vibrations analysis of plates and shells, initially, by Kerr (1969). In his paper, Kerr (1969) studied the vibrations of a uniformly stretched rectangular membrane. He showed that the final results are independent of the initial choice of the functions and that for the treated problem, the generated expressions for the eigenvalues and eigenfunctions are identical with the corresponding exact solution. Following the work of Kerr (1969), various investigators employed the EKM for the free vibration analysis of isotropic (Jones and Milne 1976, Laura *et al*. 1979, Bhat *et al*. 1993, Lee and Kim 1995, Chang 2003, and Shufrin and Eisenberger 2005, 2006) and anisotropic (Bercin 1996 and Lee *et al*. 1997) plates. Bercin (1996) utilized the Galerkin method with the EKM to obtain the natural frequencies of fully clamped orthotropic thin rectangular plates. Lee *et al*. (1997) generalized the work of Lee and Kim (1995) to analyze moderately thick laminated composite rectangular plates with symmetrically cross-ply laminations and edges elastically restrained against rotation. They presented the numerical results for the plates whose all edges subjected to the same boundary conditions.

In this study the multi-term version of EKM is conjugated to the FSDT for the free vibration of rectangular laminated plates with arbitrary boundary conditions. However, since the procedure used is simple and straightforward it can be adopted in developing higher-order shear deformation and layerwise laminated plate theories. In order to verify the convergence, accuracy, and efficiency of the proposed theory different examples are considered: cross-ply and angle-ply laminated plates under Levy's admissible boundary conditions, a symmetric cross-ply laminated plate under various boundary conditions, and finally, (θ^{o}, $2\theta^{o}$, $-\theta^{o}$) laminated plates with arbitrary boundary conditions. Also, through these examples the influences of boundary conditions and lay-ups on the vibration behavior of the laminate are studied.

2. Theoretical formulation

The mathematical formulation is based on the FSDT, in order to account the effects of both transverse shear deformation and rotary inertia. The used reference coordinate system and laminate configuration are presented in Fig. 1. The laminated plate has a total thickness h, width b in the lateral (y-) direction, and length a in the longitudinal (x-) direction, including a generally lamination.

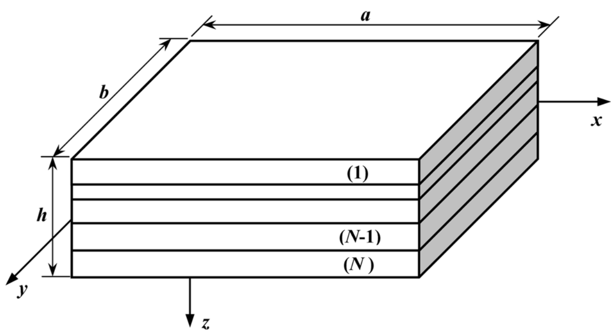


Fig. 1 The plate geometry and coordinate system

The middle plane of the plate lies on the x-y plane of the reference Cartesian coordinate system.

2.1 Displacement field

For the linear free vibrations of the plate (periodic motion), displacements are assumed as (Reddy 2004)

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y)e^{i\omega t} + z\,\psi(x,y)e^{i\omega t} \\ v(x,y,z,t) &= v_0(x,y)e^{i\omega t} + z\,\phi(x,y)e^{i\omega t} \\ w(x,y,z,t) &= w_0(x,y)e^{i\omega t}, \quad i = \sqrt{-1} \end{aligned} \tag{1}$$

where $u(x,y,z,t), v(x,y,z,t)$, and $w(x,y,z,t)$ represent the displacement components in the x, y, and z directions, respectively, of a material point in the laminate and ω denotes the frequency of vibration. Considering the technique of separation of spatial variables, the displacement field in (1) may be represented as follows

$$\begin{aligned} u(x,y,z,t) &= u_j(x)\bar{u}_j(y)e^{i\omega t} + z\,\psi_j(x)\bar{\psi}_j(y)e^{i\omega t} \\ v(x,y,z,t) &= v_j(x)\bar{v}_j(y)e^{i\omega t} + z\,\phi_j(x)\bar{\phi}_j(y)e^{i\omega t} \\ w(x,y,z,t) &= w_j(x)\bar{w}_j(y)e^{i\omega t}, \quad j = 1, 2, \ldots, n \end{aligned} \tag{2}$$

For the sake of brevity, the Einstein summation convention has been introduced – a repeated index indicates summation over all values of that index. Also $u_j(x)\bar{u}_j(y), v_j(x)\bar{v}_j(y)$, and $w_i(x)\bar{w}_i(y)$ denote the displacement of a point on the middle plane of the laminate along the x-, y-, and z-direction, respectively, $\psi_j(x)\bar{\psi}_j(y)$ and $\phi_j(x)\bar{\phi}_j(y)$ are the rotations of a transverse normal about the y- and x-axis, respectively, and n is the total number of terms which is considered in the summation.

Upon substitution of the displacement field (2) into the linear strain-displacement relations of elasticity (Fung 1965) the following strain-displacement relations will be obtained

$$\begin{aligned} &\varepsilon_x = \varepsilon_x^0 + z\kappa_x, \ \varepsilon_y = \varepsilon_y^0 + z\kappa_y, \ \varepsilon_z = 0 \\ &\gamma_{yz} = \gamma_{yz}^0, \ \gamma_{xz} = \gamma_{xz}^0, \ \gamma_{xy} = \gamma_{xy}^0 + z\kappa_{xy} \end{aligned} \tag{3}$$

where

$$\begin{aligned} &\varepsilon_x^0 = u_j\bar{u}_j' e^{i\omega t}, \quad \kappa_x = \psi_j\bar{\psi}_j' e^{i\omega t} \\ &\varepsilon_y^0 = v_j\bar{v}_j' e^{i\omega t}, \quad \kappa_y = \phi_j\bar{\phi}_j' e^{i\omega t} \\ &\gamma_{yz}^0 = (\phi_j\bar{\phi}_j + w_j\bar{w}_j')e^{i\omega t}, \quad \gamma_{xz}^0 = (\psi_j\bar{\psi}_j + w_j'\bar{w}_j)e^{i\omega t} \\ &\gamma_{xy}^0 = (u_j\bar{u}_j' + v_j'\bar{v}_j)e^{i\omega t}, \quad \kappa_{xy} = (\psi_j\bar{\psi}_j' + \phi_j'\bar{\phi}_j)e^{i\omega t} \end{aligned} \tag{4}$$

In Eq. (4) a prime indicates an ordinary derivative with respect to corresponding coordinate.

2.2 Equations of motion

Using Hamilton's principle (Reddy 2002), two sets of equations of motion and boundary conditions are obtained. If the functions $\bar{u}_j, \bar{v}_j, \bar{w}_j, \bar{\psi}_j$, and $\bar{\phi}_j$ are assumed to be known, the first set of equations of motion can be shown to be

$$\delta u_j: \quad \mathcal{N}^j_{xy1} - \frac{d\mathcal{N}^j_x}{dx} = \frac{e^{2i\omega t_2} - e^{2i\omega t_1}}{2i}\left(I_0 \mathcal{D}^{jk}_{11}\,\omega u_k + I_1 \mathcal{D}^{jk}_{14}\,\omega \psi_k\right)$$

$$\delta v_j: \quad \mathcal{N}^j_y - \frac{d\mathcal{N}^j_{xy2}}{dx} = \frac{e^{2i\omega t_2} - e^{2i\omega t_1}}{2i}\left(I_0 \mathcal{D}^{jk}_{22}\,\omega v_k + I_1 \mathcal{D}^{jk}_{25}\,\omega \phi_k\right)$$

$$\delta \psi_j: \quad \mathcal{M}^j_{xy1} - \frac{d\mathcal{M}^j_x}{dx} + \mathcal{Q}^j_{x1} = \frac{e^{2i\omega t_2} - e^{2i\omega t_1}}{2i}\left(I_1 \mathcal{D}^{jk}_{41}\,\omega u_k + I_2 \mathcal{D}^{jk}_{44}\,\omega \psi_k\right)$$

$$\delta \phi_j: \quad \mathcal{M}^j_y - \frac{d\mathcal{M}^j_{xy2}}{dx} + \mathcal{Q}^j_{y1} = \frac{e^{2i\omega t_2} - e^{2i\omega t_1}}{2i}\left(I_1 \mathcal{D}^{jk}_{52}\,\omega v_k + I_2 \mathcal{D}^{jk}_{55}\,\omega \phi_k\right)$$

$$\delta w_j: \quad \mathcal{Q}^j_{y2} - \frac{d\mathcal{Q}^j_{x2}}{dx} = \frac{e^{2i\omega t_2} - e^{2i\omega t_1}}{2i}\left(I_0 \mathcal{D}^{jk}_{33}\,\omega w_k\right) \tag{5}$$

In Eq. (5)

$$[\mathcal{D}^{jk}] = \int_0^b \{\bar{\chi}_j\}\{\bar{\chi}_k\}^T dy, \qquad \{\bar{\chi}_j\}^T = [\bar{u}_j \ \ \bar{v}_j \ \ \bar{w}_j \ \ \bar{\psi}_j \ \ \bar{\phi}_j] \tag{6}$$

and the generalized stress and moment resultants are given by

$$\begin{bmatrix} \{\mathcal{N}^j\}^T \\ \{\mathcal{M}^j\}^T \\ \{\mathcal{Q}^j\}^T \end{bmatrix} = \begin{bmatrix} \mathcal{N}^j_x & \mathcal{N}^j_y & \mathcal{N}^j_{xy1} & \mathcal{N}^j_{xy2} \\ \mathcal{M}^j_x & \mathcal{M}^j_y & \mathcal{M}^j_{xy1} & \mathcal{M}^j_{xy2} \\ \mathcal{Q}^j_{y1} & \mathcal{Q}^j_{y2} & \mathcal{Q}^j_{x1} & \mathcal{Q}^j_{x2} \end{bmatrix} = \int_0^b \begin{bmatrix} N_x \bar{u}_j & N_y \bar{v}'_j & N_{xy} \bar{u}'_j & N_{xy} \bar{v}_j \\ M_x \bar{\psi}_j & M_y \bar{\phi}'_j & M_{xy} \bar{\psi}'_j & M_{xy} \bar{\phi}_j \\ Q_y \bar{\phi}_j & Q_y \bar{w}'_j & Q_x \bar{\psi}_j & Q_x \bar{w}_j \end{bmatrix} dy \tag{7}$$

in which the stress and moment resultants are

$$(N_x, N_y, N_{xy}, Q_y, Q_x) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yz}, \sigma_{xz}) dz$$

$$(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy}) z dz \tag{8}$$

Also moments of inertia are defined as

$$(I_0, I_1, I_2) = \int_{-h/2}^{h/2} \rho(1, z, z^2) dz \tag{9}$$

where ρ is the mass density.

The boundary conditions corresponding to Eq. (5) consist of specifying the following quantities at the edges parallel to the y-axis (i.e., at $x = 0$, a)

$$\begin{aligned} &\text{Primary variables:} \quad u_j, v_j, \psi_j, \phi_j, w_j \\ &\text{Secondary variables:} \quad \mathcal{N}^j_x, \mathcal{N}^j_{xy2}, \mathcal{M}^j_x, \mathcal{M}^j_{xy2}, \mathcal{Q}^j_{x2} \end{aligned} \qquad j = 1, 2, \ldots, n \tag{10}$$

If, on the other hand, the functions u_i, v_i, w_i, ψ_i, and ϕ_i are assumed to be known, then the second set of equations of motion will be

$$\delta\bar{u}_j: \quad \bar{\mathcal{N}}^j_x - \frac{d\,\bar{\mathcal{N}}^j_{xy1}}{dy} = \frac{e^{2i\omega t_2} - e^{2i\omega t_1}}{2i}\left(I_0 \bar{\mathcal{D}}^{jk}_{11}\,\omega\,\bar{u}_k + I_1 \bar{\mathcal{D}}^{jk}_{14}\,\omega\bar{\psi}_k\right)$$

$$\delta\bar{v}_j: \quad \bar{\mathcal{N}}^j_{xy2} - \frac{d\bar{\mathcal{N}}^j_y}{dy} = \frac{e^{2i\omega t_2} - e^{2i\omega t_1}}{2i}\left(I_0 \bar{\mathcal{D}}^{jk}_{22}\,\omega\,\bar{v}_k + I_1 \bar{\mathcal{D}}^{jk}_{25}\,\omega\bar{\phi}_k\right)$$

$$\delta\bar{\psi}_j: \quad \bar{\mathcal{M}}^j_x - \frac{\bar{\mathcal{M}}^j_{xy1}}{dy} + \bar{\mathcal{Q}}^j_{x1} = \frac{e^{2i\omega t_2} - e^{2i\omega t_1}}{2i}\left(I_1 \bar{\mathcal{D}}^{jk}_{41}\,\omega\,\bar{u}_k + I_2 \bar{\mathcal{D}}^{jk}_{44}\,\omega\bar{\psi}_k\right)$$

$$\delta\bar{\phi}_j: \quad \bar{\mathcal{M}}^j_{xy2} - \frac{d\bar{\mathcal{M}}^j_y}{dy} + \bar{\mathcal{Q}}^i_{y1} = \frac{e^{2i\omega t_2} - e^{2i\omega t_1}}{2i}\left(I_1 \bar{\mathcal{D}}^{jk}_{52}\,\omega\bar{v}_k + I_2 \bar{\mathcal{D}}^{jk}_{55}\,\omega\bar{\phi}_k\right)$$

$$\delta\bar{w}_j: \quad \bar{\mathcal{Q}}^j_{x2} - \frac{d\bar{\mathcal{Q}}^j_{y2}}{dy} = \frac{e^{2i\omega t_2} - e^{2i\omega t_1}}{2i}\left(I_0 \bar{\mathcal{D}}^{jk}_{33}\,\omega\bar{w}_k\right) \tag{11}$$

where the generalized stress and moment resultants are defined as

$$\begin{bmatrix} \{\bar{\mathcal{N}}^j\}^T \\ \{\bar{\mathcal{M}}^j\}^T \\ \{\bar{\mathcal{Q}}^j\}^T \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{N}}^j_x & \bar{\mathcal{N}}^j_y & \bar{\mathcal{N}}^j_{xy1} & \bar{\mathcal{N}}^j_{xy2} \\ \bar{\mathcal{M}}^j_x & \bar{\mathcal{M}}^j_y & \bar{\mathcal{M}}^j_{xy1} & \bar{\mathcal{M}}^j_{xy2} \\ \bar{\mathcal{Q}}^j_{y1} & \bar{\mathcal{Q}}^j_{y2} & \bar{\mathcal{Q}}^j_{x1} & \bar{\mathcal{Q}}^j_{x2} \end{bmatrix} = \int_0^a \begin{bmatrix} N_x u'_j & N_y v_j & N_{xy} u_j & N_{xy} v'_j \\ M_x \psi'_j & M_y \phi_j & M_{xy}\psi_j & M_{xy}\phi'_j \\ Q_y \phi_j & Q_y w_j & Q_x \psi_j & Q_x w'_j \end{bmatrix} dx \tag{12}$$

and

$$[\bar{\mathcal{D}}^{jk}] = \int_0^b \{\chi_j\}\{\chi_k\}^T dx, \qquad \{\chi_j\}^T = [u_j \;\; v_j \;\; w_j \;\; \psi_j \;\; \phi_j] \tag{13}$$

The boundary conditions corresponding to Eq. (11) consist of specifying the following quantities at the edges parallel to the x-axis (i.e., at $y = 0$, b)

$$\begin{aligned} &\text{Primary variables:} \quad \bar{u}_j, \bar{v}_j, \bar{\psi}_j, \bar{\phi}_j, \bar{w}_j \\ &\text{Secondary variables:} \quad \bar{\mathcal{N}}^j_{xy1}, \bar{\mathcal{N}}^j_y, \bar{\mathcal{M}}^j_{xy1}, \bar{\mathcal{M}}^j_y, \bar{\mathcal{Q}}^j_{y2} \end{aligned} \qquad j = 1, 2, \ldots, n \tag{14}$$

2.3 Laminate constitutive relations

The linear plane stress constitutive relations for the kth orthotropic lamina with respect to the laminate coordinate axes (see Fig. 1) are given by Reddy (2004)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^{(k)}, \quad \begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{C}_{44} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^{(k)} \tag{15}$$

where $[\bar{Q}]^{(k)}$ is the transformed reduced stiffness matrix and $\bar{C}_{ij}^{(k)}$ (i, j = 4, 5) are the off-axis stiffness coefficients of the kth lamina. Upon substitution of Eq. (3) into Eq. (15) and the subsequent results into Eq. (8), the stress and moment resultants are obtained, which can be presented as follows

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ & & A_{66} & B_{16} & B_{26} & B_{66} \\ & & & D_{11} & D_{12} & D_{16} \\ & \text{Sym.} & & & D_{22} & D_{26} \\ & & & & & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = k^2 \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} \tag{16}$$

Here, k^2 is the shear correction factor introduced as in the first-order shear deformation plate and shell theories. Also A_{ij}, B_{ij}, and D_{ij} (i,j = 1, 2, 6) denote the extensional stiffnesses, the bending-extensional coupling stiffnesses, and the bending stiffnesses, respectively. These stiffnesses are given by

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)} (1, z, z^2) dz, \quad i,j = 1,2,6$$

$$A_{ij} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{C}_{ij}^{(k)} dz, \quad i,j = 4,5 \tag{17}$$

where N is the total number of layers. Upon substitution of Eqs. (3) into (16) and the subsequent results into Eqs. (7) and (12), the generalized stress resultants are obtained, which can be represented as follows

$$\begin{Bmatrix} \{\mathcal{N}^k\} \\ \{\mathcal{M}^k\} \end{Bmatrix} = \frac{e^{2i\omega t_2} - e^{2i\omega t_1}}{2i\omega} [\mathcal{A}^{kj}] \{\xi_j\}$$

$$\{\mathcal{Q}^k\} = \frac{e^{2i\omega t_2} - e^{2i\omega t_1}}{2i\omega} [\mathcal{B}^{kj}] \{\eta_j\}, \quad j,k = 1,2,\ldots,n \tag{18}$$

$$\begin{Bmatrix} \{\bar{\mathcal{N}}^k\} \\ \{\bar{\mathcal{M}}^k\} \end{Bmatrix} = \frac{e^{2i\omega t_2} - e^{2i\omega t_1}}{2i\omega} [\bar{\mathcal{A}}^{kj}] \{\bar{\xi}_j\}$$

$$\{\bar{\mathcal{Q}}^k\} = \frac{e^{2i\omega t_2} - e^{2i\omega t_1}}{2i\omega} [\bar{\mathcal{B}}^{kj}] \{\bar{\eta}_j\}, \quad j,k = 1,2,\ldots,n \tag{19}$$

where

$$\{\xi_j\} = [u_j' \;\; v_j \;\; u_j \;\; v_j' \;\; \psi_j' \;\; \phi_j \;\; \psi_j \;\; \phi_j']^T, \quad \{\eta_j\} = [\phi_j \;\; w_j \;\; \psi_j \;\; w_j']^T \tag{20}$$

$$\{\bar{\xi}_j\} = [\bar{u}_j \;\; \bar{v}_j' \;\; \bar{u}_j' \;\; \bar{v}_j \;\; \bar{\psi}_j \;\; \bar{\phi}_j' \;\; \bar{\psi}_j' \;\; \bar{\phi}_j]^T, \quad \{\bar{\eta}_j\} = [\bar{\phi}_j \;\; \bar{w}_j' \;\; \bar{\psi}_j \;\; \bar{w}_j]^T \tag{21}$$

and the coefficient matrices $[\mathcal{A}^{jk}], [\mathcal{B}^{jk}], [\overline{\mathcal{A}}^{jk}]$, and $[\overline{\mathcal{B}}^{jk}]$ in Eqs. (18) and (19) are defined as

$$[\mathcal{A}^{jk}] = \int_0^b ([\alpha] \otimes \{\bar{\xi}_j\}\{\bar{\xi}_k\}^T)dy, \quad [\mathcal{B}^{jk}] = \int_0^b ([\beta] \otimes \{\bar{\eta}_j\}\{\bar{\eta}_k\}^T)dy \tag{22}$$

$$[\overline{\mathcal{A}}^{jk}] = \int_0^a ([\alpha] \otimes \{\xi_j\}\{\xi_k\}^T)dx, \quad [\overline{\mathcal{B}}^{jk}] = \int_0^a ([\beta] \otimes \{\eta_j\}\{\eta_k\}^T)dx \tag{23}$$

In Eqs. (22) and (23) $[\alpha]$ and $[\beta]$ are

$$[\alpha] = \begin{bmatrix} A_{11} & A_{12} & A_{16} & A_{16} & B_{11} & B_{12} & B_{16} & B_{16} \\ & A_{22} & A_{26} & A_{26} & B_{12} & B_{22} & B_{26} & B_{26} \\ & & A_{66} & A_{66} & B_{16} & B_{26} & B_{66} & B_{66} \\ & & & A_{66} & B_{16} & B_{26} & B_{66} & B_{66} \\ & & & & D_{11} & D_{12} & D_{16} & D_{16} \\ & & \text{Sym.} & & & D_{22} & D_{26} & D_{26} \\ & & & & & & D_{66} & D_{66} \\ & & & & & & & D_{66} \end{bmatrix} \tag{24}$$

$$[\beta] = k^2 \begin{bmatrix} A_{44} & A_{44} & A_{45} & A_{45} \\ & A_{44} & A_{45} & A_{45} \\ & & A_{55} & A_{55} \\ & \text{Sym.} & & A_{55} \end{bmatrix} \tag{25}$$

It must be noted that the sign ⊗ used in Eqs. (22) and (23) is referred to *array multiplication* of two matrices.

2.4 Governing equations of motion

The equations of motion in (5) and (11) can be expressed in terms of displacements by substituting the generalized stress resultants from (18) and (19). Hence, two sets of homogeneous ordinary differential equations will be obtained as follows

$$\begin{aligned} \delta u_i : &\mathcal{A}_{11}^{ij} u_j'' + (\mathcal{A}_{13}^{ij} - \mathcal{A}_{31}^{ij}) u_j' + (\omega^2 I_0 \mathcal{D}_{11}^{ij} - \mathcal{A}_{33}^{ij}) u_j + \mathcal{A}_{14}^{ij} v_j'' + (\mathcal{A}_{12}^{ij} - \mathcal{A}_{34}^{ij}) v_j' - \mathcal{A}_{32}^{ij} v_j + \\ &\mathcal{A}_{15}^{ij} \psi_j'' + (\mathcal{A}_{17}^{ij} - \mathcal{A}_{35}^{ij}) \psi_j' + (\omega^2 I_1 \mathcal{D}_{14}^{ij} - \mathcal{A}_{37}^{ij}) \psi_j + \mathcal{A}_{18}^{ij} \phi_j'' + (\mathcal{A}_{16}^{ij} - \mathcal{A}_{38}^{ij}) \phi_j' - \mathcal{A}_{36}^{ij} \phi_j = 0 \end{aligned}$$

$$\begin{aligned} \delta v_i : &\mathcal{A}_{41}^{ij} u_j'' + (\mathcal{A}_{43}^{ij} - \mathcal{A}_{21}^{ij}) u_j' - \mathcal{A}_{23}^{ij} u_j + \mathcal{A}_{44}^{ij} v_j'' + (\mathcal{A}_{42}^{ij} - \mathcal{A}_{24}^{ij}) v_j' + (\omega^2 I_0 \mathcal{D}_{22}^{ij} - \mathcal{A}_{22}^{ij}) v_j + \\ &\mathcal{A}_{45}^{ij} \psi_j'' + (\mathcal{A}_{47}^{ij} - \mathcal{A}_{25}^{ij}) \psi_j' - \mathcal{A}_{27}^{ij} \psi_j + \mathcal{A}_{48}^{ij} \phi_j'' + (\mathcal{A}_{46}^{ij} - \mathcal{A}_{28}^{ij}) \phi_j' + (\omega^2 I_1 \mathcal{D}_{25}^{ij} \mathcal{A}_{26}^{ij}) \phi_j = 0 \end{aligned}$$

$$\delta\psi_i : \mathcal{A}_{51}^{ij}u''_j + (\mathcal{A}_{53}^{ij} - \mathcal{A}_{71}^{ij})u'_j + (\omega^2 I_1\mathcal{D}_{41}^{ij} + \mathcal{A}_{73}^{ij})u_j + \mathcal{A}_{54}^{ij}v''_j + (\mathcal{A}_{52}^{ij} - \mathcal{A}_{74}^{ij})v'_j - \mathcal{A}_{72}^{ij}v_j +$$
$$\mathcal{A}_{55}^{ij}\psi''_j + (\mathcal{A}_{57}^{ij} - \mathcal{A}_{75}^{ij})\psi'_j - (\mathcal{A}_{77}^{ij} + \mathcal{B}_{33}^{ij} - \omega^2 I_2\mathcal{D}_{44}^{ij})\psi_j + \mathcal{A}_{58}^{ij}\phi''_j + (\mathcal{A}_{56}^{ij} - \mathcal{A}_{78}^{ij})\phi'_j -$$
$$(\mathcal{A}_{76}^{ij} + \mathcal{B}_{31}^{ij})\phi_j - \mathcal{B}_{34}^{ij}w'_j - \mathcal{B}_{32}^{ij}w_j = 0$$

$$\delta\phi_i : \mathcal{A}_{81}^{ij}u''_j + (\mathcal{A}_{83}^{ij} - \mathcal{A}_{61}^{ij})u'_j - \mathcal{A}_{63}^{ij}u_j + \mathcal{A}_{84}^{ij}v''_j + (\mathcal{A}_{82}^{ij} - \mathcal{A}_{64}^{ij})v'_j + (\omega^2 I_1\mathcal{D}_{52}^{ij} - \mathcal{A}_{62}^{ij})v_j +$$
$$\mathcal{A}_{85}^{ij}\psi''_j + (\mathcal{A}_{87}^{ij} - \mathcal{A}_{65}^{ij})\psi'_j - (\mathcal{A}_{67}^{ij} + \mathcal{B}_{13}^{ij})\psi_j + \mathcal{A}_{88}^{ij}\phi''_j + (\mathcal{A}_{86}^{ij} - \mathcal{A}_{68}^{ij})\phi'_j -$$
$$(\mathcal{A}_{66}^{ij} + \mathcal{B}_{11}^{ij} - \omega^2 I_2\mathcal{D}_{55}^{ij})\phi_j - \mathcal{B}_{14}^{ij}w'_j - \mathcal{B}_{12}^{ij}w_j = 0$$

$$\delta w_i : \mathcal{B}_{43}^{ij}\psi'_j - \mathcal{B}_{23}^{ij}\psi_j + \mathcal{B}_{41}^{ij}\phi'_j - \mathcal{B}_{21}^{ij}\phi_j + \mathcal{B}_{44}^{ij}w''_j + (\mathcal{B}_{42}^{ij} - \mathcal{B}_{24}^{ij})w'_j + (\mathcal{B}_{22}^{ij} - \omega^2 I_0\mathcal{D}_{33}^{ij})w_j = 0 \tag{26}$$

and

$$\delta\bar{u}_i : \bar{\mathcal{A}}_{33}^{ij}\bar{u}''_j + (\bar{\mathcal{A}}_{31}^{ij} - \bar{\mathcal{A}}_{13}^{ij})\bar{u}'_j + (\omega^2 I_0\bar{\mathcal{D}}_{11}^{ij} - \bar{\mathcal{A}}_{11}^{ij})\bar{u}_j + \bar{\mathcal{A}}_{32}^{ij}\bar{v}''_j + (\bar{\mathcal{A}}_{34}^{ij} - \bar{\mathcal{A}}_{12}^{ij})\bar{v}'_j - \bar{\mathcal{A}}_{14}^{ij}\bar{v}_j +$$
$$\bar{\mathcal{A}}_{37}^{ij}\bar{\psi}''_j + (\bar{\mathcal{A}}_{35}^{ij} - \bar{\mathcal{A}}_{17}^{ij})\bar{\psi}'_j + (\omega^2 I_1\bar{\mathcal{D}}_{14}^{ij} - \bar{\mathcal{A}}_{15}^{ij})\bar{\psi}_j + \bar{\mathcal{A}}_{36}^{ij}\bar{\phi}''_j + (\bar{\mathcal{A}}_{38}^{ij} - \bar{\mathcal{A}}_{16}^{ij})\bar{\phi}'_j - \bar{\mathcal{A}}_{18}^{ij}\bar{\phi}_j = 0$$

$$\delta\bar{v}_i : \bar{\mathcal{A}}_{23}^{ij}\bar{u}''_j + (\bar{\mathcal{A}}_{21}^{ij} - \bar{\mathcal{A}}_{43}^{ij})\bar{u}'_j - \bar{\mathcal{A}}_{41}^{ij}\bar{u}_j + \bar{\mathcal{A}}_{22}^{ij}\bar{v}''_j + (\bar{\mathcal{A}}_{24}^{ij} - \bar{\mathcal{A}}_{42}^{ij})\bar{v}'_j + (\omega^2 I_0\bar{\mathcal{D}}_{22}^{ij} - \bar{\mathcal{A}}_{44}^{ij})\bar{v}_j +$$
$$\bar{\mathcal{A}}_{27}^{ij}\bar{\psi}''_j + (\bar{\mathcal{A}}_{25}^{ij} - \bar{\mathcal{A}}_{47}^{ij})\bar{\psi}'_j - \bar{\mathcal{A}}_{45}^{ij}\bar{\psi}_j + \bar{\mathcal{A}}_{26}^{ij}\bar{\phi}''_j + (\bar{\mathcal{A}}_{28}^{ij} - \bar{\mathcal{A}}_{46}^{ij})\bar{\phi}'_j + (\omega^2 I_1\bar{\mathcal{D}}_{25}^{ij} - \bar{\mathcal{A}}_{48}^{ij})\bar{\phi}_j = 0$$

$$\delta\bar{\psi}_i : \bar{\mathcal{A}}_{73}^{ij}\bar{u}''_j + (\bar{\mathcal{A}}_{71}^{ij} - \bar{\mathcal{A}}_{53}^{ij})\bar{u}'_j + (\omega^2 I_1\bar{\mathcal{D}}_{41}^{ij} - \bar{\mathcal{A}}_{51}^{ij})\bar{u}_j + \bar{\mathcal{A}}_{72}^{ij}\bar{v}''_j + (\bar{\mathcal{A}}_{74}^{ij} - \bar{\mathcal{A}}_{52}^{ij})\bar{v}'_j - \bar{\mathcal{A}}_{54}^{ij}\bar{v}_j +$$
$$\bar{\mathcal{A}}_{77}^{ij}\bar{\psi}''_j + (\bar{\mathcal{A}}_{75}^{ij} - \bar{\mathcal{A}}_{57}^{ij})\bar{\psi}'_j - (\bar{\mathcal{A}}_{55}^{ij} + \bar{\mathcal{B}}_{33}^{ij} - \omega^2 I_2\bar{\mathcal{D}}_{44}^{ij})\bar{\psi}_j + \bar{\mathcal{A}}_{76}^{ij}\bar{\phi}''_j + (\bar{\mathcal{A}}_{78}^{ij} - \bar{\mathcal{A}}_{56}^{ij})\bar{\phi}'_j -$$
$$(\bar{\mathcal{A}}_{58}^{ij} + \bar{\mathcal{B}}_{31}^{ij})\bar{\phi}_j - \bar{\mathcal{B}}_{32}^{ij}\bar{w}'_j - \bar{\mathcal{B}}_{34}^{ij}\bar{w}_j = 0$$

$$\delta\bar{\phi}_i : \bar{\mathcal{A}}_{63}^{ij}\bar{u}''_j + (\bar{\mathcal{A}}_{61}^{ij} - \bar{\mathcal{A}}_{83}^{ij})\bar{u}'_j - \bar{\mathcal{A}}_{81}^{ij}\bar{u}_j + \bar{\mathcal{A}}_{62}^{ij}\bar{v}''_j + (\bar{\mathcal{A}}_{64}^{ij} - \bar{\mathcal{A}}_{82}^{ij})\bar{v}'_j + (\omega^2 I_1\bar{\mathcal{D}}_{52}^{ij} - \bar{\mathcal{A}}_{84}^{ij})\bar{v}_j +$$
$$\bar{\mathcal{A}}_{67}^{ij}\bar{\psi}''_j + (\bar{\mathcal{A}}_{65}^{ij} - \bar{\mathcal{A}}_{87}^{ij})\bar{\psi}'_j - (\bar{\mathcal{A}}_{85}^{ij} + \bar{\mathcal{B}}_{13}^{ij})\bar{\psi}_j + \bar{\mathcal{A}}_{66}^{ij}\bar{\phi}''_j + (\bar{\mathcal{A}}_{68}^{ij} - \bar{\mathcal{A}}_{86}^{ij})\bar{\phi}'_j -$$
$$(\bar{\mathcal{A}}_{88}^{ij} + \bar{\mathcal{B}}_{11}^{ij} - \omega^2 I_2\bar{\mathcal{D}}_{55}^{ij})\bar{\phi}_j - \bar{\mathcal{B}}_{12}^{ij}\bar{w}'_j - \bar{\mathcal{B}}_{14}^{ij}\bar{w}_j = 0$$

$$\delta\bar{w}_i : \bar{\mathcal{B}}_{23}^{ij}\bar{\psi}'_j - \bar{\mathcal{B}}_{43}^{ij}\bar{\psi}_j + \bar{\mathcal{B}}_{21}^{ij}\bar{\phi}'_j - \bar{\mathcal{B}}_{41}^{ij}\bar{\phi}_j + \bar{\mathcal{B}}_{22}^{ij}\bar{w}'' + (\bar{\mathcal{B}}_{24}^{ij} - \bar{\mathcal{B}}_{42}^{ij})\bar{w}'_j + (\omega^2 I_0\bar{\mathcal{D}}_{33}^{ij} - \bar{\mathcal{B}}_{44}^{ij})\bar{w}_j = 0 \tag{27}$$

3. Solution of the governing equations of motion

Here, we employ the state-space approach to solve the equations of motion obtained in the previous section. Taking the state-space vectors as

$$\begin{aligned}
&\{X_1\} = \{u(x)\}, \quad \{X_2\} = \{u'(x)\}, \quad \{X_3\} = \{v(x)\}, \quad \{X_4\} = \{v'(x)\} \\
&\{X_5\} = \{\psi(x)\}, \quad \{X_6\} = \{\psi'(x)\}, \quad \{X_7\} = \{\phi(x)\}, \quad \{X_8\} = \{\phi'(x)\} \\
&\{X_9\} = \{w(x)\}, \quad \{X_{10}\} = \{w'(x)\}
\end{aligned} \tag{28}$$

reduces the system of Eq. (26) to a system of coupled first-order ordinary differential equations which, on the other hand, may be presented as

$$\{X'\} = [T]\{X\} \tag{29}$$

Note that in the above relations each of the state variables, intrinsically, is a $n \times 1$ vector (e.g., $\{\bar{u}'\} = [\bar{u}'_1 \ \ \bar{u}'_2 \ \ldots \ \bar{u}'_n]$). In order to solve Eq. (29), let us assume that $\bar{u}_i(y), \bar{v}_i(y), \ldots$ and $\bar{w}_i(y)$ are chosen. Next, the coefficient matrices $[\mathcal{A}^{ij}]$ and $[\mathcal{B}^{ij}]$ are found. Since these coefficients are constant, the coefficient matrix $[T]$ in Eq. (29) is a matrix of constant elements and therefore the solution of matrix differential Eq. (29) may be obtained analytically as follows (Franklin 1968)

$$\{X(x)\} = [\Lambda][E]\{K\} \tag{30}$$

where $[\Lambda]$ is the matrix of distinct eigenvectors of matrix $[T]$ and $\{K\}$ is a vector of unknown constants to be found by imposing the boundary conditions at the edges $x = 0$, a. Also the diagonal matrix $[E]$ is defined as

$$[E] = \begin{bmatrix} e^{\lambda_1 x} & & & 0 \\ & e^{\lambda_2 x} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_{10n} x} \end{bmatrix} \tag{31}$$

where $\lambda_k (k = 1, 2, \ldots, 10n)$ are the eigenvalues of the coefficient matrix $[T]$ which, in general, must be regarded to have complex values. Substitution of Eq. (30) into the set of arbitrary boundary conditions at the edges $x = 0$, a results in a homogeneous system of equations

$$[L]\{K\} = 0 \tag{32}$$

For a nontrivial solution, the determinant of the coefficient matrix $[L]$ in Eq. (32) should be zero

$$|L_{ij}| = 0, \quad i,j = 1, 2, \ldots, 10n \tag{33}$$

The roots of the above equation are – the square of – the frequencies of natural vibration and substitution of the normalized solutions for $\{K\}$ obtained from Eq. (32) into Eq. (30) give the corresponding mode shapes (displacement fields).

Next, we can substitute the general solution of $u_i(x), v_i(x), \ldots, \ldots$, and $w_i(x)$ into Eq. (23) to find the coefficient matrices $[\bar{\mathcal{A}}^{ij}]$ and $[\bar{\mathcal{B}}^{ij}]$ which, here, will be constant. The solution procedure of Eq. (27) is completely analogous to the one presented for Eq. (26) and therefore, for the sake of brevity will not be taken up here. This procedure (solving the coupled systems of ordinary differential equations) will be continued until the discrepancy between the natural frequencies generated by solving the systems of Eqs. (26) and (27) get a value of order ε, where ε is a prescribed relative error convergence criterion.

It is to be remarked, generally, the initial guesses to start iterative procedure are arbitrary functions and do not require to satisfy any of the boundary conditions. This latitude for selection of initial assumed functions is related to this fact that the boundary conditions are automatically satisfied in the subsequent iterations. Also, the iterative essence of the method causes that for a specified value of n, the final form and preciseness of the converged solution is independent of the form of the initial guess of the mode shapes.

4. Numerical results

Based on the mathematical procedure discussed in the preceding sections, a computer program was provided to solve the vibration problems of laminated plates. Three different numerical examples are studied in this section to demonstrate the validity and accuracy of the presented formulation and the capability of the method to analyze laminated plates with various laminations and boundary conditions. The results obtained from this theory are compared with those obtained by the Levy method, for the cases that Levy's solution exists (i.e., for cross-ply and antisymmetric angle-ply laminates with at least two simply supported opposite edges). For other cases that there exist no Levy-type solutions, the present results are compared with those of finite element analysis as well as those presented by Liew (1996) and Lanhe *et al*. (2005).

In all examples, each lamina is assumed to be of the same thickness and density. The numerical results are presented for two types of material chosen same for all laminae, whose properties are

$$\text{Material I: } E_1 = 40E_2,\ G_{12} = G_{13} = 0.6E_2,\ G_{23} = 0.5E_2,\ \nu_{12} = 0.25$$

$$\text{Material II: } E_1 = 25E_2,\ G_{12} = G_{13} = 0.5E_2,\ G_{23} = 0.2E_2,\ \nu_{12} = 0.25 \tag{34}$$

where E, G, and ν denote Young's modulus, shear modulus, and Poisson's ratio, respectively, and the subscripts 1, 2, and 3 indicate the on-axis material coordinates. Unless otherwise stated, the shear correction factor (k^2) takes the value of 5/6. For convenience, the natural frequencies are defined in term of the following non-dimensional frequency parameter

$$\overline{\omega} = \omega(a^2/h)\sqrt{\rho/E_2} \tag{36}$$

To designate the boundary conditions on the four edges of the plate, a 4-word notation such as SFSC, in which S, C, and F delegate simply supported, clamped, and free boundary conditions, is employed. The 1-4th word indicates the boundary conditions on edges $x = 0$, $y = 0$, $x = a$, and $y = b$ respectively. In order to compare the numerical results with the results presented in the other works three types of simply supported boundary conditions will be used. These simple support conditions in the first-order shear deformation laminated plate theory are defined, say at $x = 0$, a, as below

$$\text{S1:} \quad u_0 = N_{xy} = M_x = \phi = w = 0 \tag{37}$$

$$\text{S2:} \quad N_x = v_0 = M_x = \phi = w = 0 \tag{38}$$

$$\text{S3:} \quad u_0 = v_0 = M_x = \phi = w = 0 \tag{39}$$

Moreover, other boundary conditions which will be used in the following examples (i.e., clamped and free boundary conditions) are defined, say at $x = 0$, a, as

$$\text{C:} \quad u_0 = v_0 = \psi = \phi = w = 0 \tag{40}$$

$$\text{F:} \quad N_x = N_{xy} = M_x = M_{xy} = Q_x = 0 \tag{41}$$

It is to be noted that throughout the solution procedure of the presented numerical examples the value of parameter ε has been selected 10^{-7}.

Example 1: Cross-ply and angle-ply laminated plates with opposite two edges simply supported

The main intention of the present example is to perform a numerical comparison between the present results with those obtained from the Levy method. For this purpose, square laminated plates with either cross-ply or antisymmetric angle-ply lamination will be examined. The plates have an aspect ratio $a/h = 10$ and lamina material properties of type I. As previously mentioned, Levy's analytical solution exists only for antisymmetric angle-ply and any cross-ply laminated plates if at least two parallel opposite edges of the plates have simple supports. More specifically, for antisymmetric angle-ply and cross-ply laminates the simple support conditions (in the FSDT) must, say at $x = 0$, a, be of type S1 and S2, respectively.

In Table 1 the dimensionless fundamental frequencies of $(45^o, -45^o)_5$ and $(30^o, -30^o)_5$ ten-layer laminates subjected to various boundary conditions on edges $y = 0$, b, are tabulated. Similarly, Table 2 shows the results for $(0^o, 90^o)$ and $(0^o, 90^o)_5$ antisymmetric cross-ply laminated plates. It clearly can be seen that there are excellent agreements between the results of the present method and those of the Levy method. The influence of increasing the number of terms summed in the initial assumed function on the preciseness of the numerical results, for the special specimens investigated in this example, has been examined. In Table 1 the fundamental frequency parameters associated with $(45^o, -45^o)_5$ lay-up for values of $n = 1$ and $n = 2$ are instanced. It is observed that this increment has no effect on the results –which are presented herein for an accuracy of four decimal points. To rationalize this fact, it can be said that for the special cases which have been studied (laminated

Table 1 Fundamental frequency parameters of antisymmetric angle-ply $(\theta^o, -\theta^o)_5$ laminated plates under various boundary conditions

		SSSS	SSSC	SCSC	SFSF	SFSS	SFSC
$\theta = 45°$							
Present method	$n = 1$	19.3804	20.2650	21.2502	6.5655	10.6042	10.8810
	$n = 2$	19.3804	20.2650	21.2502	6.5655	10.6042	10.8810
Levy method (Reddy 2004)		19.38	20.27	21.25	6.57	10.60	10.88
$\theta = 30°$							
Present method		18.505	19.110	19.808	10.107	12.329	12.478
Levy method (Reddy 2004)		18.51	19.11	19.81	10.11	12.33	12.48

Table 2 Fundamental frequency parameters of $(0^o, 90^o)$ and $(0^o, 90^o)_5$ laminated plates under various boundary conditions

	SSSS	SSCS	CSCS	FSFS	FSSS	FSCS
$(0°, 90°)$						
Present method	10.473	12.609	15.151	6.881	7.215	7.741
Levy method (Reddy 2004)	10.473	12.610	15.152	6.881	7.215	7.741
$(0°, 90°)_5$						
Present method	15.779	18.044	20.471	10.900	11.079	11.862
Levy method (Reddy 2004)	15.779	18.044	20.471	10.900	11.079	11.862

plates with Levy's admissible boundary conditions), transverse deformations of the plate during the natural vibration occur such that the lines of extremum points are placed-exactly or nearly- parallel to the x- and/or y-axis. These conditions permit us to treat the problems as spreadable models and consequently, using single-term initial guesses may be terminated in solutions with an excellent accuracy. It is worth to be noted, by taking the initial assumed functions in the form of trigonometric series satisfying simply supported boundary conditions S1 or S2 on parallel edges of the plate, the proposed method can be led to the analytical solution of Levy. However, to demonstrate the generality of the present solution and likewise its accuracy, some simple polynomial functions were utilized as the initial guesses in the solution procedure.

As an interesting model, which with help of it the influence of increasing the number of terms in the initial assumed functions on the computed natural frequencies may be seen more distinctly, $(45^o, -45^o)_s$ laminated plate with SFSC boundary conditions is studied (simple supports are of type S2). The lowest three natural frequency parameters of the plate calculated for various values of n are listed in Table 3. Since for the problem in hand, no analytical solution exists the numerical results are verified by those obtained from finite element analysis. Finite element analyses were carried out using commercial package of ANSYS with SHELL99 element and sufficiently fine mesh generation. Table 3 reveals that the single-term theory predicts the natural frequencies values inaccurately. But Table 3 also shows that the increase in the number of terms in the initial guess improves rapidly preciseness of the results. Moreover, Table 3 indicates that there is quite good agreement between the results of the present method and FEM. The transverse deformations of vibrations of the plate in its three first natural modes are illustrated by Fig. 2. Inspection of Fig. 2 and the previous paragraph explanations, one can easily find the reason of unsuccessful of the single-term theory to appropriate prediction of the free vibration behavior of the plate; specially, in the higher modes. In Fig. 3 the second transverse mode shape of the plate is plotted for n = 1 and n = 2. It is observed that for the specimen studied here, the single-term approach cannot suggest a proper mode shape, whereas the multi-term approach -even for n = 2- gives a perfectly right configuration.

Table 3 Convergence and comparison studies of the lowest three frequency parameters for $(45^o, -45^o)_s$ square laminated plate under SFSC boundary conditions

Mode number	n					FEM
	1	2	3	4	5	
1	9.932	9.465	9.438	9.435	9.433	9.475
2	25.971	22.067	21.852	21.836	21.833	22.035
3	38.448	27.752	27.299	27.239	27.234	27.431

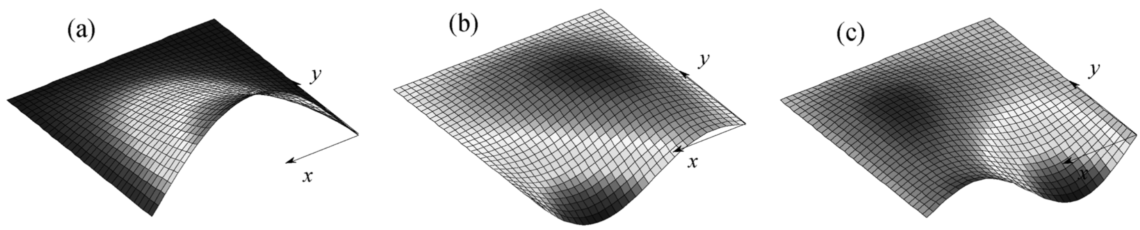


Fig. 2 Transverse mode shapes of $(45°, -45°)_s$ square laminated plate under SFSC boundary conditions (a) the first, (b) second, (c) third mode

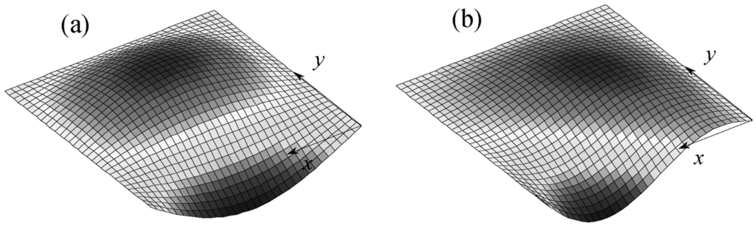


Fig. 3 The second transverse mode shape of (45°, −45°)$_s$ square laminated plate under SFSC boundary conditions obtained using (a) one-term, (b) two-term initial functions

Table 4 Comparison of frequency parameters for (0º, 90º, 0º) square laminated plate under various boundary conditions

Mode number		CCCC	CFCF	CCFF	CFFF	SSFF*
1	Present method	21.131	18.953	5.882	5.469	1.266
	Liew (1996)	21.131	18.953	5.882	5.469	1.266
	Lanhe *et al*. (2005)	21.191	18.953	5.888	5.472	1.275
2	Present method	29.633	19.087	11.255	5.995	7.614
	Liew (1996)	29.634	19.087	11.254	5.996	7.616
	Lanhe *et al*. (2005)	29.651	19.084	11.249	6.002	7.627
3	Present method	39.670	21.712	22.366	11.941	20.665
	Liew (1996)	39.671	21.713	22.366	11.941	20.666
	Lanhe *et al*. (2005)	39.799	21.727	22.352	11.904	20.624
4	Present method	43.993	31.521	24.463	22.117	20.859
	Liew (1996)	43.993	31.522	24.464	22.118	20.860
	Lanhe *et al*. (2005)	44.102	31.570	24.581	22.118	20.872
5	Present method	45.067	38.391	25.285	22.698	23.015
	Liew (1996)	45.068	38.390	25.286	22.700	22.916
	Lanhe *et al*. (2005)	45.159	38.661	25.300	22.799	22.936

*The simple supports are of type S3.

Example 2: Symmetric cross-ply laminated plates under various boundary conditions

Consider a (0º, 90º, 0º) square laminated plate with aspect ratio $a/h = 10$ and lamina material properties of type I. Liew (1996) and Lanhe *et al*. (2005) educed the natural frequencies of the plate under different boundary conditions using *p*-Ritz and moving least squares differential quadrature, respectively. The numerical results of the present method are compared with those reported by Liew (1996) and Lanhe *et al*. (2005) in Table 4 for the lowest five frequency parameters. It is noted that both Liew (1996) and Lanhe *et al*. (2005) performed their analyses in the framework of the FSDT. In this example, the shear correction factor is taken to be $k^2 = \pi^2/12$. It is obvious that the presented results match very well with the results of the two other methods, especially with those given by Liew (1996).

A convergence study is carried out through Table 5 in which frequency parameters pertinent to

Table 5 Convergence study of the lowest three frequency parameters for (0º, 90º, 0º) square laminated plate with CCCC and SSFF boundary conditions

Supports	Mode number	n			
		1	2	3	4
CCCC	1	21.1308	21.1306	21.1306	21.1306
	2	29.6335	29.6332	29.6332	29.6332
	3	39.6703	39.6703	39.6703	39.6702
SSFF	1	1.2696	1.2667	1.2665	1.2665
	2	7.6216	7.6151	7.6144	7.6144
	3	20.6972	20.6662	20.6662	20.6662

CCCC and SSFF boundary conditions are tabulated versus n. From Table 5 it is implied that the convergence rate varies with the variation of the mode sequence and combination of boundary conditions. However, for the present model, choosing $n = 4$ provides a reliable accuracy with four decimal points.

Example 3: (θº, 2θº, −θº) laminated plates with arbitrary boundary conditions

Eventually, the free vibration behavior of (θº, 2θº, −θº) square laminated plates subjected to arbitrary boundary conditions is studied. The plate has aspect ratio a/h = 100 and material properties of type II. Also, the kind of all simply supported boundary conditions applied to the edges of the plate is assumed to be S1. The lowest three natural frequencies of (45º, 90º, −45º) laminated plate under SSCC, SSSS, CCCF, SSFC, CFSF, and SSFF boundary conditions obtained from the present theory and results of finite element analysis are listed in Table 6. There is quite good agreement between all results of the two methods, which may be outcome of smallish thickness of the plate. Table 7 shows the fundamental frequency parameters of (45º, 90º, −45º) laminated plate under various boundary conditions for the increasing values of n. It is seen that as the number of terms in the initial guessed functions are increased, from one to seven, the natural frequencies turn gradually into smaller and more accurate values. Table 7 also reveals that depending on the type of applied boundary conditions, employing functions with two to seven terms can lead to the numerical results with an accuracy of three significant digits. Furthermore, it can be concluded that whatever the symmetry of boundary conditions with respect to x and y axes being the more, generally, the number of terms required to reach a certain level of accuracy is less.

Table 6 Frequency parameters of (45º, 90º, −45º) laminated plate with various boundary conditions

Mode number		SSCC	SSSS	CCCF	SSFC	CFSF	SSFF
1	Present method	16.209	14.173	11.440	9.698	6.429	2.452
	FEM	16.211	14.175	11.442	9.698	6.430	2.453
2	Present method	32.056	25.642	22.319	19.554	11.566	9.270
	FEM	32.063	25.647	22.326	19.553	11.568	9.278
3	Present method	34.236	28.713	28.242	26.831	20.502	10.693
	FEM	34.245	28.734	28.249	26.834	20.506	10.704

Table 7 Convergence and comparison studies of the fundamental frequency parameters for (45°, 90°, −45°) laminated plate under various boundary conditions

Supports	n							FEM
	1	2	3	4	5	6	7	
CCCC	19.2016	19.1803	19.1799	19.1799	19.1798	19.1798	19.1798	19.182
CFSC	9.3583	8.9680	8.9476	8.9443	8.9436	8.9432	8.9430	8.944
CCFF	4.0477	3.9674	3.9375	3.9315	3.9298	3.9291	3.9291	3.930
CSFF	3.4882	3.3095	3.2907	3.2873	3.2858	3.2856	3.2854	3.286
CFFF	1.4531	1.4272	1.4229	1.4222	1.4215	1.4215	1.4215	1.424

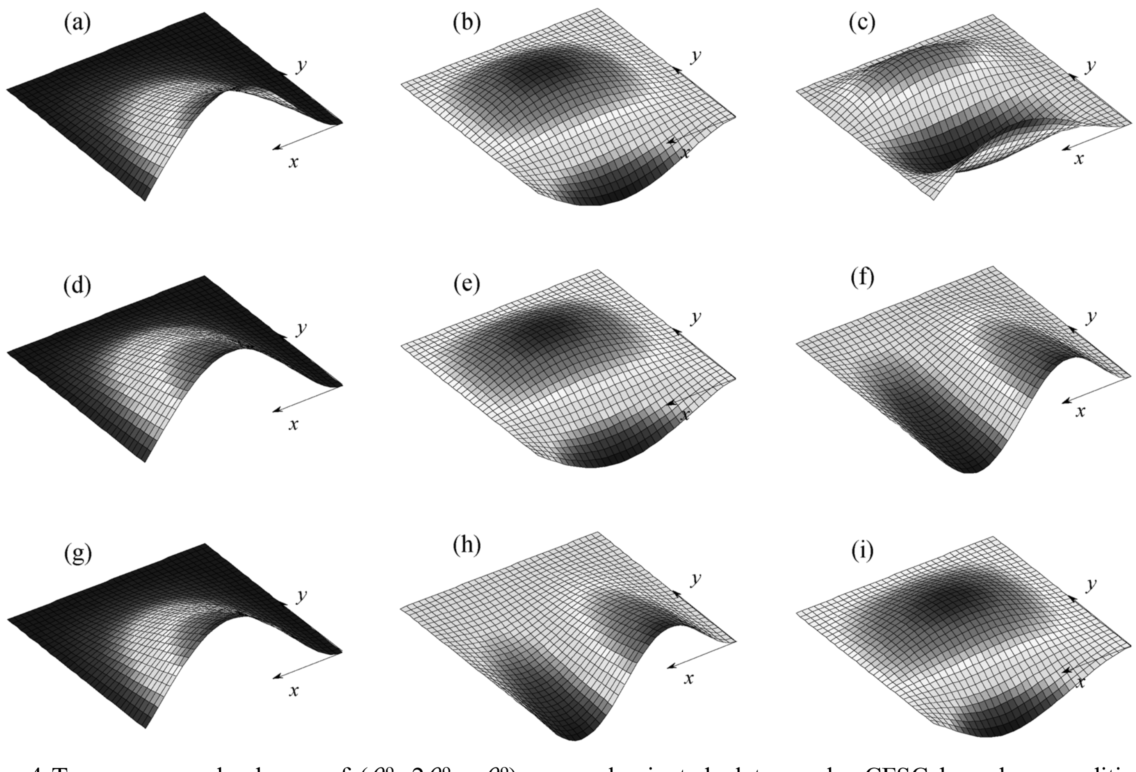


Fig. 4 Transverse mode shapes of (θ°, 2θ°, −θ°) square laminated plates under CFSC boundary conditions (a) 1st mode, $\theta = 15°$, (b) 2nd mode, $\theta = 15°$, (c) 3rd mode, $\theta = 15°$, (d) 1st mode, $\theta = 45°$, (e) 2nd mode, $\theta = 45°$; (f) 3rd mode, $\theta = 45°$, (g) 1st mode, $\theta = 75°$, (h) 2nd mode, $\theta = 75°$, (i) 3rd mode, $\theta = 75°$

The transverse component of the mode shape, corresponding to the lowest three natural frequencies of (θ°, 2θ°, −θ°) laminated plates under CFSC and CSFF boundary conditions is depicted in Figs. 4 and 5, respectively. It is seen that due to asymmetrically boundary conditions, the distributions of the lines of extremum points are basically complicated and skewed with respect to x and y axes. Therefore, as Table 7 shows, among the support conditions listed in Table 7, CFSC and CSFF boundary conditions need the most values of n to attain a specified degree of accuracy.

Fig. 6 displays the variations of the fundamental natural frequencies against lamination angle, θ,

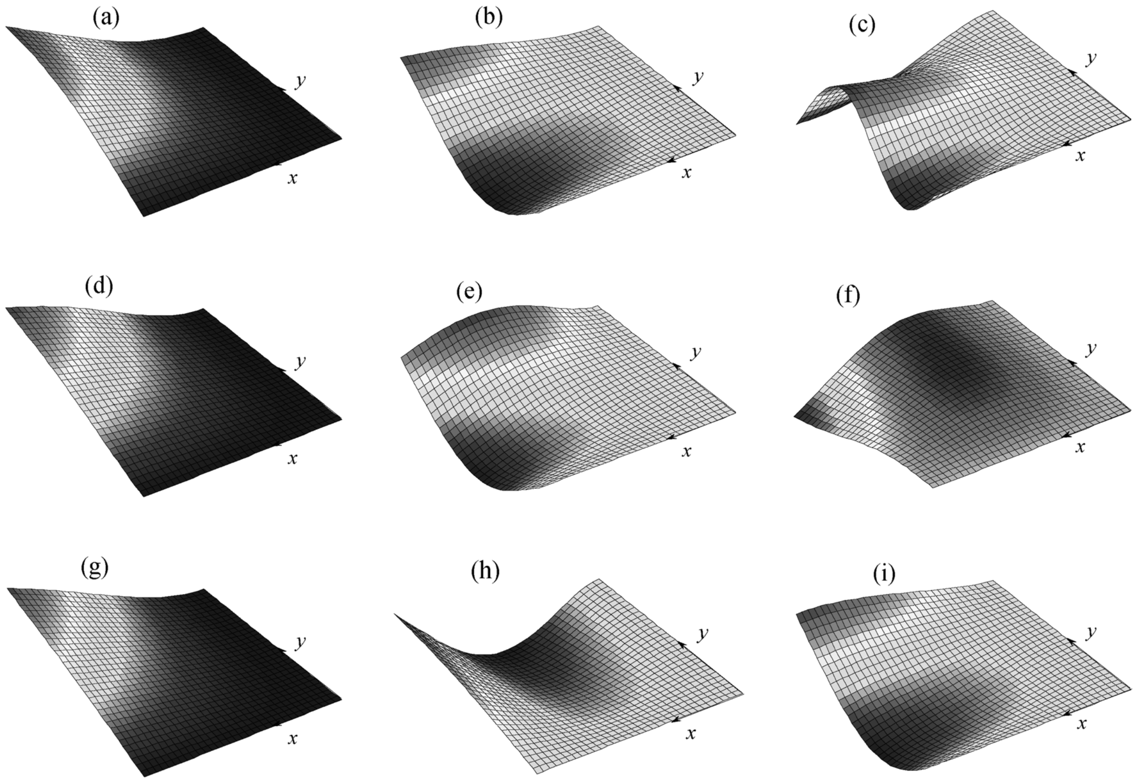


Fig. 5 Transverse mode shapes of ($\theta°$, $2\theta°$, $-\theta°$) square laminated plates under CSFF boundary conditions (a) 1st mode, $\theta = 15°$, (b) 2nd mode, $\theta = 15°$, (c) 3rd mode, $\theta = 15°$, (d) 1st mode, $\theta = 45°$, (e) 2nd mode, $\theta = 45°$, (f) 3rd mode, $\theta = 45°$, (g) 1st mode, $\theta = 75°$, (h) 2nd mode, $\theta = 75°$, (i) 3rd mode, $\theta = 75°$

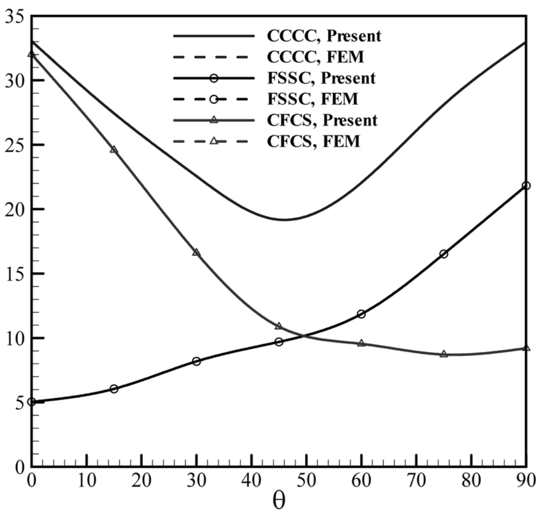
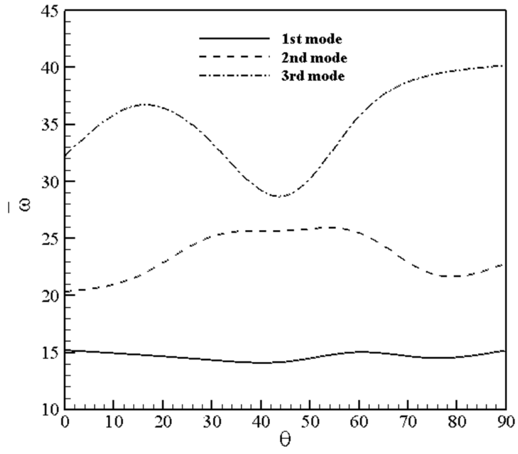
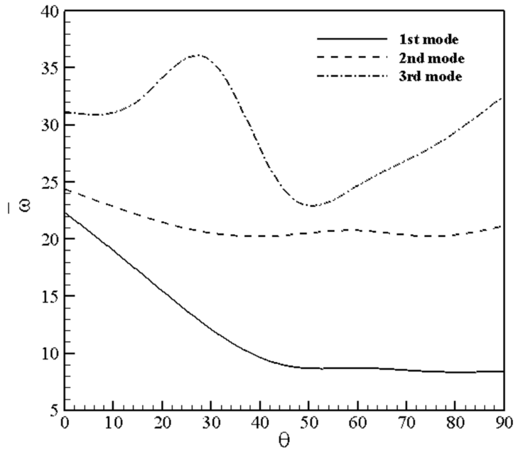


Fig. 6 Variations of the fundamental frequency parameters versus θ, for ($\theta°$, $2\theta°$, $-\theta°$) square laminated plates under CCCC, FSSC, and CFCS boundary conditions



(a)



(b)

Fig. 7 Variations of the lowest three natural frequency parameters versus θ, for (θ°, $2\theta^{\circ}$, $-\theta^{\circ}$) square laminated plates under (a) SSSS, (b) CFSC boundary conditions

for (θ°, $2\theta^{\circ}$, $-\theta^{\circ}$) laminated plates under CCCC, FSSC, and CFCS boundary conditions obtained from the present method and FEM. By comparing the results of the two methods, it is found that they are all in very close agreement. It is also obviously seen that the kind of boundary conditions applied on the edges of the plate may have a considerable effect on the amount and trend of the natural frequencies variations for different values of angle θ.

Fig. 7 shows the variations of the lowest three natural frequencies versus θ for (θ°, $2\theta^{\circ}$, $-\theta^{\circ}$) laminated plates with SSSS and CFSC boundary conditions. A first look at Fig. 7 indicate that the variation trends of the frequencies in the different natural modes do not comply specified fashions. For example, although the variations of the first frequency for SSSS boundary conditions, meanwhile the fiber angle changes from 0° to 90°, are negligible; but the variations of the third frequency are quite tangible. In addition, Fig. 7 illustrates that the maximum difference between the two first natural frequencies occurs when the distance between the second and third natural frequencies is the minimum at the same fiber angle (e.g., for SSSS boundary conditions it happens at about $\theta = 45°$).

5. Conclusions

An accurate solution based on idea of the EKM is developed to study the vibration behavior of laminated composite plates with arbitrary lamination and boundary conditions. To incorporate the effects of transverse shear deformation and rotary inertia the formulation is exploited in the framework of the FSDT. The procedure used is simple and straightforward and, therefore, it can be easily adopted in developing higher-order shear deformation and layerwise laminated plate theories. Several numerical examples, including laminated plates with cross-ply, symmetric and antisymmetric angle-ply, and general laminations with various sets of boundary condition, are studied. The numerical results are compared with those of the Levy-type solutions and also with those of the published results and finite element analysis when there exist no analytical solutions.

All the numerical results demonstrate the capability of the proposed method for the analysis of laminated plates with arbitrary lamination and boundary conditions as well as its excellent accuracy. Some convergence studies are performed to investigate the influence of the number of summed terms in the initial assumed functions (n) on the preciseness of the numerical results obtained from the present method. It is found that increasing n improves monotonically the accuracy of the results. It is further seen, the single-term theory not only may result in producing the natural frequency values with a poor approximation, but also it may predict the corresponding mode shapes improperly.

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