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Shear deformation effect in flexural-torsional buckling analysis of beams of arbitrary cross section by BEM

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Abstract. In this paper a boundary element method is developed for the general flexural-torsional buckling analysis of Timoshenko beams of arbitrarily shaped cross section. The beam is subjected to a compressive centrally applied concentrated axial load together with arbitrarily axial, transverse and torsional distributed loading, while its edges are restrained by the most general linear boundary conditions. The resulting boundary value problem, described by three coupled ordinary differential equations, is solved employing a boundary integral equation approach. All basic equations are formulated with respect to the principal shear axes coordinate system, which does not coincide with the principal bending one in a nonsymmetric cross section. To account for shear deformations, the concept of shear deformation coefficients is used. Six coupled boundary value problems are formulated with respect to the transverse displacements, to the angle of twist, to the primary warping function and to two stress functions and solved using the Analog Equation Method, a BEM based method. Several beams are analysed to illustrate the method and demonstrate its efficiency and wherever possible its accuracy. The range of applicability of the thin-walled theory and the significant influence of the boundary conditions and the shear deformation effect on the buckling load are investigated through examples with great practical interest.

Keywords: flexural-torsional buckling; nonuniform torsion; elastic stability; warping; flexural; bar; beam; twist; boundary element method; shear deformation.

1. Introduction

Elastic stability of beams is one of the most important criteria in the design of structures subjected to compressive loads. This beam buckling analysis becomes much more complicated in the case the cross section's centroid does not coincide with its shear center (asymmetric beams), leading to the formulation of the flexural-torsional buckling problem. Moreover, unless the beam is very "thin" the error incurred from the ignorance of the effect of shear deformation is substantial, and an accurate analysis requires its inclusion in it.

The first published work on elastic stability of structures appeared in 1759 by Euler (1759) who studied the flexural buckling of axially loaded, simply supported columns. His treatise has been the initial theoretical basis for the development of flexural-torsional buckling, a more general case of elastic stability which can appear in axially or transversely loaded beams. It is worth noting that torsional (Barsoum and Gallagher 1970, Szymczak 1980) and flexural buckling due to compressive

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axial loading can be treated as limited cases of flexural-torsional buckling. Since then, important development has been achieved, regarding elastic stability of structures, as presented in various books written by Timoshenko and Gere (1961), Bazant and Cedolin (1991), Trahair (1993) and Simitses and Hodges (2006). Furthermore, research efforts focusing on flexural-torsional buckling have been presented especially for thin-walled structures employing the assumptions of the thin-walled theory developed by Vlasov (1961), such as the work published by Kounadis (1998), Ioannidis and Kounadis (1999) and Mohri *et al.* (2001) on the evaluation of the critical flexural torsional buckling load of simply supported beams and the prediction of their postbuckling behavior. Vlasov effects consist in taking into account the nonuniform torsion arising from restrained warping while an important role is played by the so-called middle surface of the beam. More specifically, the warping function (needed for the evaluation of the torsion and warping constants), employing Vlasov theory is assumed to vary along only the middle surface of the cross section (Vlasov 1961) and arises from the assumption that the shearing strain in the middle surface is absent.

More recently, flexural buckling of multi step bars with varying geometric cross section properties taking into account shear deformation (Li 2003) or ignoring it and assuming varying material properties (Gadalla and Abdalla 2006) or of structures partially embedded in soil such as columns, piles (Catal and Catal 2006, Rajasekaran 2008) has also been studied. To the authors' knowledge publications on the solution to the general flexural-torsional buckling analysis of Timoshenko beams of arbitrarily shaped cross section do not exist.

In this investigation, an integral equation technique is developed for the solution of the aforementioned problem. The beam is subjected to a compressive centrally applied concentrated axial load together with arbitrarily axial, transverse and torsional distributed loading, while its edges are restrained by the most general linear boundary conditions. The resulting boundary value problem, described by three coupled ordinary differential equations, is solved employing the concept of the analog equation (Katsikadelis 2002). According to this method, the three coupled fourth order hyperbolic partial differential equations are replaced by three uncoupled ones subjected to fictitious load distributions under the same boundary conditions. All basic equations are formulated with respect to the principal shear axes coordinate system, which does not coincide with the principal bending one in a nonsymmetric cross section. To account for shear deformations, the concept of shear deformation coefficients is used. Six boundary value problems are formulated with respect to the transverse displacements, to the angle of twist, to the primary warping function and to two stress functions and solved using the Analog Equation Method (Katsikadelis 2002), a BEM based method. Vlasov's theory assumptions are not adopted in the proposed analysis, independently of the thin- or thick- walled character of the cross section. The proposed method for both thin- or thick- walled beams takes into account the nonuniform torsion arising from restrained warping, by employing the analysis presented by Sapountzakis and Mokos (2004), where the primary warping function is derived from a 2-D elasticity formulation. It is worth here noting that the invalidity of the Vlasov's assumptions is demonstrated by Sapountzakis and Mokos (2004) presenting that the warping function cannot be regarded as constant along the thickness of the cross section. An alternative method for the evaluation of the warping function has been developed by Yu *et al.* (2005), according to which the 3-D warping field has been recovered through a first-order warping analysis based on the generalized strain measures of the classical beam theory. In both of these two latter research efforts (Sapountzakis and Mokos 2004, Yu *et al.* 2005), the exact mode of cross-sectional deformation (which dominates the exponentially decaying behavior near the boundaries, especially in closed section beams) has not been accurately taken into account. Nevertheless, for the flexural-torsional buckling problem, this issue does not mainly affect the buckling loads, as it is proved from the accuracy of the results (compared with 3D solutions) of the present paper's examples.

The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows.

i. The proposed method can be applied to beams having an arbitrary simply or multiply connected constant cross section and not to a necessarily thin-walled one.
ii. All basic equations are formulated with respect to the principal shear axes coordinate system, which does not necessarily coincide with the principal bending one.
iii. For the first time in the literature, shear deformation effect is taken into account on the flexural-torsional buckling analysis of a beam of a non-symmetric constant cross section.
iv. Torsional warping arising from nonuniform torsion is taken into account.
v. The beam is supported by the most general linear boundary conditions including elastic support or restraint.
vi. The shear deformation coefficients are evaluated using an energy approach, instead of Timoshenko's (1921) and Cowper's (1966) definitions, for which several authors (Schramm *et al*. 1994, 1997) have pointed out that one obtains unsatisfactory results or definitions given by other researchers (Stephen 1980, Hutchinson 2001), for which these factors take negative values.
vii. The effect of the material's Poisson ratio ν is taken into account.
viii. The proposed method employs a pure BEM approach (requiring only boundary discretization) resulting in line or parabolic elements instead of area elements of the FEM solutions (requiring the whole cross section to be discretized into triangular or quadrilateral area elements), while a small number of line elements are required to achieve high accuracy.

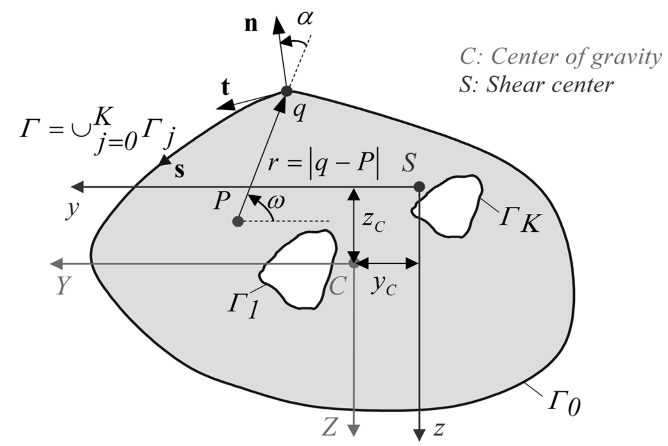
Several beams are analysed to illustrate the method and demonstrate its efficiency and wherever possible its accuracy. The range of applicability of the thin-walled theory and the significant influence of the boundary conditions and the shear deformation effect on the buckling load are investigated through examples with great practical interest.

2. Statement of the problem

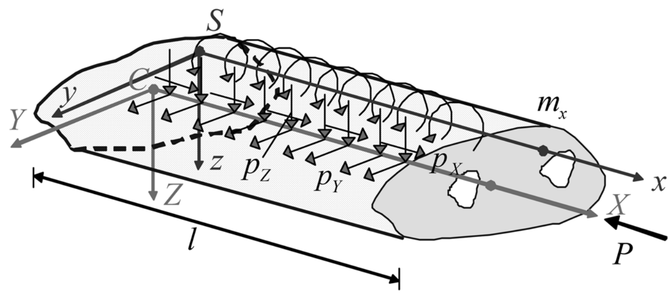
Let us consider a prismatic beam of length l (Fig. 1), of constant arbitrary cross-section of area A. The homogeneous isotropic and linearly elastic material of the beam cross-section, with modulus of elasticity E, shear modulus G and Poisson's ratio ν occupies the two dimensional multiply connected region Ω of the y, z plane and is bounded by the Γ_j $(j = 1, 2, \ldots, K)$ boundary curves, which are piecewise smooth, i.e., they may have a finite number of corners. In Fig. 1(a) CYZ is the principal shear system of axes through the cross section's centroid C, while y_C, z_C are its coordinates with respect to Syz system of axes through the cross section's shear center S, with axes parallel to those of CYZ. The beam is subjected to a compressive load $-P$, to the combined action of the arbitrarily distributed axial loading $p_X = p_X(X)$, transverse loading $p_Y = p_Y(X)$, $p_Z = p_Z(X)$ acting in the Y and Z directions, respectively and to the arbitrarily distributed twisting moment $m_x = m_x(x)$ (Fig. 1(b)).

Under the aforementioned loading the displacement field of the beam is given as

$$\bar{u}(x,y,z) = u(x) + \theta_Y(x)Z - \theta_Z(x)Y + \frac{d\theta_x(x)}{dx}\phi_S^P(y,z) + \phi_S^S(x,y,z) \tag{1a}$$



(a)



(b)

Fig. 1 (a) Prismatic element of an arbitrarily shaped constant cross section occupying region Ω (b) subjected in bending and torsional loading

$$\bar{v}(x,y,z) = v(x) - z\theta_x \tag{1b}$$

$$\bar{w}(x,y,z) = w(x) + y\theta_x \tag{1c}$$

and therefore the displacement components of the cross section's centroid can be written as

$$u_C = u(x) + \frac{d\theta_x}{dx}\phi_S^P \tag{2a}$$

$$v_C = v(x) - z_C\theta_x \tag{2b}$$

$$w_C = w(x) + y_C\theta_x \tag{2c}$$

where $v(x), w(x)$ are the beam transverse displacements of the shear center S with respect to y, z axes, respectively, $u(x)$ denotes the average longitudinal displacement of the cross section (Attard 1986), θ_Y, θ_Z are the angles of rotation of the cross-section due to bending, $d\theta_x/dx$ denotes the rate of change of the angle of twist θ_x regarded as the torsional curvature and ϕ_S^P, ϕ_S^S are the primary and secondary warping functions with respect to the shear center S of the cross section of the beam (Sapountzakis and Mokos 2003).

Moreover, according to the linear stability theory of beams (small deflections), the angles of rotation of the deflection line with respect to the centroid (β) in the *x-z* and *x-y* planes of the beam subjected to the aforementioned loading and taking into account shear deformation effect satisfy the following relations

$$\cos\beta_y \approx 1 \qquad \cos\beta_z \approx 1 \tag{3a,b}$$

$$\sin\beta_y \simeq -\frac{dw_C}{dx} \qquad \sin\beta_z \simeq -\frac{dv_C}{dx} \tag{3c,d}$$

while employing the stress-strain relations of the three-dimensional elasticity, the arising shear stress resultants Q_z, Q_y are given as

$$Q_z = \int_\Omega \tau_{xz} d\Omega = \int_\Omega G\gamma_{xz} d\Omega = GA_Z\left(\frac{dw}{dx} + \theta_Y\right) \tag{4a}$$

$$Q_y = \int_\Omega \tau_{xy} d\Omega = \int_\Omega G\gamma_{xy} d\Omega = GA_Y\left(\frac{dv}{dx} - \theta_Z\right) \tag{4b}$$

where γ_{xz}, γ_{xy} are the additional angles of rotation of the cross-section due to shear deformation (Fig. 2(a)) and GA_Z, GA_Y are the cross-section's shear rigidities of the Timoshenko's beam theory, where

$$A_Z = \kappa_Z A = \frac{1}{a_Z}A \qquad A_Y = \kappa_Y A = \frac{1}{a_Y}A \tag{5a,b}$$

are the shear areas with respect to *Z*, *Y* axes, respectively with κ_Z, κ_Y the shear correction factors, a_Z, a_Y the shear deformation coefficients and *A* the cross section area.

Referring to Fig. 2, the stress resultants R_x, R_y, R_z acting in the x, y, z directions, respectively, are related to the axial *N* and the shear Q_y, Q_z forces as

$$R_x = N\cos\beta + Q_z\sin\beta_y + Q_y\sin\beta_z \tag{6a}$$

$$R_y = -N\cos\beta_z + Q_y\cos\beta_z \tag{6b}$$

$$R_z = -N\sin\beta_y + Q_z\cos\beta_y \tag{6c}$$

which by virtue of the small deflection theory and Eqs. (2), (3) become

$$R_x = N - Q_z\frac{dw_C}{dx} - Q_y\frac{dv_C}{dx} \tag{7a}$$

$$R_y = N\frac{dv_C}{dx} + Q_y = N\left(\frac{dv}{dx} - z_C\frac{d\theta_x}{dx}\right) + Q_y \tag{7b}$$

$$R_z = N\frac{dw_C}{dx} + Q_z = N\left(\frac{dw}{dx} + y_C\frac{d\theta_x}{dx}\right) + Q_z \tag{7c}$$

The second and third terms in the right hand side of Eq. (7a), express the influence of the shear forces Q_y, Q_z on the horizontal stress resultant R_x. However, these terms can be neglected since Q_y, Q_z are much smaller than *N* and thus Eq. (7a) can be written as

$$R_x \approx N \tag{8}$$

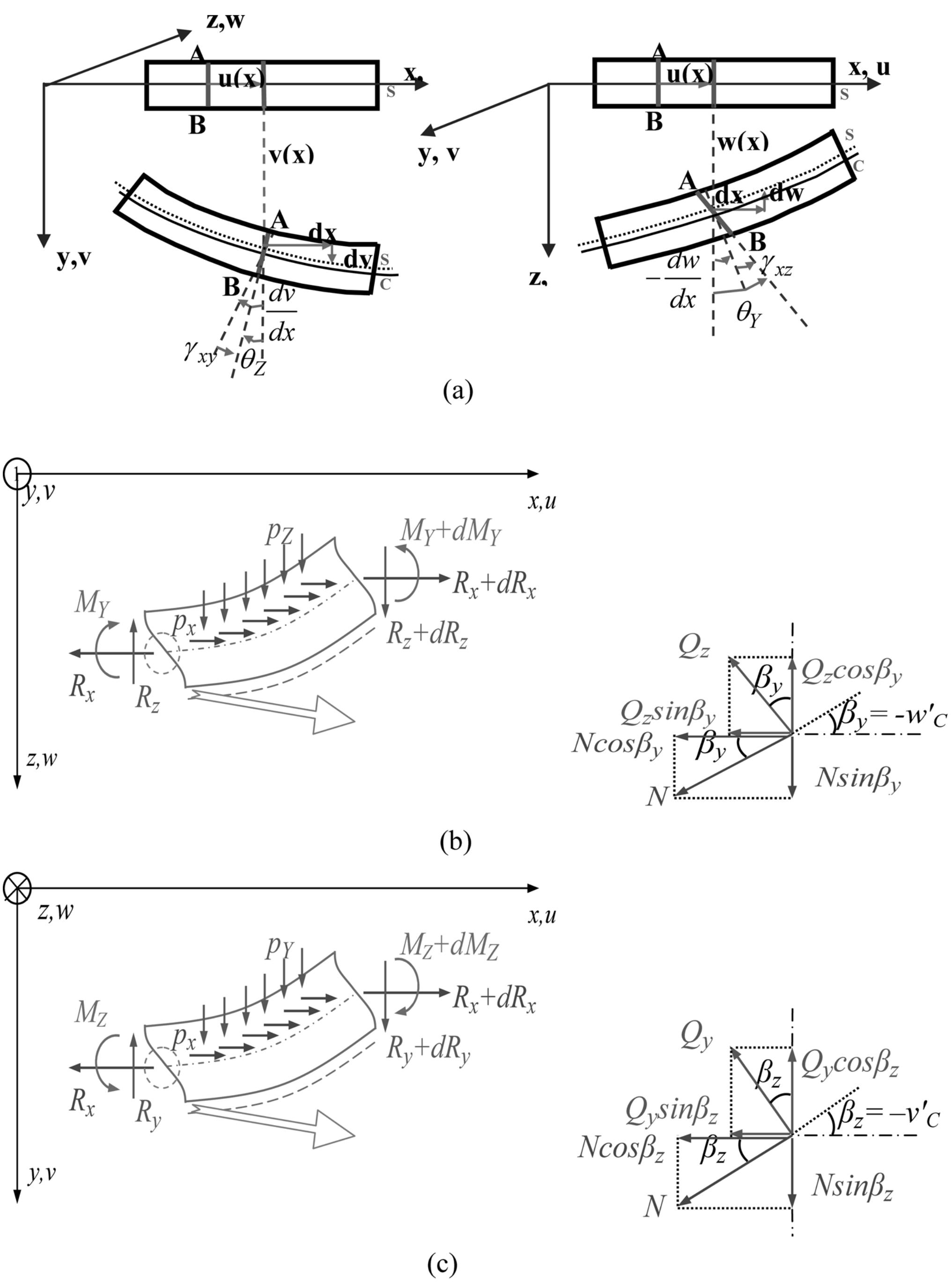


Fig. 2 Displacements (a) and equilibrium of an element in the *xz* (b) and *xy* (c) planes

Employing Eq. (1a) to the strain - displacement equations of the three-dimensional elasticity and ignoring axial deformations and the effect of secondary warping function, the normal strain component ε_x can be written as

$$\varepsilon_x = \frac{d\theta_Y}{dx}Z - \frac{d\theta_Z}{dx}Y + \frac{d^2\theta_x}{dx^2}\phi_S^P \tag{9}$$

and the arising bending moments M_Y, M_Z are given as

$$M_Y = \int_\Omega E\varepsilon_x Z d\Omega = EI_{YY}\frac{d\theta_Y}{dx} - EI_{YZ}\frac{d\theta_Z}{dx} \tag{10a}$$

$$M_Z = -\int_\Omega E\varepsilon_x Y d\Omega = EI_{ZZ}\frac{d\theta_Z}{dx} - EI_{YZ}\frac{d\theta_Y}{dx} \tag{10b}$$

where I_{YY}, I_{ZZ}, I_{YZ} are the moments and the product of inertia of the cross-section with respect to its centroid *C*. Substituting Eqs. (4a,b) in Eqs. (10a,b) the bending moments M_Y, M_Z can be written as

$$M_Y = EI_{YY}\left(\frac{1}{GA_Z}\frac{dQ_z}{dx} - \frac{d^2w}{dx^2}\right) - EI_{YZ}\left(\frac{d^2v}{dx^2} - \frac{1}{GA_Y}\frac{dQ_y}{dx}\right) \tag{11a}$$

$$M_Z = EI_{ZZ}\left(\frac{d^2v}{dx^2} - \frac{1}{GA_Y}\frac{dQ_y}{dx}\right) - EI_{YZ}\left(\frac{1}{GA_Z}\frac{dQ_z}{dx} - \frac{d^2w}{dx^2}\right) \tag{11b}$$

The governing equations of the problem at hand will be derived by considering the equilibrium of the deformed element. Thus, referring to Fig. 2 we obtain

$$\frac{dR_x}{dx} + p_X = 0 \qquad \frac{dR_y}{dx} + p_Y = 0 \qquad \frac{dR_z}{dx} + p_Z = 0 \tag{12a,b,c}$$

$$\frac{dM_Y}{dx} - Q_z = 0 \qquad \frac{dM_Z}{dx} + Q_y = 0 \tag{12d,e}$$

Substituting Eqs. (8), (7b), (7c) into Eqs. (12a), (12b),(12c) we obtain

$$\frac{dN}{dx} = -p_X \tag{13a}$$

$$\frac{dQ_y}{dx} + \frac{dN}{dx}\left(\frac{dv}{dx} - z_C\frac{d\theta_x}{dx}\right) + N\left(\frac{d^2v}{dx^2} - z_C\frac{d^2\theta_x}{dx^2}\right) + p_Y = 0 \tag{13b}$$

$$\frac{dQ_z}{dx} + \frac{dN}{dx}\left(\frac{dw}{dx} + y_C\frac{d\theta_x}{dx}\right) + N\left(\frac{d^2w}{dx^2} + y_C\frac{d^2\theta_x}{dx^2}\right) + p_Z = 0 \tag{13c}$$

Substituting Eqs. (13b,c) into Eqs. (11a,b) we obtain the expressions of the bending moments M_Y, M_Z as

$$\begin{aligned} M_Y = &-EI_{YY}\frac{d^2w}{dx^2} - \alpha_Z\frac{EI_{YY}}{GA}\left(p_Z - p_X\left(\frac{dw}{dx} + y_C\frac{d\theta_x}{dx}\right) + N\left(\frac{d^2w}{dx^2} + y_C\frac{d^2\theta_x}{dx^2}\right)\right) \\ &- EI_{YZ}\frac{d^2v}{dx^2} - \alpha_Y\frac{EI_{YZ}}{GA}\left(p_Y - p_X\left(\frac{dv}{dx} - z_C\frac{d\theta_x}{dx}\right) + N\left(\frac{d^2v}{dx^2} - z_C\frac{d^2\theta_x}{dx^2}\right)\right) \end{aligned} \tag{14a}$$

$$M_Z = EI_{ZZ}\frac{d^2v}{dx^2} + \alpha_Y \frac{EI_{ZZ}}{GA}\left(p_Y - p_X\left(\frac{dv}{dx} - z_C\frac{d\theta_x}{dx}\right) + N\left(\frac{d^2v}{dx^2} - z_C\frac{d^2\theta_x}{dx^2}\right)\right)$$

$$+ EI_{YZ}\frac{d^2w}{dx^2} + \alpha_Z \frac{EI_{YZ}}{GA}\left(p_Z - p_X\left(\frac{dw}{dx} + y_C\frac{d\theta_x}{dx}\right) + N\left(\frac{d^2w}{dx^2} + y_C\frac{d^2\theta_x}{dx^2}\right)\right) \tag{14b}$$

subsequently the expressions of the shear forces Q_y, Q_z employing Eqs. (12d,e) as

$$Q_y = -EI_{ZZ}\frac{d^3v}{dx^3} - EI_{YZ}\frac{d^3w}{dx^3}$$

$$-\alpha_Y\frac{EI_{ZZ}}{GA}\left(\frac{dp_Y}{dx} - \frac{dp_X}{dx}\left(\frac{dv}{dx} - z_C\frac{d\theta_x}{dx}\right) - 2p_X\left(\frac{d^2v}{dx^2} - z_C\frac{d^2\theta_x}{dx^2}\right) + N\left(\frac{d^3v}{dx^3} - z_C\frac{d^3\theta_x}{dx^3}\right)\right)$$

$$-\alpha_Z\frac{EI_{YZ}}{GA}\left(\frac{dp_Z}{dx} - \frac{dp_X}{dx}\left(\frac{dw}{dx} + y_C\frac{d\theta_x}{dx}\right) - 2p_X\left(\frac{d^2w}{dx^2} + y_C\frac{d^2\theta_x}{dx^2}\right) + N\left(\frac{d^3w}{dx^3} + y_C\frac{d^3\theta_x}{dx^3}\right)\right) \tag{15a}$$

$$Q_z = -EI_{YY}\frac{d^3w}{dx^3} - EI_{YZ}\frac{d^3v}{dx^3}$$

$$-\alpha_Z\frac{EI_{YY}}{GA}\left(\frac{dp_Z}{dx} - \frac{dp_X}{dx}\left(\frac{dw}{dx} + y_C\frac{d\theta_x}{dx}\right) - 2p_X\left(\frac{d^2w}{dx^2} + y_C\frac{d^2\theta_x}{dx^2}\right) + N\left(\frac{d^3w}{dx^3} + y_C\frac{d^3\theta_x}{dx^3}\right)\right)$$

$$-\alpha_Y\frac{EI_{YZ}}{GA}\left(\frac{dp_Y}{dx} - \frac{dp_X}{dx}\left(\frac{dv}{dx} - z_C\frac{d\theta_x}{dx}\right) - 2p_X\left(\frac{d^2v}{dx^2} - z_C\frac{d^2\theta_x}{dx^2}\right) + N\left(\frac{d^3v}{dx^3} - z_C\frac{d^3\theta_x}{dx^3}\right)\right) \tag{15b}$$

and eliminating these forces from Eqs. (13b,c) we obtain the first two coupled partial differential equations of the problem of the beam under consideration subjected to the combined action of axial, bending and torsional loading as

$$EI_{ZZ}\frac{d^4v}{dx^4} + EI_{YZ}\frac{d^4w}{dx^4} + \alpha_Y\frac{EI_{ZZ}}{GA}\left[\frac{d^2p_Y}{dx^2} - \frac{d^2p_X}{dx^2}\left(\frac{dv}{dx} - z_C\frac{d\theta_x}{dx}\right) - 3\frac{dp_X}{dx}\left(\frac{d^2v}{dx^2} - z_C\frac{d^2\theta_x}{dx^2}\right)\right.$$

$$\left. -3p_X\left(\frac{d^3v}{dx^3} - z_C\frac{d^3\theta_x}{dx^3}\right) + N\left(\frac{d^4v}{dx^4} - z_C\frac{d^4\theta_x}{dx^4}\right)\right] + \alpha_Z\frac{EI_{YZ}}{GA}\left[\frac{d^2p_Z}{dx^2} - \frac{d^2p_X}{dx^2}\left(\frac{dw}{dx} + y_C\frac{d\theta_x}{dx}\right)\right.$$

$$\left. -3\frac{dp_X}{dx}\left(\frac{d^2w}{dx^2} + y_C\frac{d^2\theta_x}{dx^2}\right) - 3p_X\left(\frac{d^3w}{dx^3} + y_C\frac{d^3\theta_x}{dx^3}\right) + N\left(\frac{d^4w}{dx^4} + y_C\frac{d^4\theta_x}{dx^4}\right)\right]$$

$$- p_Y + p_X\left(\frac{dv}{dx} - z_C\frac{d\theta_x}{dx}\right) - N\left(\frac{d^2v}{dx^2} - z_C\frac{d^2\theta_x}{dx^2}\right) = 0 \tag{16a}$$

$$EI_{YY}\frac{d^4w}{dx^4}+EI_{YZ}\frac{d^4v}{dx^4}+\alpha_Z\frac{EI_{YY}}{GA}\left[\frac{d^2p_Z}{dx^2}-\frac{d^2p_X}{dx^2}\left(\frac{dw}{dx}+y_C\frac{d\theta_x}{dx}\right)-3\frac{dp_X}{dx}\left(\frac{d^2w}{dx^2}+y_C\frac{d^2\theta_x}{dx^2}\right)\right.$$
$$\left.-3p_X\left(\frac{d^3w}{dx^3}+y_C\frac{d^3\theta_x}{dx^3}\right)+N\left(\frac{d^4w}{dx^4}+y_C\frac{d^4\theta_x}{dx^4}\right)\right]+\alpha_Y\frac{EI_{YZ}}{GA}\left[\frac{d^2p_Y}{dx^2}-\frac{d^2p_X}{dx^2}\left(\frac{dv}{dx}-z_C\frac{d\theta_x}{dx}\right)\right.$$
$$\left.-3\frac{dp_X}{dx}\left(\frac{d^2v}{dx^2}-z_C\frac{d^2\theta_x}{dx^2}\right)-3p_X\left(\frac{d^3v}{dx^3}-z_C\frac{d^3\theta_x}{dx^3}\right)+N\left(\frac{d^4v}{dx^4}-z_C\frac{d^4\theta_x}{dx^4}\right)\right]$$
$$-p_Z+p_X\left(\frac{dw}{dx}+y_C\frac{d\theta_x}{dx}\right)-N\left(\frac{d^2w}{dx^2}+y_C\frac{d^2\theta_x}{dx^2}\right)=0 \tag{16b}$$

Finally, the angles of rotation of the cross-section due to bending θ_Y, θ_Z are given from Eqs. (4a,b) as

$$\theta_Y=-\frac{dw}{dx}-\alpha_Z\frac{EI_{YY}}{GA}\frac{d^3w}{dx^3}-\alpha_Z\frac{EI_{YZ}}{GA}\frac{d^3v}{dx^3}$$
$$-\alpha_Z^2\frac{EI_{YY}}{G^2A^2}\left[\frac{dp_Z}{dx}-\frac{dp_X}{dx}\left(\frac{dw}{dx}+y_C\frac{d\theta_x}{dx}\right)-2p_X\left(\frac{d^2w}{dx^2}+y_C\frac{d^2\theta_x}{dx^2}\right)+N\left(\frac{d^3w}{dx^3}+y_C\frac{d^3\theta_x}{dx^3}\right)\right]$$
$$-\alpha_Y\alpha_Z\frac{EI_{YZ}}{G^2A^2}\left[\frac{dp_Y}{dx}-\frac{dp_X}{dx}\left(\frac{dv}{dx}-z_C\frac{d\theta_x}{dx}\right)-2p_X\left(\frac{d^2v}{dx^2}-z_C\frac{d^2\theta_x}{dx^2}\right)+N\left(\frac{d^3v}{dx^3}-z_C\frac{d^3\theta_x}{dx^3}\right)\right] \tag{17a}$$

$$\theta_Z=\frac{dv}{dx}+\alpha_Y\frac{EI_{ZZ}}{GA}\frac{d^3v}{dx^3}+\alpha_Y\frac{EI_{YZ}}{GA}\frac{d^3w}{dx^3}$$
$$+\alpha_Y^2\frac{EI_{ZZ}}{G^2A^2}\left[\frac{dp_Y}{dx}-\frac{dp_X}{dx}\left(\frac{dv}{dx}-z_C\frac{d\theta_x}{dx}\right)-2p_X\left(\frac{d^2v}{dx^2}-z_C\frac{d^2\theta_x}{dx^2}\right)+N\left(\frac{d^3v}{dx^3}-z_C\frac{d^3\theta_x}{dx^3}\right)\right]$$
$$+\alpha_Z\alpha_Y\frac{EI_{YZ}}{G^2A^2}\left[\frac{dp_Z}{dx}-\frac{dp_X}{dx}\left(\frac{dw}{dx}+y_C\frac{d\theta_x}{dx}\right)-2p_X\left(\frac{d^2w}{dx^2}+y_C\frac{d^2\theta_x}{dx^2}\right)+N\left(\frac{d^3w}{dx^3}+y_C\frac{d^3\theta_x}{dx^3}\right)\right] \tag{17b}$$

Equilibrium of torsional moments along x axis of the beam element (Sapountzakis and Mokos 2003), after taking into account the additional shear stresses due to the presence of the axial force N (Timoshenko and Gere 1961), which employing Eq. (2) are written as

$$\tau_{xz}=\frac{N}{A}\left(\frac{d\overline{w}}{dx}\right)=\frac{N}{A}\left(\frac{dw}{dx}+y\frac{d\theta_x}{dx}\right) \tag{18a}$$

$$\tau_{xy}=\frac{N}{A}\left(\frac{d\overline{v}}{dx}\right)=\frac{N}{A}\left(\frac{dv}{dx}-z\frac{d\theta_x}{dx}\right) \tag{18b}$$

and the corresponding arising additional twisting moment

$$M_{t,add}=\int_\Omega(\tau_{xz}y-\tau_{xy}z)d\Omega=Ny_C\frac{dw}{dx}-Nz_C\frac{dv}{dx}+N\frac{I_S}{A}\frac{d\theta_x}{dx} \tag{19}$$

leads to the third (coupled with the previous two) partial differential equation of the problem of the beam under consideration as

$$EC_S\frac{d^4\theta_x}{dx^4}-GI_t\frac{d^2\theta_x}{dx^2}-N\left(y_C\frac{d^2w}{dx^2}-z_C\frac{d^2v}{dx^2}+\frac{I_S}{A}\frac{d^2\theta_x}{dx^2}\right)=m_x+p_Zy_C-p_Yz_C-p_X\left(y_C\frac{dw}{dx}-z_C\frac{dv}{dx}\right)-p_X\frac{I_S}{A}\frac{d\theta_x}{dx} \quad (20)$$

where I_S is the polar moment of inertia with respect to the shear center S, EC_S and GI_t are the cross section's warping and torsional rigidities, respectively, with C_S, I_t being its warping and torsion constants, respectively, given as (Sapountzakis and Mokos 2003)

$$C_S = \int_\Omega (\varphi_S^P)^2 d\Omega \quad (21)$$

$$I_t = \int_\Omega \left(y^2+z^2+y\frac{\partial\varphi_S^P}{\partial z}-z\frac{\partial\varphi_S^P}{\partial y}\right)d\Omega \quad (22)$$

It is worth here noting that the primary warping function $\varphi_S^P(y,z)$ can be established by solving independently the Neumann problem (Sapountzakis and Mokos 2003)

$$\nabla^2\varphi_S^P = 0 \qquad \text{in} \quad \Omega \quad (23a)$$

$$\frac{\partial\varphi_S^P}{\partial n} = \frac{1}{2}\frac{\partial(r_S^2)}{\partial s} \qquad \text{on} \quad \Gamma_j \quad (j=1,2,\ldots,K) \quad (23b)$$

where $\nabla^2 = \partial^2/\partial y^2+\partial^2/\partial z^2$ is the Laplace operator; $r_S = \sqrt{y^2+z^2}$ is the distance of a point on the boundary Γ_j from the shear center S; $\partial/\partial n$ denotes the directional derivative normal to the boundary Γ_j and $\partial/\partial s$ denotes differentiation with respect to its arc length s.

As it is already mentioned, Eqs. (16a), (16b), (20) constitute the governing equations of the beam subjected to the combined action of axial, bending and torsional loading taking into account shear deformation effect.

The differential equations of equilibrium for the flexural-torsional buckling problem of an axially compressed beam result from the aforementioned differential equations after setting $N=-P$ and $p_X=p_{X,x}=p_{X,xx}=0$, $p_Y=p_{Y,xx}=0$, $p_Z=p_{Z,xx}=0$, $m_x=0$ as

$$EI_{ZZ}\frac{d^4v}{dx^4}-\alpha_Y\frac{EI_{ZZ}}{GA}P\left(\frac{d^4v}{dx^4}-z_C\frac{d^4\theta_x}{dx^4}\right)+EI_{YZ}\frac{d^4w}{dx^4}$$
$$-\alpha_Z\frac{EI_{YZ}}{GA}P\left(\frac{d^4w}{dx^4}+y_C\frac{d^4\theta_x}{dx^4}\right)+P\left(\frac{d^2v}{dx^2}-z_C\frac{d^2\theta_x}{dx^2}\right)=0 \quad (24a)$$

$$EI_{YY}\frac{d^4w}{dx^4}-\alpha_Z\frac{EI_{YY}}{GA}P\left(\frac{d^4w}{dx^4}+y_C\frac{d^4\theta_x}{dx^4}\right)+EI_{YZ}\frac{d^4v}{dx^4}+$$
$$-\alpha_Y\frac{EI_{YZ}}{GA}P\left(\frac{d^4v}{dx^4}-z_C\frac{d^4\theta_x}{dx^4}\right)+P\left(\frac{d^2w}{dx^2}+y_C\frac{d^2\theta_x}{dx^2}\right)=0 \quad (24b)$$

$$EC_S\frac{d^4\theta_x}{dx^4}-GI_t\frac{d^2\theta_x}{dx^2}+P\left(y_c\frac{d^2w}{dx^2}-z_c\frac{d^2v}{dx^2}+\frac{I_S}{A}\frac{d^2\theta_x}{dx^2}\right)=0 \quad (24c)$$

In the case of a nonsymmetric cross section beam, the obtained buckling mode from the solution of the previous system of equations corresponds to a flexural-torsional coupled one. In the presence of an axis of symmetry (e.g., z axis), the principal bending system of axes coincides with the principal shear one. In this case, $y_C = 0$, $I_{YZ} = 0$ and Eq. (24b) becomes uncoupled to the other two differential Eq. (24a,c). Thus, the buckling modes will be either flexural or flexural - torsional. Finally, in the special case of a doubly symmetric cross section beam, the buckling modes will be either flexural or torsional.

The aforementioned equations are also subjected to the pertinent boundary conditions of the problem, which are given as

$$\alpha_1 v(x) + \alpha_2 R_y(x) = \alpha_3 \qquad \bar{\alpha}_1 \theta_Z(x) + \bar{\alpha}_2 M_Z(x) = \bar{\alpha}_3 \tag{25a,b}$$

$$\beta_1 w(x) + \beta_2 R_z(x) = \beta_3 \qquad \bar{\beta}_1 \theta_Y(x) + \bar{\beta}_2 M_Y(x) = \bar{\beta}_3 \tag{26a,b}$$

$$\gamma_1 \theta_x(x) + \gamma_2 M_t(x) = \gamma_3 \qquad \bar{\gamma}_1 \frac{d\theta_x(x)}{dx} + \bar{\gamma}_2 M_w(x) = \bar{\gamma}_3 \tag{27a,b}$$

at the beam ends $x = 0, l$, where R_y, R_z and M_Z, M_Y are the reactions and bending moments with respect to y and z axes, respectively, obtained from Eqs. (7b,c), (14a,b), (15a,b) as

$$\begin{aligned} R_y = & -EI_{ZZ}\frac{d^3v}{dx^3} - \alpha_Y \frac{EI_{ZZ}}{GA} N\left(\frac{d^3v}{dx^3} - z_C \frac{d^3\theta_x}{dx^3}\right) - EI_{YZ}\frac{d^3w}{dx^3} - \\ & \alpha_Z \frac{EI_{YZ}}{GA} N\left(\frac{d^3w}{dx^3} + y_C \frac{d^3\theta_x}{dx^3}\right) + N\left(\frac{dv}{dx} - z_C \frac{d\theta_x}{dx}\right) \end{aligned} \tag{28}$$

$$\begin{aligned} R_z = & -EI_{YY}\frac{d^3w}{dx^3} - \alpha_Z \frac{EI_{YY}}{GA} N\left(\frac{d^3w}{dx^3} + y_C \frac{d^3\theta_x}{dx^3}\right) - EI_{YZ}\frac{d^3v}{dx^3} - \\ & \alpha_Y \frac{EI_{YZ}}{GA} N\left(\frac{d^3v}{dx^3} - z_C \frac{d^3\theta_x}{dx^3}\right) + N\left(\frac{dw}{dx} + y_C \frac{d\theta_x}{dx}\right) \end{aligned} \tag{29}$$

$$M_Y = -EI_{YY}\frac{d^2w}{dx^2} - \alpha_Z \frac{EI_{YY}}{GA} N\left(\frac{d^2w}{dx^2} + y_C \frac{d^2\theta_x}{dx^2}\right) - EI_{YZ}\frac{d^2v}{dx^2} - \alpha_Y \frac{EI_{YZ}}{GA} N\left(\frac{d^2v}{dx^2} - z_C \frac{d^2\theta_x}{dx^2}\right) \tag{30}$$

$$M_Z = EI_{ZZ}\frac{d^2v}{dx^2} + \alpha_Y \frac{EI_{ZZ}}{GA} N\left(\frac{d^2v}{dx^2} - z_C \frac{d^2\theta_x}{dx^2}\right) + EI_{YZ}\frac{d^2w}{dx^2} + \alpha_Z \frac{EI_{YZ}}{GA} N\left(\frac{d^2w}{dx^2} + y_C \frac{d^2\theta_x}{dx^2}\right) \tag{31}$$

the angles of rotation due to bending θ_Y, θ_Z are evaluated from Eq. (17) as

$$\begin{aligned} \theta_Y = & -\frac{dw}{dx} - \alpha_Z \frac{EI_{YY}}{GA}\frac{d^3w}{dx^3} - \alpha_Z^2 \frac{EI_{YY}}{G^2A^2} N\left(\frac{d^3w}{dx^3} + y_C \frac{d^3\theta_x}{dx^3}\right) - \alpha_Z \frac{EI_{YZ}}{GA}\frac{d^3v}{dx^3} \\ & -\alpha_Y \alpha_Z \frac{EI_{YZ}}{G^2A^2} N\left(\frac{d^3v}{dx^3} - z_C \frac{d^3\theta_x}{dx^3}\right) \end{aligned} \tag{32a}$$

$$\theta_Z = \frac{dv}{dx} + \alpha_Y \frac{EI_{ZZ}}{GA}\frac{d^3v}{dx^3} + \alpha_Y^2 \frac{EI_{ZZ}}{G^2A^2} N\left(\frac{d^3v}{dx^3} - z_C\frac{d^3\theta_x}{dx^3}\right) + \alpha_Y \frac{EI_{YZ}}{GA}\frac{d^3w}{dx^3}$$
$$+ \alpha_Z\alpha_Y \frac{EI_{YZ}}{G^2A^2} N\left(\frac{d^3w}{dx^3} + y_C\frac{d^3\theta_x}{dx^3}\right) \tag{32b}$$

while in Eq. (27) M_t and M_w at the beam ends are the torsional and warping moments, respectively, given as

$$M_t = -EC_S\frac{d^3\theta_x}{dx^3} + GI_t\frac{d\theta_x}{dx} + N\left(y_C\frac{dw}{dx} - z_C\frac{dv}{dx} + \frac{I_S}{A}\frac{d\theta_x}{dx}\right) \tag{33a}$$

$$M_w = -EC_S\frac{d^2\theta_x}{dx^2} \tag{33b}$$

Finally, $\alpha_k, \bar{\alpha}_k, \beta_k, \bar{\beta}_k, \gamma_k, \bar{\gamma}_k$ $(k = 1, 2, 3)$ are functions specified at the beam ends $x = 0, l$. Eqs. (25)-(27) describe the most general linear boundary conditions associated with the problem at hand and can include elastic support or restraint. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived from these equations by specifying appropriately these functions (e.g., for a clamped edge it is $\alpha_1 = \beta_1 = \gamma_1 = 1$, $\bar{\alpha}_1 = \bar{\beta}_1 = \bar{\gamma}_1 = 1$, $\alpha_2 = \alpha_3 = \beta_2 = \beta_3 = \gamma_2 = \gamma_3 = \bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\beta}_2 = \bar{\beta}_3 = \bar{\gamma}_2 = \bar{\gamma}_3 = 0$).

The solution of the boundary value problem given from Eqs. (16), (20) subjected to the boundary conditions (25)-(27) which represents the flexural-torsional buckling of beams, presumes the evaluation of the shear deformation coefficients a_Y, a_Z, corresponding to the principal shear axes coordinate system. These coefficients are established equating the approximate formula of the shear strain energy per unit length (Schramm *et al.* 1997)

$$U_{appr.} = \frac{a_Y Q_y^2}{2AG} + \frac{a_Z Q_z^2}{2AG} \tag{34}$$

with the exact one given from

$$U_{exact} = \int_\Omega \frac{(\tau_{xz})^2 + (\tau_{xy})^2}{2G} d\Omega \tag{35}$$

and are obtained as (Sapountzakis and Mokos 2005)

$$a_Y = \frac{1}{\kappa_Y} = \frac{A}{\Delta^2}\int_\Omega [(\nabla\Theta) - \boldsymbol{e}]\cdot[(\nabla\Theta) - \boldsymbol{e}]d\Omega \tag{36a}$$

$$a_Z = \frac{1}{\kappa_Z} = \frac{A}{\Delta^2}\int_\Omega [(\nabla\Phi) - \boldsymbol{d}]\cdot[(\nabla\Phi) - \boldsymbol{d}]d\Omega \tag{36b}$$

where $(\tau_{xz})_j, (\tau_{xy})_j$ are the transverse (direct) shear stress components, $(\nabla) \equiv \boldsymbol{i_Y}(\partial/\partial Y) + \boldsymbol{i_Z}(\partial/\partial Z)$ is a symbolic vector with $\boldsymbol{i_Y}, \boldsymbol{i_Z}$ the unit vectors along Y and Z axes, respectively, Δ is given from

$$\Delta = 2(1+\nu)(I_{YY}I_{ZZ} - I_{YZ}^2) \tag{37}$$

ν is the Poisson ratio of the cross section material, $\boldsymbol{e}$ and $\boldsymbol{d}$ are vectors defined as

$$\boldsymbol{e} = \left[\nu\left(I_Y\frac{Y^2-Z^2}{2}-I_{YZ}YZ\right)\right]\boldsymbol{i_Y}+\left[\nu\left(I_YYZ+I_{YZ}\frac{Y^2-Z^2}{2}\right)\right]\boldsymbol{i_Z} \tag{38a}$$

$$\boldsymbol{d} = \left[\nu\left(I_ZYZ-I_{YZ}\frac{Y^2-Z^2}{2}\right)\right]\boldsymbol{i_Y}+\left[-\nu\left(I_Z\frac{Y^2-Z^2}{2}+I_{YZ}YZ\right)\right]\boldsymbol{i_Z} \tag{38b}$$

and $\Theta(Y,Z), \Phi(Y,Z)$ are stress functions, which are evaluated from the solution of the following Neumann type boundary value problems (Sapountzakis and Mokos 2005)

$$\nabla^2\Theta = 2(I_{YZ}Z-I_YY) \quad \text{in} \quad \Omega \tag{39a}$$

$$\frac{\partial\Theta}{\partial n} = \boldsymbol{n}\cdot\boldsymbol{e} \qquad \text{on} \quad \Gamma = \bigcup_{j=1}^{K+1}\Gamma_j \tag{39b}$$

$$\nabla^2\Phi = 2(I_{YZ}Y-I_ZZ) \quad \text{in} \quad \Omega \tag{40a}$$

$$\frac{\partial\Phi}{\partial n} = \boldsymbol{n}\cdot\boldsymbol{d} \qquad \text{on} \quad \Gamma = \bigcup_{j=1}^{K+1}\Gamma_j \tag{40b}$$

where $\boldsymbol{n}$ is the outward normal vector to the boundary Γ. In the case of negligible shear deformations $a_Z = a_Y = 0$. It is also worth here noting that the boundary conditions (23b), (39b), (40b) have been derived from the physical consideration that the traction vector in the direction of the normal vector $\boldsymbol{n}$ vanishes on the free surface of the beam.

3. Integral representations - numerical solution

3.1 For the transverse v, w displacements and the angle of twist θ_x

According to the precedent analysis, the flexural-torsional buckling problem of axially loaded beams reduces in establishing the critical value of axial load for which the displacement components $v(x), w(x)$ and $\theta_x(x)$ of a beam, under no other external loading, having continuous derivatives up to the fourth order with respect to x, satisfying the coupled governing Eqs. (16), (20) inside the beam and the boundary conditions (25)-(27) at the beam ends $x = 0, l$ are not equal to zero.

Eqs. (16), (20) are solved using the Analog Equation Method (Katsikadelis 2002) as it is developed for hyperbolic differential equations (Sapountzakis and Katsikadelis 2000). This method is applied for the problem at hand as follows. Let $v(x), w(x)$ and $\theta_x(x)$ be the sought solution of the aforementioned boundary value problem. Setting as $u_1(x) = v(x)$, $u_2(x) = w(x)$, $u_3(x) = \theta_x(x)$ and differentiating these functions four times with respect to x yields

$$\frac{d^4u_i}{dx^4} = q_i(x) \qquad (i = 1, 2, 3) \tag{41}$$

Eq. (41) indicate that the solution of Eqs. (16), (20) can be established by solving Eq. (41) under the same boundary conditions (25)-(27), provided that the fictitious load distributions $q_i(x)$ $(i = 1, 2, 3)$ are first established. These distributions can be determined using BEM as follows.

The solution of Eq. (41) is given in integral form as

$$u_i(x) = \int_0^l q_iu^*d\xi-\left[u^*\frac{d^3u_i}{dx^3}-\frac{du^*}{dx}\frac{d^2u_i}{dx^2}+\frac{d^2u^*}{dx^2}\frac{du_i}{dx}-\frac{d^3u^*}{dx^3}u_i\right]_0^l \tag{42}$$

where u^* is the fundamental solution given as

$$u^* = \frac{1}{12}l^3\left(2+\left|\frac{r}{l}\right|^3-3\left|\frac{r}{l}\right|^2\right) \tag{43}$$

with $r = x-\xi$, x, ξ points of the beam, which is a particular singular solution of the equation

$$\frac{d^4u^*}{dx^4} = \delta(x-\xi) \tag{44}$$

Employing Eq. (43) the integral representation (42) can be written as

$$u_i(x) = \int_0^l q_i\Lambda_4(r)d\xi - \left[\Lambda_4(r)\frac{d^3u_i}{dx^3}+\Lambda_3(r)\frac{d^2u_i}{dx^2}+\Lambda_2(r)\frac{du_i}{dx}+\Lambda_1(r)u_i\right]_0^l \tag{45}$$

where the kernels $\Lambda_j(r)$, $(j = 1, 2, 3, 4)$ are given as

$$\Lambda_1(r) = -\frac{1}{2}\mathrm{sgn}\frac{r}{l} \tag{46a}$$

$$\Lambda_2(r) = -\frac{1}{2}l\left(1-\left|\frac{r}{l}\right|\right) \tag{46b}$$

$$\Lambda_3(r) = -\frac{1}{4}l^2\left|\frac{r}{l}\right|\left(\left|\frac{r}{l}\right|-2\right)\mathrm{sgn}\frac{r}{l} \tag{46c}$$

$$\Lambda_4(r) = \frac{1}{12}l^3\left(2+\left|\frac{r}{l}\right|^3-3\left|\frac{r}{l}\right|^2\right) \tag{46d}$$

Notice that in Eq. (45) for the line integral it is $r = x-\xi$, x, ξ points inside the beam, whereas for the rest terms it is $r = x-\zeta$, x inside the beam, ζ at the beam ends 0, l. Differentiating Eq. (45) with respect to x, results in the integral representations of the derivatives of u_i as

$$\frac{du_i(x)}{dx} = \int_0^l q_i\Lambda_3(r)d\xi - \left[\Lambda_3(r)\frac{d^3u_i}{dx^3}+\Lambda_2(r)\frac{d^2u_i}{dx^2}+\Lambda_1(r)\frac{du_i}{dx}\right]_0^l \tag{47a}$$

$$\frac{d^2u_i(x)}{dx^2} = \int_0^l q_i\Lambda_2(r)d\xi - \left[\Lambda_2(r)\frac{d^3u_i}{dx^3}+\Lambda_1(r)\frac{d^2u_i}{dx^2}\right]_0^l \tag{47b}$$

$$\frac{d^3u_i(x)}{dx^3} = \int_0^l q_i\Lambda_1(r)d\xi - \left[\Lambda_1(r)\frac{d^3u_i}{dx^3}\right]_0^l \tag{47c}$$

$$\frac{d^4u_i(x)}{dx^4} = q_i(x) \tag{47d}$$

The integral representations (45) and (47a), when applied for the beam ends (0, l), together with the boundary conditions (25)-(27) are employed to express the unknown boundary quantities $u_i(\zeta), u_{i,x}(\zeta), u_{i,xx}(\zeta)$ and $u_{i,xxx}(\zeta)$ $(\zeta = 0, l)$ in terms of q_i. This is accomplished numerically as follows.

The interval (0, l) is divided into L equal elements (Fig. 3), on which $q_i(x)$ is assumed to vary according to certain law (constant, linear, parabolic etc). The constant element assumption is

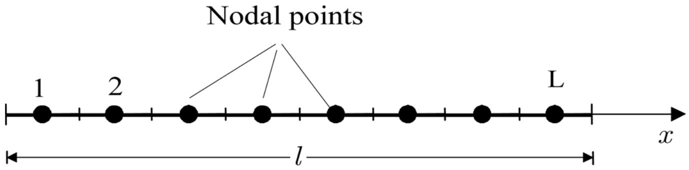


Fig. 3 Discretization of the beam interval and distribution of the nodal points

employed here as the numerical implementation becomes very simple and the obtained results are very good. Employing the aforementioned procedure for the coupled boundary conditions (25), (26) the following set of linear equations is obtained

$$\begin{bmatrix} \mathbf{D}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{14} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{18} \\ \mathbf{0} & \mathbf{D}_{22} & \mathbf{D}_{23} & \mathbf{D}_{24} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{27} & \mathbf{D}_{28} \\ \mathbf{E}_{31} & \mathbf{E}_{32} & \mathbf{E}_{33} & \mathbf{E}_{34} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_{42} & \mathbf{E}_{43} & \mathbf{E}_{44} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{54} & \mathbf{D}_{55} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{58} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_{63} & \mathbf{D}_{64} & \mathbf{0} & \mathbf{D}_{66} & \mathbf{D}_{67} & \mathbf{D}_{68} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{E}_{31} & \mathbf{E}_{32} & \mathbf{E}_{33} & \mathbf{E}_{34} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{E}_{42} & \mathbf{E}_{43} & \mathbf{E}_{44} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{u}}_1 \\ \hat{\mathbf{u}}_{1,x} \\ \hat{\mathbf{u}}_{1,xx} \\ \hat{\mathbf{u}}_{1,xxx} \\ \hat{\mathbf{u}}_2 \\ \hat{\mathbf{u}}_{2,x} \\ \hat{\mathbf{u}}_{2,xx} \\ \hat{\mathbf{u}}_{2,xxx} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\alpha}_3 \\ \bar{\boldsymbol{\alpha}}_3 \\ \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{\beta}_3 \\ \bar{\boldsymbol{\beta}}_3 \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{F}_3 \\ \mathbf{F}_4 \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{q}_1 + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{F}_3 \\ \mathbf{F}_4 \end{bmatrix} \mathbf{q}_2 \tag{48}$$

while for the boundary conditions (27) we have

$$\begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} & \mathbf{0} & \mathbf{E}_{14} \\ \mathbf{0} & \mathbf{E}_{22} & \mathbf{E}_{23} & \mathbf{0} \\ \mathbf{E}_{31} & \mathbf{E}_{32} & \mathbf{E}_{33} & \mathbf{E}_{34} \\ \mathbf{0} & \mathbf{E}_{42} & \mathbf{E}_{43} & \mathbf{E}_{44} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{u}}_3 \\ \hat{\mathbf{u}}_{3,x} \\ \hat{\mathbf{u}}_{3,xx} \\ \hat{\mathbf{u}}_{3,xxx} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\gamma}_3 \\ \bar{\boldsymbol{\gamma}}_3 \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{F}_3 \\ \mathbf{F}_4 \end{bmatrix} \mathbf{q}_3 \tag{49}$$

where $\mathbf{D}_{11}$, $\mathbf{D}_{14}$, $\mathbf{D}_{18}$, $\mathbf{D}_{22}$, $\mathbf{D}_{23}$, $\mathbf{D}_{24}$, $\mathbf{D}_{27}$, $\mathbf{D}_{28}$, $\mathbf{D}_{54}$, $\mathbf{D}_{55}$, $\mathbf{D}_{58}$, $\mathbf{D}_{63}$, $\mathbf{D}_{64}$, $\mathbf{D}_{66}$, $\mathbf{D}_{67}$, $\mathbf{D}_{68}$, $\mathbf{E}_{22}$, $\mathbf{E}_{23}$, $\mathbf{E}_{1j}$, ($j = 1, 2, 4$) are 2×2 known square matrices including the values of the functions $a_j, \bar{a}_j, \beta_j, \bar{\beta}_j$, $\gamma_j, \bar{\gamma}_j$ ($j = 1, 2$) of Eqs. (25)-(27); $\boldsymbol{\alpha}_3$, $\bar{\boldsymbol{\alpha}}_3, \boldsymbol{\beta}_3, \bar{\boldsymbol{\beta}}_3, \boldsymbol{\gamma}_3, \bar{\boldsymbol{\gamma}}_3$ are 2×1 known column matrices including the boundary values of the functions a_3, $\bar{a}_3, \beta_3, \bar{\beta}_3, \gamma_3, \bar{\gamma}_3$ of Eqs. (25)-(27); $\boldsymbol{E}_{jk}$ ($j = 3, 4$, $k = 1, 2, 3, 4$) are square 2×2 known coefficient matrices resulting from the values of the kernels $\Lambda_j(r)$ ($j = 1, 2, 3, 4$) at the beam ends and $\mathbf{F}_j$ ($j = 3, 4$) are $2 \times L$ rectangular known matrices originating from the integration of the kernels on the axis of the beam. Moreover,

$$\hat{\mathbf{u}}_i = \{u_i(0) \;\; u_i(l)\}^T \tag{50a}$$

$$\hat{\mathbf{u}}_{i,x} = \left\{ \frac{du_i(0)}{dx} \;\; \frac{du_i(l)}{dx} \right\}^T \tag{50b}$$

$$\hat{\mathbf{u}}_{i,xx} = \left\{ \frac{d^2u_i(0)}{dx^2} \;\; \frac{d^2u_i(l)}{dx^2} \right\}^T \tag{50c}$$

$$\hat{\mathbf{u}}_{i,xxx} = \left\{ \frac{d^3u_i(0)}{dx^3} \;\; \frac{d^3u_i(l)}{dx^3} \right\}^T \tag{50d}$$

are vectors including the two unknown boundary values of the respective boundary quantities and $\mathbf{q}_i = \{q_1^i \; q_2^i \; \dots \; q_L^i\}^T$ $(i = 1, 2, 3)$ is the vector including the L unknown nodal values of the fictitious load.

Discretization of Eqs. (45), (47) and application to the L collocation points yields

$$\mathbf{u}_i = \mathbf{C}_4\mathbf{q}_i - (\mathbf{H}_1\hat{\mathbf{u}}_i + \mathbf{H}_2\mathbf{u}_{i,x} + \mathbf{H}_3\hat{\mathbf{u}}_{i,xx} + \mathbf{H}_4\mathbf{u}_{i,xxx}) \tag{51a}$$

$$\mathbf{u}_{i,x} = \mathbf{C}_3\mathbf{q}_i - (\mathbf{H}_1\hat{\mathbf{u}}_{i,x} + \mathbf{H}_2\mathbf{u}_{i,xx} + \mathbf{H}_3\mathbf{u}_{i,xxx}) \tag{51b}$$

$$\mathbf{u}_{i,xx} = \mathbf{C}_2\mathbf{q}_i - (\mathbf{H}_1\hat{\mathbf{u}}_{i,xx} + \mathbf{H}_2\mathbf{u}_{i,xxx}) \tag{51c}$$

$$\mathbf{u}_{i,xxx} = \mathbf{C}_1\mathbf{q}_i - \mathbf{H}_1\hat{\mathbf{u}}_{i,xxx} \tag{51d}$$

$$\mathbf{u}_{i,xxxx} = \mathbf{q}_i \tag{51e}$$

where $\mathbf{C}_j$ $(j = 1, 2, 3, 4)$ are $L \times L$ known matrices; $\mathbf{H}_j$ $(j = 1, 2, 3, 4)$ are $L \times 2$ also known matrices and $\mathbf{u}_i, \mathbf{u}_{i,x}, \mathbf{u}_{i,xx}, \mathbf{u}_{i,xxx}, \mathbf{u}_{i,xxxx}$ are vectors including the values of $u_i(x)$ and their derivatives at the L nodal points.

The above equations, after eliminating the boundary quantities employing Eqs. (48) and (49), can be written as

$$\mathbf{u}_i = \mathbf{T}_i\mathbf{q}_i + \mathbf{T}_{ij}\mathbf{q}_j + \mathbf{t}_i \qquad i,j = 1,2 \qquad i \neq j \tag{52a}$$

$$\mathbf{u}_3 = \mathbf{T}_3\mathbf{q}_3 + \mathbf{t}_3 \tag{52b}$$

$$\mathbf{u}_{i,x} = \mathbf{T}_{ix}\mathbf{q}_i + \mathbf{T}_{ijx}\mathbf{q}_j + \mathbf{t}_{ix} \qquad i,j = 1,2 \qquad i \neq j \tag{52c}$$

$$\mathbf{u}_{3,x} = \mathbf{T}_{3x}\mathbf{q}_3 + \mathbf{t}_{3x} \tag{52d}$$

$$\mathbf{u}_{i,xx} = \mathbf{T}_{ixx}\mathbf{q}_i + \mathbf{T}_{ijxx}\mathbf{q}_j + \mathbf{t}_{ixx} \qquad i,j = 1,2 \qquad i \neq j \tag{52e}$$

$$\mathbf{u}_{3,xx} = \mathbf{T}_{3xx}\mathbf{q}_3 + \mathbf{t}_{3xx} \tag{52f}$$

$$\mathbf{u}_{i,xxx} = \mathbf{T}_{ixxx}\mathbf{q}_i + \mathbf{T}_{ijxxx}\mathbf{q}_j + \mathbf{t}_{ixxx} \qquad i,j = 1,2 \qquad i \neq j \tag{52g}$$

$$\mathbf{u}_{3,xxx} = \mathbf{T}_{3xxx}\mathbf{q}_3 + \mathbf{t}_{3xxx} \tag{52h}$$

$$\mathbf{u}_{1,xxxx} = \mathbf{q}_i \qquad i = 1,2,3 \tag{52i}$$

where $\mathbf{T}_i, \mathbf{T}_{ix}, \mathbf{T}_{ixx}, \mathbf{T}_{ixxx}, \mathbf{T}_{ij}, \mathbf{T}_{ijx}, \mathbf{T}_{ijxx}, \mathbf{T}_{ijxxx}$ are known $L \times L$ matrices and $\mathbf{t}_i, \mathbf{t}_{ix}, \mathbf{t}_{ixx}, \mathbf{t}_{ixxx}$ are known $L \times 1$ matrices. It is worth here noting that for homogeneous boundary conditions ($\alpha_3 = \bar{\alpha}_3 = \beta_3 = \bar{\beta}_3 = \gamma_3 = \bar{\gamma}_3 = 0$) it is $\mathbf{t}_i = \mathbf{t}_{ix} = \mathbf{t}_{ixx} = \mathbf{t}_{ixxx} = \mathbf{0}$.

In the conventional BEM, the load vectors $\mathbf{q}_i$ are known and Eq. (52) are used to evaluate $u_i(x)$ and their derivatives at the L nodal points. This, however, can not be done here since $\mathbf{q}_i$ are unknown. For this purpose, $3L$ additional equations are derived, which permit the establishment of $\mathbf{q}_i$. These equations result by applying Eqs. (16), (20) to the L collocation points, leading to the formulation of the following set of $3L$ simultaneous equations

$$(\mathbf{A}-N\mathbf{B}+\mathbf{C})\begin{Bmatrix}\mathbf{q}_1\\ \mathbf{q}_2\\ \mathbf{q}_3\end{Bmatrix}=\mathbf{f} \tag{53}$$

where the $3L\times 3L$ matrices $\mathbf{A},\mathbf{B},\mathbf{N},\mathbf{C}$ are given as

$$\mathbf{A}=\begin{bmatrix}\mathbf{EI}_{ZZ} & \mathbf{EI}_{YZ} & \mathbf{0}\\ \mathbf{EI}_{YZ} & \mathbf{EI}_{YY} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{EC}_S-\mathbf{GI}_t\mathbf{T}_{3xx}\end{bmatrix} \tag{54a}$$

$$\mathbf{B}=\begin{bmatrix}-\frac{\alpha_Y}{GA}\mathbf{EI}_{ZZ}+\mathbf{T}_{1xx} & -\frac{\alpha_Z}{GA}\mathbf{EI}_{YZ}+\mathbf{T}_{12xx} & \frac{\alpha_Y z_C}{GA}\mathbf{EI}_{ZZ}-\frac{\alpha_Z y_C}{GA}\mathbf{EI}_{YZ}-z_C\mathbf{T}_{3xx}\\ -\frac{\alpha_Y}{GA}\mathbf{EI}_{YZ}+\mathbf{T}_{21xx} & -\frac{\alpha_Z}{GA}\mathbf{EI}_{YY}+\mathbf{T}_{2xx} & \frac{\alpha_Y z_C}{GA}\mathbf{EI}_{YZ}-\frac{\alpha_Z y_C}{GA}\mathbf{EI}_{YY}+y_C\mathbf{T}_{3xx}\\ y_C\mathbf{T}_{21xx}-z_C\mathbf{T}_{1xx} & y_C\mathbf{T}_{2xx}-z_C\mathbf{T}_{12xx} & \frac{I_S}{A}\mathbf{T}_{3xx}\end{bmatrix} \tag{54b}$$

$$\mathbf{N}=\begin{bmatrix}N_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & N_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & N_L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & N_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & N_2 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_L & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_L\end{bmatrix} \tag{54c}$$

$$\mathbf{C}=\begin{bmatrix}\mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{C}_{13}\\ \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{C}_{23}\\ \mathbf{C}_{31} & \mathbf{C}_{32} & \mathbf{C}_{33}\end{bmatrix} \tag{54d}$$

the $\mathbf{C}_{ij}$ $L\times L$ matrices are evaluated from the expressions

$$\mathbf{C}_{11}=\Big[\mathbf{p}_X\mathbf{T}_{1x}-a_Y\frac{EI_{ZZ}}{GA}\big(\mathbf{p}_{X,xx}\mathbf{T}_{1x}+3\mathbf{p}_{X,x}\mathbf{T}_{1xx}+3\mathbf{p}_X\mathbf{T}_{1xxx}\big)$$
$$-a_Z\frac{EI_{YZ}}{GA}\big(\mathbf{p}_{X,xx}\mathbf{T}_{21x}+3\mathbf{p}_{X,x}\mathbf{T}_{21xx}+3\mathbf{p}_X\mathbf{T}_{21xxx}\big)\Big] \tag{55a}$$

$$\mathbf{C}_{12} = \Big[\mathbf{p}_X \mathbf{T}_{12x} - a_Y \frac{EI_{ZZ}}{GA} \left(\mathbf{p}_{X,xx} \mathbf{T}_{12x} + 3\mathbf{p}_{X,x} \mathbf{T}_{12xx} + 3\mathbf{p}_X \mathbf{T}_{12xxx} \right)$$
$$- a_Z \frac{EI_{YZ}}{GA} \left(\mathbf{p}_{X,xx} \mathbf{T}_{2x} + 3\mathbf{p}_{X,x} \mathbf{T}_{2xx} + 3\mathbf{p}_X \mathbf{T}_{2xxx} \right) \Big] \tag{55b}$$

$$\mathbf{C}_{13} = \Big[-z_C \mathbf{p}_X \mathbf{T}_{3x} + a_Y \frac{EI_{ZZ}}{GA} \left(z_C \mathbf{p}_{X,xx} \mathbf{T}_{3x} + 3z_C \mathbf{p}_{X,x} \mathbf{T}_{3xx} + 3z_C \mathbf{p}_X \mathbf{T}_{3xxx} \right)$$
$$- a_Z \frac{EI_{YZ}}{GA} \left(y_C \mathbf{p}_{X,xx} \mathbf{T}_{3x} + 3y_C \mathbf{p}_{X,x} \mathbf{T}_{3xx} + 3y_C \mathbf{p}_X \mathbf{T}_{3xxx} \right) \Big] \tag{55c}$$

$$\mathbf{C}_{21} = \Big[\mathbf{p}_X \mathbf{T}_{21x} - a_Z \frac{EI_{YY}}{GA} \left(\mathbf{p}_{X,xx} \mathbf{T}_{21x} + 3\mathbf{p}_{X,x} \mathbf{T}_{21xx} + 3\mathbf{p}_X \mathbf{T}_{21xxx} \right)$$
$$- a_Y \frac{EI_{YZ}}{GA} \left(\mathbf{p}_{X,xx} \mathbf{T}_{1x} + 3\mathbf{p}_{X,x} \mathbf{T}_{1xx} + 3\mathbf{p}_X \mathbf{T}_{1xxx} \right) \Big] \tag{55d}$$

$$\mathbf{C}_{22} = \Big[\mathbf{p}_X \mathbf{T}_{2x} - a_Z \frac{EI_{YY}}{GA} \left(\mathbf{p}_{X,xx} \mathbf{T}_{2x} + 3\mathbf{p}_{X,x} \mathbf{T}_{2xx} + 3\mathbf{p}_X \mathbf{T}_{2xxx} \right)$$
$$- a_Y \frac{EI_{YZ}}{GA} \left(\mathbf{p}_{X,xx} \mathbf{T}_{12x} + 3\mathbf{p}_{X,x} \mathbf{T}_{12xx} + 3\mathbf{p}_X \mathbf{T}_{12xxx} \right) \Big] \tag{55e}$$

$$\mathbf{C}_{23} = \Big[y_C \mathbf{p}_X \mathbf{T}_{3x} - a_Z \frac{EI_{YY}}{GA} \left(y_C \mathbf{p}_{X,xx} \mathbf{T}_{3x} + 3y_C \mathbf{p}_{X,x} \mathbf{T}_{3xx} + 3y_C \mathbf{p}_X \mathbf{T}_{3xxx} \right)$$
$$+ a_Y \frac{EI_{YZ}}{GA} \left(z_C \mathbf{p}_{X,xx} \mathbf{T}_{3x} + 3z_C \mathbf{p}_{X,x} \mathbf{T}_{3xx} + 3z_C \mathbf{p}_X \mathbf{T}_{3xxx} \right) \Big] \tag{55f}$$

$$\mathbf{C}_{31} = \left[y_C \mathbf{p}_X \mathbf{T}_{21x} - z_C \mathbf{p}_X \mathbf{T}_{1x} \right] \tag{55g}$$

$$\mathbf{C}_{32} = \left[y_C \mathbf{p}_X \mathbf{T}_{2x} - z_C \mathbf{p}_X \mathbf{T}_{12x} \right] \tag{55h}$$

$$\mathbf{C}_{33} = \left[\frac{I_S}{A} \mathbf{p}_X \mathbf{T}_{3x} \right] \tag{55i}$$

and the $3L \times 1$ column matrix $\mathbf{f}$ is given as

$$\mathbf{f} = \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ GI_t \mathbf{t}_{3xx} \end{Bmatrix} + N \begin{Bmatrix} \mathbf{t}_{1xx} - z_C \mathbf{t}_{3xx} \\ \mathbf{t}_{2xx} + y_C \mathbf{t}_{3xx} \\ y_C \mathbf{t}_{2xx} - z_C \mathbf{t}_{1xx} + \dfrac{I_S}{A} \mathbf{t}_{3xx} \end{Bmatrix} \tag{56}$$

with

$$\mathbf{f}_1 = \mathbf{p}_Y - \mathbf{p}_X \left(\mathbf{t}_{1x} - z_C \mathbf{t}_{3x} \right) - a_Y \frac{EI_{ZZ}}{GA} \Big(\mathbf{p}_{Y,xx} - \mathbf{p}_{X,xx} \left(\mathbf{t}_{1x} - z_C \mathbf{t}_{3x} \right)$$
$$- 3\mathbf{p}_{X,x} \left(\mathbf{t}_{1xx} - z_C \mathbf{t}_{3xx} \right) - 3\mathbf{p}_X \left(\mathbf{t}_{1xxx} - z_C \mathbf{t}_{3xxx} \right) \Big) - a_Z \frac{EI_{YZ}}{GA} \Big(\mathbf{p}_{Z,xx}$$
$$- \mathbf{p}_{X,xx} \left(\mathbf{t}_{2x} + y_C \mathbf{t}_{3x} \right) - 3\mathbf{p}_{X,x} \left(\mathbf{t}_{2xx} + y_C \mathbf{t}_{3xx} \right) - 3\mathbf{p}_X \left(\mathbf{t}_{2xxx} + y_C \mathbf{t}_{3xxx} \right) \Big) \tag{57a}$$

$$\mathbf{f}_2 = \mathbf{p}_Z - \mathbf{p}_X\left(\mathbf{t}_{2x} + y_C\mathbf{t}_{3x}\right) - a_Z\frac{EI_{YY}}{GA}\Big(\mathbf{p}_{Z,xx} - \mathbf{p}_{X,xx}\left(\mathbf{t}_{2x} + y_C\mathbf{t}_{3x}\right)$$
$$-3\mathbf{p}_{X,x}\left(\mathbf{t}_{2xx} + y_C\mathbf{t}_{3xx}\right) - 3\mathbf{p}_X\left(\mathbf{t}_{2xxx} + y_C\mathbf{t}_{3xxx}\right)\Big) - a_Y\frac{EI_{YZ}}{GA}\Big(\mathbf{p}_{Y,xx}$$
$$-\mathbf{p}_{X,xx}\left(\mathbf{t}_{1x} - z_C\mathbf{t}_{3x}\right) - 3\mathbf{p}_{X,x}\left(\mathbf{t}_{1xx} - z_C\mathbf{t}_{3xx}\right) - 3\mathbf{p}_X\left(\mathbf{t}_{1xxx} - z_C\mathbf{t}_{3xxx}\right)\Big) \tag{57b}$$

$$\mathbf{f}_3 = \mathbf{m}_x + \mathbf{p}_Z y_C - \mathbf{p}_Y z_C + z_C\mathbf{p}_X\mathbf{t}_{1x} - y_C\mathbf{p}_X\mathbf{t}_{2x} - \frac{I_S}{A}\mathbf{p}_X\mathbf{t}_{3x} \tag{57c}$$

In the above set of equations the matrices $\mathbf{EI}_{YY}, \mathbf{EI}_{ZZ}, \mathbf{EI}_{YZ}, \mathbf{EC}_S, \mathbf{GI}_t$ are $L \times L$ diagonal matrices including the values of the corresponding quantities, respectively, at the L nodal points. Moreover, $\mathbf{p}_X, \mathbf{p}_{X,x}, \mathbf{p}_{X,xx}$ are diagonal matrices and $\mathbf{p}_Y, \mathbf{p}_{Y,xx}, \mathbf{p}_Z, \mathbf{p}_{Z,xx}$ and $\mathbf{m}_x$ are vectors containing the values of the external loading and their derivatives at these points.

Solving the linear system of Eq. (53) for the fictitious load distributions $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ the displacements and their derivatives in the interior of the beam are computed using Eq. (52).

Buckling equation

In this case it is $\alpha_3 = \bar{\alpha}_3 = \beta_3 = \bar{\beta}_3 = \gamma_3 = \bar{\gamma}_3 = 0$ (homogeneous boundary conditions) and $\mathbf{p}_X = \mathbf{p}_{X,x} = \mathbf{p}_{X,xx} = \mathbf{p}_Y = \mathbf{p}_{Y,xx} = \mathbf{p}_Z = \mathbf{p}_{Z,xx} = \mathbf{m}_x = \mathbf{0}$, $N = -P$. Thus, Eq. (53) becomes

$$(\mathbf{A} + P\mathbf{B})\begin{Bmatrix}\mathbf{q}_1\\ \mathbf{q}_2\\ \mathbf{q}_3\end{Bmatrix} = \mathbf{0} \tag{58}$$

The condition that Eq. (58) has a non-trivial solution yields the buckling equation

$$det(\mathbf{A} + P\mathbf{B}) = 0 \tag{59}$$

3.2 For the primary warping function φ_S^P

The integral representations and the numerical solution for the evaluation of the buckling load assume that the warping C_S and torsion I_t constants given from Eqs. (21), (22) are already established. Eqs. (21), (22) indicate that the evaluation of the aforementioned constants presumes that the primary warping function φ_S^P at any interior point of the domain Ω of the cross section of the beam is known. Once φ_S^P is established, C_S and I_t constants are evaluated by converting the domain integrals into line integrals along the boundary employing the following relations

$$C_S = -\int_\Gamma B\frac{\partial\varphi_S^P}{\partial n}ds \qquad \text{on} \quad \Gamma = \bigcup_{j=1}^{K+1}\Gamma_j \tag{60a}$$

$$I_t = \int_\Gamma [(yz^2 - z\varphi_S^P)n_y + (y^2z + y\varphi_S^P)n_z]ds \qquad \text{on} \quad \Gamma = \bigcup_{j=1}^{K+1}\Gamma_j \tag{60b}$$

and using constant boundary elements for the approximation of these line integrals. In Eqs. (60a,b) n_y, n_z are the direction cosines, while $B(y,z)$ is a fictitious function defined as the solution of the following Neumann problem

$$\nabla^2 B = \varphi_S^P \quad \text{in} \quad \Omega \tag{61}$$

$$\frac{\partial B}{\partial n} = 0 \qquad \text{on} \quad \Gamma = \bigcup_{j=1}^{K+1} \Gamma_j \tag{62}$$

The evaluation of the primary warping function φ_S^P and the fictitious function $B(y,z)$ is accomplished using BEM as this is presented in Sapountzakis (2001) and in Sapountzakis and Mokos (2003), respectively.

3.3 For the stress functions $\Theta(Y, Z)$ and $\Phi(Y, Z)$

The evaluation of the stress functions $\Theta(Y, Z)$ and $\Phi(Y, Z)$ is accomplished using BEM as this is presented in Sapountzakis and Mokos (2005).

Moreover, since the flexural-torsional buckling problem of beams is solved by the BEM, the domain integrals for the evaluation of the area, the bending moments of inertia and the shear deformation coefficients (Eqs. (36a,b)) have to be converted to boundary line integrals, in order to maintain the pure boundary character of the method. This can be achieved using integration by parts, the Gauss theorem and the Green identity. Thus, the moments, the product of inertia and the cross section area can be written as

$$I_Y = \int_\Gamma (YZ^2 n_Y) ds \tag{63a}$$

$$I_Z = \int_\Gamma (ZY^2 n_Z) ds \tag{63b}$$

$$I_{YZ} = \frac{1}{2}\int_\Gamma (ZY^2 n_Y) ds \tag{63c}$$

$$A = \frac{1}{2}\int_\Gamma (Yn_Y + Zn_Z) ds \tag{63d}$$

while the shear deformation coefficients a_Y and a_Z are obtained from the relations

$$a_Y = \frac{A}{\Delta^2}\left((4v+2)(I_Y I_{\Theta Y} - I_{YZ} I_{\Theta Z}) + \frac{1}{4}v^2(I_Y^2 + I_{YZ}^2)I_{ed} - I_{\Theta e}\right) \tag{64a}$$

$$a_Z = \frac{A}{\Delta^2}\left((4v+2)(I_Z I_{\Phi Z} - I_{YZ} I_{\Phi Y}) + \frac{1}{4}v^2(I_Z^2 + I_{YZ}^2)I_{ed} - I_{\Phi d}\right) \tag{64b}$$

where

$$I_{\Theta e} = \int_\Gamma \Theta(\mathbf{n}\cdot\mathbf{e}) ds \tag{65a}$$

$$I_{\Phi d} = \int_\Gamma \Phi(\mathbf{n}\cdot\mathbf{d}) ds \tag{65b}$$

$$I_{\Phi Z} = \frac{1}{6}\int_\Gamma [(I_{YZ}Z^3Y^2 - 2I_Z Z^4 Y)n_Y + (3\Phi n_Z - Z(\mathbf{n}\cdot\mathbf{d}))Z^2] ds \tag{65c}$$

$$I_{\Theta Y} = \frac{1}{6}\int_\Gamma [(I_{YZ}Y^3Z^2 - 2I_Y Y^4 Z)n_Z + (3\Theta n_Y - Y(\mathbf{n}\cdot\mathbf{e}))Y^2] ds \tag{65d}$$

$$I_{\Theta Z} = \frac{1}{6}\int_{\Gamma}[(2I_{YZ}Z^4Y - I_YZ^3Y^2)n_Y + (3\Theta n_Z - Z(\mathbf{n}\cdot\mathbf{e}))Z^2]ds \tag{65e}$$

$$I_{\Phi Y} = \frac{1}{6}\int_{\Gamma}[(2I_{YZ}Y^4Z - I_ZY^3Z^2)n_Z + (3\Phi n_Y - Y(\mathbf{n}\cdot\mathbf{d}))Y^2]ds \tag{65f}$$

$$I_{\Theta Z} = \frac{1}{6}\int_{\Gamma}[(I_{YZ}Z^3Y^2 - 2I_ZZ^4Y)\cos\beta + (3\Phi\sin\beta - Z(\mathbf{n}\cdot\mathbf{d}))Z^2]ds \tag{65g}$$

4. Numerical examples

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and representative examples have been studied to demonstrate the efficiency, wherever possible the accuracy and the range of applications of the developed method. In all the examples treated, each cross section has been analysed employing 300 constant boundary elements along the boundary of the cross section, which are enough to ensure convergence at the calculation of the sectional constants, while the beam interval is divided into L = 50 constant equal elements.

Example 1

For comparison reasons, the simply supported beam of Fig. 4 ($A = 0.051\ \text{m}^2$, $E = 3.0 \times 10^7\ \text{kN/m}^2$, $\nu = 0.2$, $C_s = 4.696 \times 10^{-6}\ \text{m}^6$, $I_t = 2.693 \times 10^{-4}\ \text{m}^4$, $I_S = 1.582 \times 10^{-3}\ \text{m}^4$) has been studied for the cases of length, $l = 3.0$ m and $l = 4.0$ m. Since the proposed method requires the coordinate system CYZ through the cross section's centroid C to be the principal shear one, in the first column of Table 1 the geometric, the inertia constants and the shear deformation coefficients of the examined cross section are given with respect to an original coordinate system $C\tilde{Y}\tilde{Z}$, followed by the evaluation of the angle of rotation θ^S (Sapountzakis and Mokos 2005) giving the final coordinate system CYZ and the new geometric, inertia constants and shear deformation coefficients presented

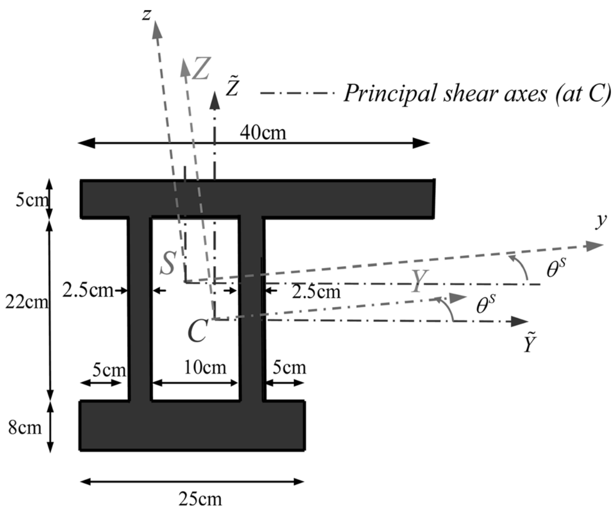


Fig. 4 Cross section of the thin-walled beam of example 1

Table 1 Geometric, inertia constants and shear deformation coefficients of the cross section of example 1

Coordinate system $C\tilde{Y}\tilde{Z}$	Coordinate system CYZ
$I_{\tilde{Y}\tilde{Y}} = 8.719 \times 10^{-4}$ m^4	$I_{YY} = 9.055 \times 10^{-4}$ m^4
$I_{\tilde{Z}\tilde{Z}} = 4.828 \times 10^{-4}$ m^4	$I_{ZZ} = 4.492 \times 10^{-4}$ m^4
$I_{\tilde{Y}\tilde{Z}} = 2.113 \times 10^{-4}$ m^4	$I_{YZ} = 1.746 \times 10^{-4}$ m^4
$\alpha_{\tilde{y}} = 1.88$	$\alpha_y = 1.87$
$\alpha_{\tilde{z}} = 3.14$	$\alpha_z = 3.15$
$\alpha_{\tilde{y}\tilde{z}} = 0.11$	$\alpha_{yz} = 0.0$
$\tilde{y}_C = 4.71 \times 10^{-2}$ m	$y_C = 5.10 \times 10^{-2}$ m
$\tilde{z}_C = -4.73 \times 10^{-2}$ m	$z_C = -4.30 \times 10^{-2}$ m
$\theta^S = -0.087$ rad	-

in the second column of the aforementioned table. Moreover, in the case of a simply supported beam an analytical solution can be obtained (Timoshenko and Gere 1961) by setting

$$v = A_1 \sin\frac{\pi x}{l}, \quad w = A_2 \sin\frac{\pi x}{l}, \quad \theta_x = A_3 \sin\frac{\pi x}{l} \tag{66a,b,c}$$

in Eqs. (16), (20) leading to the formulation of the following homogeneous system of equations with respect to A_1, A_2 and A_3

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{67}$$

where

$$K_{11} = EI_{ZZ}\frac{\pi^2}{l^2} - P\left(a_Y \frac{EI_{ZZ}}{GA}\frac{\pi^2}{l^2} + 1\right) \qquad K_{12} = EI_{YZ}\frac{\pi^2}{l^2} - Pa_Z \frac{EI_{YZ}}{GA}\frac{\pi^2}{l^2} \tag{68a,b}$$

$$K_{13} = -P\left(-z_C + y_C a_Z \frac{EI_{YZ}}{GA}\frac{\pi^2}{l^2} - z_C a_Y \frac{EI_{ZZ}}{GA}\frac{\pi^2}{l^2}\right) \tag{68c}$$

$$K_{21} = EI_{YZ}\frac{\pi^2}{l^2} - Pa_Y \frac{EI_{YZ}}{GA}\frac{\pi^2}{l^2} \qquad K_{22} = EI_{YY}\frac{\pi^2}{l^2} - P\left(a_Z \frac{EI_{YY}}{GA}\frac{\pi^2}{l^2} + 1\right) \tag{68d,e}$$

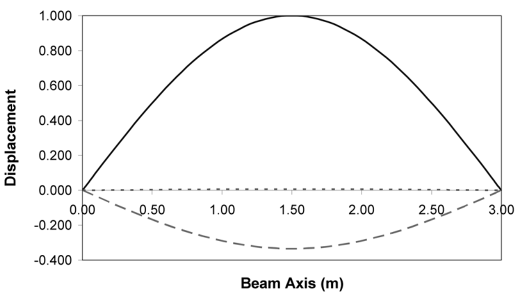
$$K_{23} = -P\left(y_C - z_C a_Y \frac{EI_{YZ}}{GA}\frac{\pi^2}{l^2} + y_C a_Z \frac{EI_{YY}}{GA}\frac{\pi^2}{l^2}\right) \tag{68f}$$

$$K_{31} = Pz_C \qquad K_{32} = -Py_C \qquad K_{33} = EC_S\frac{\pi^2}{l^2} + GI_t - P\frac{I_S}{A} \tag{68g,h,i}$$

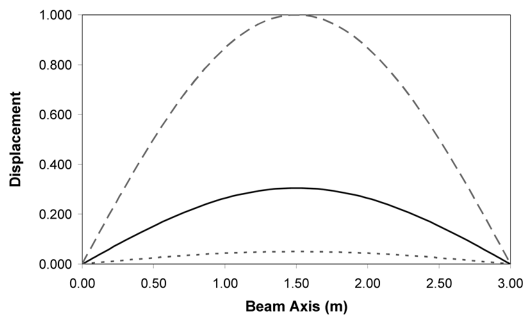
For a non-trivial solution, the determinant of the above system must be equal to zero. Thus, a cubic equation is obtained, leading to three positive roots P_y, P_z, P_θ, from which the smallest one is of importance in engineering design. In Table 2 the computed buckling loads P_y, P_z, P_θ of the aforementioned beams are presented as compared with those obtained from the analytical solution, in which the shear deformation effect is excluded and included in turn. The accuracy of the obtained results using the proposed method is remarkable. Moreover, in Fig. 5 the corresponding buckling modes are also presented, demonstrating that the main component is the displacement along y axis (v) in mode 1, the displacement along z axis (w) in mode 2 and the angle of twist (θ_x) in mode 3. In Table 3 the computed buckling load (the smallest root of the buckling equation) of

Table 2 Buckling loads (kN) of the simply supported beams of example 1

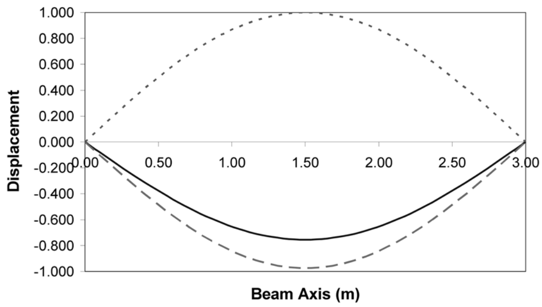
	Without Shear Deformation		With Shear Deformation	
	analytical	computed	analytical	computed
		$l = 3.0$ m		
P_y	12800	12802	12308	12309
P_z	30373	30378	26591	26589
P_θ	138808	138810	137865	137865
		$l = 4.0$ m		
P_y	7209	7209	7049	7050
P_z	17451	17453	16124	16124
P_θ	133138	133138	132880	132880



Buckling Mode for $P_y = 12309$



Buckling Mode for $P_z = 26589$



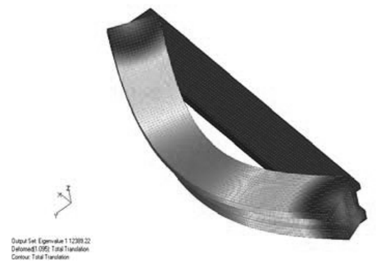
Buckling Mode for $P_\theta = 137865$

Fig. 5 Displacement components (displ. y ——, displ. z – – – rot. x ·······) of the modes corresponding to buckling loads P_y P_z, P_θ of the simply supported beam of length $l = 3.0$ m of example 1

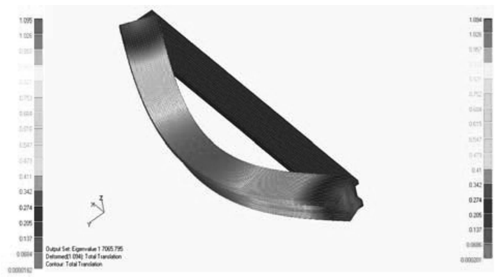
Table 3 Buckling loads (kN) of the beams of example 1 for various boundary conditions

Fixed-Hinged			Fixed-Fixed		
Without Shear Deformation	With Shear Deformation		Without Shear Deformation	With Shear Deformation	
Present study	Present study	FEM*	Present study	Present study	FEM*
Length $l = 3.0$ m					
26115	23948	24474	50694	43820	45613
Length $l = 4.0$ m					
14729	14015	14185	28713	26351	27096

*MSC/NASTRAN (1999)



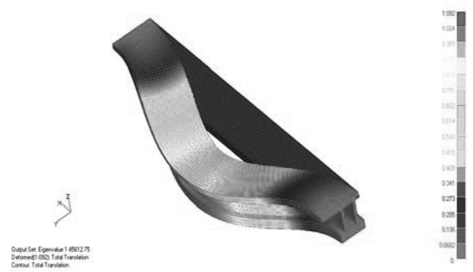
Hinged – Hinged
($l = 3$ m, $P = 12389$ kN)



Hinged – Hinged
($l = 4$ m, $P = 7066$ kN)

Fixed – Hinged
($l = 3$ m, $P = 24474$ kN)

Fixed – Hinged
($l = 4$ m, $P = 14185$ kN)



Fixed – Fixed
($l = 3$ m, $P = 45613$ kN)

Fixed – Fixed
($l = 4$ m, $P = 27096$ kN)

Fig. 6 3-D views of the buckling mode shapes of the FEM solution (MSC/NASTRAN 1999) of the beams of example 1 (numbers in parentheses correspond to the FEM solution)

Table 4 Buckling loads (kN) of the beam of example 1 with length $l = 4.0$ m for various beam elements and boundary conditions

Elements	Hinged-Hinged		Fixed-Hinged		Fixed-Fixed	
	Without Shear Deform.	With Shear Deform.	Without Shear Deform.	With Shear Deform.	Without Shear Deform.	With Shear Deform.
10	7238	7077	14846	14120	29152	26719
20	7216	7056	14755	14038	28811	26433
30	7212	7052	14738	14023	28747	26379
40	7210	7051	14732	14017	28724	26359
50	7209	7050	14729	14015	28713	26351
60	7209	7050	14729	14013	28707	26346

the beams with fixed-hinged and fixed-fixed boundary conditions taking into account or ignoring shear deformation effect are presented as compared with those obtained from a FEM solution (MSC/NASTRAN 1999) employing 9600 and 12800 solid brick elements for the beam length $l = 3.0$ m and $l = 4.0$ m, respectively (the buckling mode shapes of the latter are presented in Fig. 6). From the obtained results the influence of the shear deformation effect and the accuracy of the proposed method are remarkable. Finally, in Table 4 the variation of the aforementioned buckling loads with the number of beam elements is presented demonstrating the convergence of the proposed method using a relatively small number of beam elements. It is worth here noting the major merit of the aforementioned accuracy of the proposed method compared with the 3-D FEM solution using solid elements, arising from the disadvantages of the latter due to the difficulties of

- support modelling
- discretization of a complex structure despite the existing element generators
- discretization of a structure including thin walled members (shear-locking, membrane-locking (Knothe and Wessels 1992))
- increased number of degrees of freedom leading to severe or unrealistic computational time especially for structures consisting of many elements
- reduced oversight of the 3-D FEM solution compared with that of the beam-like srtuctures employing stress resultants

while the use of shell elements cannot give accurate results since the warping of the walls of a cross section cannot be taken into account (midline model).

Example 2

The elastic stability of the beam ($E = 3.0 \times 10^7$ kN/m^2, $G = 1.25 \times 10^7$ kN/m^2, $b_1 = h_1 = 0.4$ m) of Fig. 7, with length $l = 3.0$ m has been studied. Three different types of cross-section starting from a thin-walled one and ending with a thick-walled one are considered, that is (i) $b_2 = h_2 = 0.02$ m ($I_{YY} = 2.476 \times 10^{-4}$ m^4, $I_{ZZ} = 1.069 \times 10^{-4}$ m^4, $A = 1.56 \times 10^{-2}$ m^2, $I_S = 4.992 \times 10^{-4}$ m^4, $I_t = 2.127 \times 10^{-6}$ m^4, $C_s = 1.656 \times 10^{-8}$ m^6, $z_C = 9.63 \times 10^{-2}$ m, $a_Y = 2.30$, $a_Z = 2.49$), (ii) $b_2 = h_2 = 0.08$ m ($I_{YY} = 8.044 \times 10^{-4}$ m^4, $I_{ZZ} = 4.403 \times 10^{-4}$ m^4, $I_S = 1.588 \times 10^{-3}$ m^4, $I_t = 1.221 \times 10^{-4}$ m^4, $C_s = 7.703 \times 10^{-7}$ m^6, $A = 5.76 \times 10^{-2}$ m^2, $z_C = 7.72 \times 10^{-2}$ m, $a_Y = 1.93$, $a_Z = 2.23$) and (iii) $b_2 = h_2 = 0.20$ m ($I_{YY} = 1.467 \times 10^{-3}$ m^4, $I_{ZZ} = 1.200 \times 10^{-3}$ m^4, $I_S = 2.726 \times 10^{-3}$ m^4, $I_t = 1.484 \times 10^{-3}$ m^4, $C_s = 2.675 \times 10^{-6}$ m^6, $A = 1.2 \times 10^{-1}$ m^2, $z_C = 2.22 \times 10^{-2}$ m, $a_Y = 1.36$, $a_Z = 1.48$). In Tables 5 through 7 the computed values of the buckling load P for the cases of hinged-hinged, fixed-hinged and fixed-

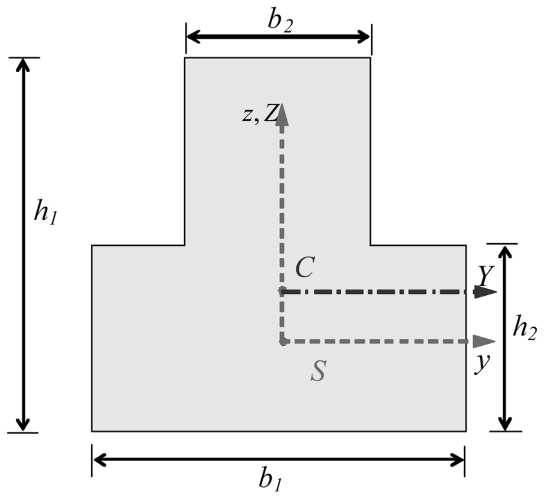


Fig. 7 Cross section of the beam of Example 2

Table 5 Buckling load P(kN) for the hinged-hinged beam of Example 2

	without shear deformation		with shear deformation	
	TWT	Present study	Present study	FEM*
$b_2 = h_2 = 0.02$ m	771	783	780	815
$b_2 = h_2 = 0.08$ m	13365	13558	13089	13288
$b_2 = h_2 = 0.20$ m	39183	39442	38082	38301

*MSC/NASTRAN (1999)

Table 6 Buckling load P(kN) for the fixed-hinged beam of Example 2

	without shear deformation		with shear deformation	
	TWT	Present study	Present study	FEM*
$b_2 = h_2 = 0.02$ m	821	834	831	841
$b_2 = h_2 = 0.08$ m	24625	25306	23646	24550
$b_2 = h_2 = 0.20$ m	79505	80601	74580	76766

*MSC/NASTRAN (1999)

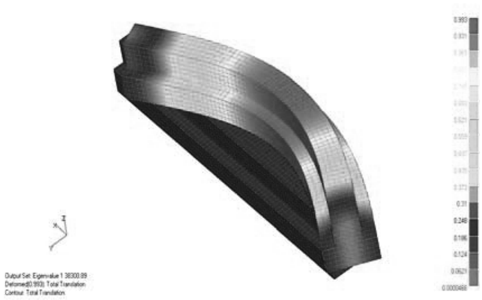
Table 7 Buckling load P(kN) for the fixed-fixed beam of Example 2

	without shear deformation		with shear deformation	
	TWT	Present study	Present study	FEM*
$b_2 = h_2 = 0.02$ m	869	882	879	849
$b_2 = h_2 = 0.08$ m	38515	39929	36892	38117
$b_2 = h_2 = 0.20$ m	152949	157223	137653	147284

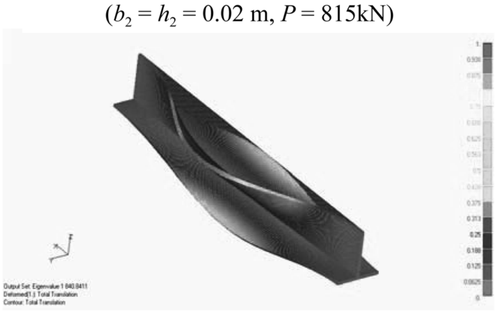
*MSC/NASTRAN (1999)

fixed boundary conditions are presented taking into account or ignoring shear deformation effect as compared with those obtained from the thin-walled theory (TWT) (Vlasov 1961, Timoshenko and Goodier 1984), from which the following values of the previous constants for the three different types of cross-section are obtained (i) $I_t = 2.107 \times 10^{-6}$ m^4, $C_s = 1.674 \times 10^{-8}$ m^6, $z_C = 9.74 \times 10^{-2}$ m, $I_S = 5.026 \times 10^{-4}$ m^4 (ii) $I_t = 1.297 \times 10^{-4}$ m^4, $C_s = 8.911 \times 10^{-7}$ m^6, $z_C = 8.89 \times 10^{-2}$ m, $I_S = 1.700 \times 10^{-3}$ m^4 and (iii) $I_t = 1.867 \times 10^{-3}$ m^4, $C_s = 9.556 \times 10^{-6}$ m^6, $z_C = 6.67 \times 10^{-2}$ m, $I_S = 3.200 \times 10^{-3}$ m^4. As expected, the utilization of the thin- walled theory proves to be prohibitive in thick walled sections, while the resulting inaccuracy even in thin walled sections is remarkable. To demonstrate the accuracy of the proposed method, the obtained results are also compared with those obtained from a FEM solution (MSC/NASTRAN 1999) employing 46800, 21600 and 5600 solid brick

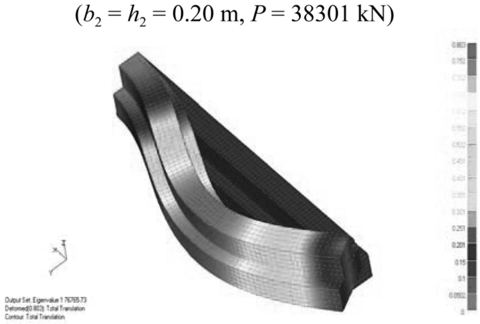
Hinged – Hinged
($b_2 = h_2 = 0.02$ m, $P = 815$kN)



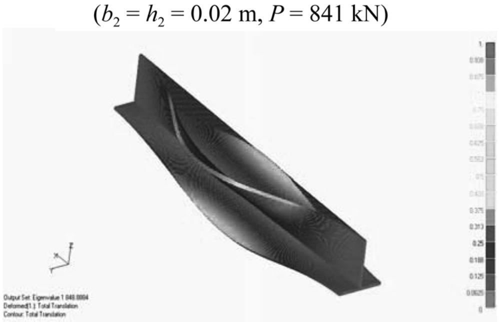
Hinged – Hinged
($b_2 = h_2 = 0.20$ m, $P = 38301$ kN)



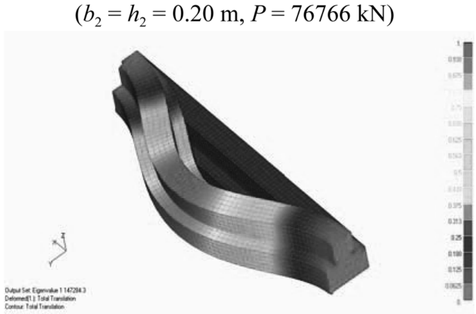
Fixed – Hinged
($b_2 = h_2 = 0.02$ m, $P = 841$ kN)



Fixed – Hinged
($b_2 = h_2 = 0.20$ m, $P = 76766$ kN)



Fixed – Fixed
($b_2 = h_2 = 0.02$ m, $P = 849$ kN)



Fixed – Fixed
($b_2 = h_2 = 0.20$ m, $P = 147284$ kN)

Fig. 8 3-D views of the buckling mode shapes of the FEM solution (MSC/NASTRAN 1999) of the beams of example 2 (numbers in parentheses correspond to the FEM solution)

elements for the three cases, respectively. The corresponding buckling mode shapes of the latter method are presented in Fig. 8. Moreover, the significant influence of both the boundary conditions and the shear deformation effect (especially in thick-walled cross sections) on the buckling load is verified.

Example 3

The elastic stability of the steel L-beam ($E = 2.1 \times 10^8$ kN/m^2, $A = 2.5 \times 10^{-3}$ m^2, $\nu = 0.3$, $C_s = 1.199 \times 10^{-10}$ m^6, $I_t = 8.275 \times 10^{-8}$ m^4, $I_S = 14.44 \times 10^{-6}$ m^4) of unequal legs and uniform

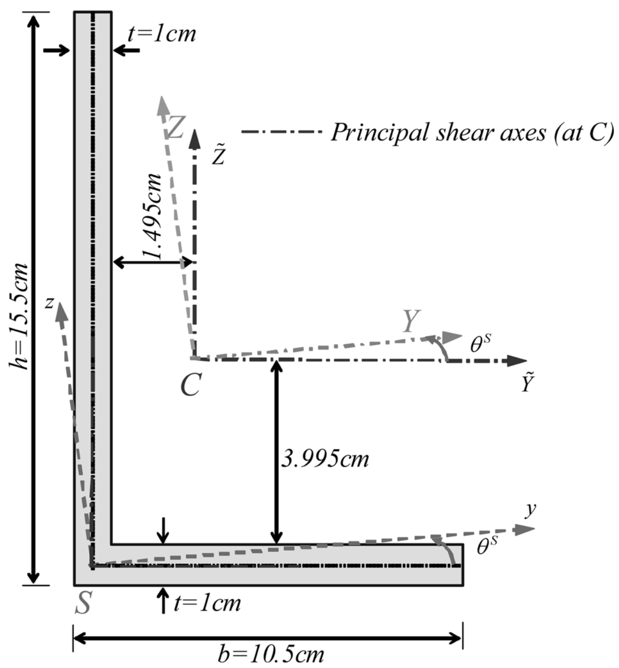


Fig. 9 L-shaped cross section of unequal legs of the beam of Example 3

Table 8 Geometric, inertia constants and shear deformation coefficients of the cross section of Fig. 9 (t = 1 cm)

Coordinate system $C\tilde{Y}\tilde{Z}$	Coordinate system CYZ
$I_{\tilde{Y}\tilde{Y}} = 6.207 \times 10^{-6}$ m^4	$I_{YY} = 6.352 \times 10^{-6}$ m^4
$I_{\tilde{Z}\tilde{Z}} = 2.351 \times 10^{-6}$ m^4	$I_{ZZ} = 2.206 \times 10^{-6}$ m^4
$I_{\tilde{Y}\tilde{Z}} = -2.242 \times 10^{-6}$ m^4	$I_{YZ} = -2.109 \times 10^{-6}$ m^4
$\alpha_{\tilde{y}} = 3.067$	$\alpha_y = 3.068$
$\alpha_{\tilde{z}} = 1.900$	$\alpha_z = 1.899$
$\alpha_{\tilde{y}\tilde{z}} = 0.039$	$\alpha_{yz} = 0.0$
$\tilde{y}_C = 2.00 \times 10^{-2}$ m	$y_C = 2.14 \times 10^{-2}$ m
$\tilde{z}_C = 4.43 \times 10^{-2}$ m	$z_C = 4.36 \times 10^{-2}$ m
$\theta^S = 0.033$ rad	-

Table 9 Buckling load P (kN) of the beam of Fig. 9 (t = 1 cm)

l	Hinged-Hinged			Fixed-Hinged			Fixed-Fixed		
	without shear deformation	with shear deformation		without shear deformation	with shear deformation		without shear deformation	with shear deformation	
	Present Study	Present Study	FEM*	Present Study	Present Study	FEM*	Present Study	Present Study	FEM*
1.00	1062	1053	1179	1178	1167	1262	1291	1281	1288
1.20	980	970	1056	1123	1112	1183	1225	1216	1220
1.40	884	875	932	1070	1060	1117	1178	1168	1171
1.60	779	771	806	1015	1004	1051	1138	1128	1131
1.80	675	669	688	953	943	981	1100	1091	1093
2.00	580	575	587	885	875	905	1062	1053	1055

*MSC/NASTRAN (1999)

Table 10 Geometric, inertia constants and shear deformation coefficients of the L-shaped cross section with uniform thickness t = 4 cm

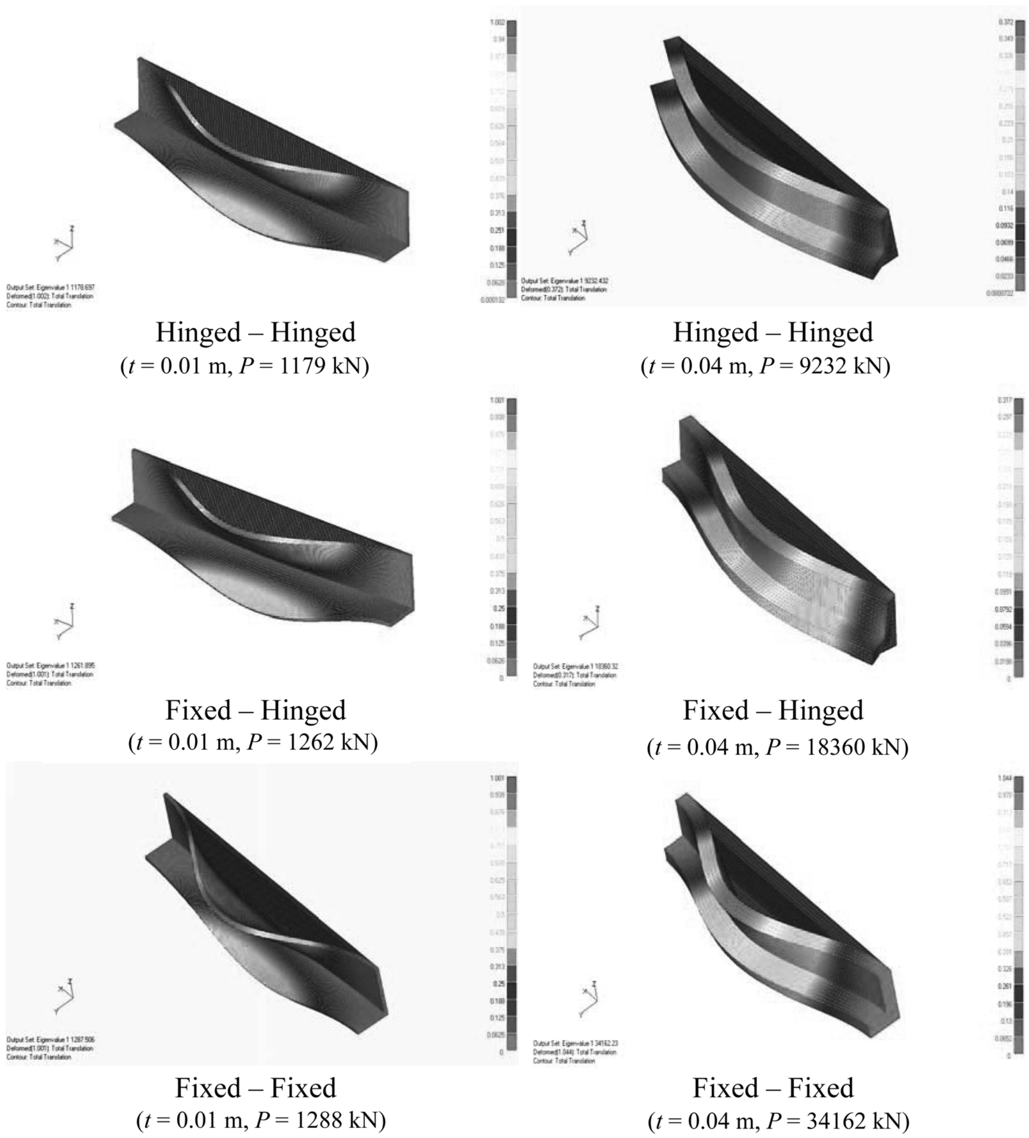
Coordinate system $C\tilde{Y}\tilde{Z}$	Coordinate system CYZ
$I_{\tilde{Y}\tilde{Y}} = 1.882 \times 10^{-5}$ m^4	$I_{YY} = 1.659 \times 10^{-5}$ m^4
$I_{\tilde{Z}\tilde{Z}} = 6.790 \times 10^{-6}$ m^4	$I_{ZZ} = 9.018 \times 10^{-6}$ m^4
$I_{\tilde{Y}\tilde{Z}} = -5.530 \times 10^{-6}$ m^4	$I_{YZ} = -7.240 \times 10^{-6}$ m^4
$\alpha_{\tilde{y}} = 2.082$	$\alpha_y = 2.096$
$\alpha_{\tilde{z}} = 1.618$	$\alpha_z = 1.604$
$\alpha_{\tilde{y}\tilde{z}} = -0.083$	$\alpha_{yz} = 0.0$
$\tilde{y}_C = 1.53 \times 10^{-2}$ m	$y_C = 1.00 \times 10^{-2}$ m
$\tilde{z}_C = 2.95 \times 10^{-2}$ m	$z_C = 3.17 \times 10^{-2}$ m
$\theta^S = -0.173$ rad	-

Table 11 Buckling load P (*kN*) of the L-shaped cross section beam with uniform thickness t = 4 cm

l	Hinged-Hinged			Fixed-Hinged			Fixed-Fixed		
	without shear deformation	with shear deformation		without shear deformation	with shear deformation		without shear deformation	with shear deformation	
	Present Study	Present Study	FEM*	Present Study	Present Study	FEM*	Present Study	Present Study	FEM*
1.00	9472	9229	9232	19052	17997	18360	35816	32600	34162

*MSC/NASTRAN (1999)

thickness $t = 1$ cm, as shown in Fig. 9, has been studied. Following the same procedure as in example 1, in Table 8 the geometric, the inertia constants and the shear deformation coefficients of the examined cross section are given with respect to the original $C\tilde{Y}\tilde{Z}$ and the final CYZ coordinate systems. Three types of boundary conditions, namely hinged-hinged, fixed-hinged and fixed-fixed are considered. In Table 9, the variation with the beam length of the buckling load P taking into account or ignoring shear deformation effect for the aforementioned cases of boundary conditions is shown. The influence of the support conditions on the buckling load is pronounced, while as expected due to the thin-walled character of the cross section the influence of the shear deformation



Hinged – Hinged ($t = 0.01$ m, $P = 1179$ kN)

Hinged – Hinged ($t = 0.04$ m, $P = 9232$ kN)

Fixed – Hinged ($t = 0.01$ m, $P = 1262$ kN)

Fixed – Hinged ($t = 0.04$ m, $P = 18360$ kN)

Fixed – Fixed ($t = 0.01$ m, $P = 1288$ kN)

Fixed – Fixed ($t = 0.04$ m, $P = 34162$ kN)

Fig. 10 3-D views of the buckling mode shapes of the FEM solution (MSC/NASTRAN 1999) of the beams of example 3 (numbers in parentheses correspond to the FEM solution)

effect could be ignored. Finally, in Table 11 the necessity of encountering this effect with the increase of the thickness of the cross section is demonstrated by presenting the buckling load P taking into account or ignoring shear deformation effect of the steel beam with length $l = 1.0$ m and L-shaped cross section with uniform thickness $t = 4$ cm ($A = 8.8 \times 10^{-3}\ \mathrm{m}^2$, $C_s = 4.170 \times 10^{-9}\ \mathrm{m}^6$, $I_t = 4.340 \times 10^{-6}\ \mathrm{m}^4$, $I_S = 3.536 \times 10^{-5}\ \mathrm{m}^4$) whose constants and coefficients with respect to the original $C\tilde{Y}\tilde{Z}$ and the final CYZ coordinate systems are given in Table 10, for the aforementioned three types of boundary conditions. The influence of the shear deformation effect on the buckling load in this case should not be ignored. The obtained results of all the aforementioned cases are compared with those obtained from a FEM solution (MSC/NASTRAN 1999), employing 20000 and 8800 solid brick elements for the two cases of the cross section, respectively, while the corresponding buckling mode shapes are presented in Fig. 10.

5. Conclusions

In this paper a boundary element method is developed for the general flexural-torsional buckling analysis of Timoshenko beams of arbitrarily shaped cross section subjected to a compressive centrally applied concentrated axial load, while its edges are restrained by the most general linear boundary conditions. The main conclusions that can be drawn from this investigation are

a. The numerical technique presented in this investigation is well suited for computer aided analysis for homogeneous beams of arbitrary cross section, subjected to any linear boundary conditions.

b. Accurate results are obtained using a relatively small number of beam elements.

c. The displacements components of the buckling modes can be computed at any cross-section of the beam using the respective integral representations as mathematical formulae.

d. The significant influence of the boundary conditions on the buckling load is remarkable.

e. In the proposed method all basic equations are formulated with respect to the principal shear coordinate system.

f. As expected, the utilization of the thin-walled theory proves to be prohibitive in thick-walled sections.

g. The discrepancy of the obtained results arising from the ignorance of shear deformation especially in thick-walled cross sections is remarkable and necessitates its inclusion in these cases.

h. The developed procedure retains the advantages of a BEM solution over a pure domain discretization method since it requires only boundary discretization.

Acknowledgements

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