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Technical Note

Free vibration analysis of a beam with arbitrarily distributed rigid beam segments using elastic-and-rigid-combined beam element

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1. Introduction

The literature concerning free vibration characteristics of beams carrying multiple masses is abundant. However, in most of the literature, the mass carried by the beam was regarded as a (*point*) *lumped* mass rather than a *distributed* mass. For the cases that the dynamic effects due to size of the mass cannot be neglected, the mass carried by the beam must be regarded as a *distributed* mass. Based on this concept, Chan *et al.* (1998), Kopmaz and Telli (2002) and Banerjee and Sobey (2003) have determined the natural frequencies of a two-part beam-mass system, where the distributed mass is connected by two elastic beam segments at its two ends. Because the techniques of the above-mentioned literature are developed based on analytical approaches, they are not easy to be extended to the problem of beams, with various boundary conditions, carrying multiple distributed masses with each distributed mass being connected by two elastic beam segments at its two ends. To solve the last problem, this paper presents an *elastic-and-rigid-combined beam element* such that the natural frequencies and mode shapes of a hybrid beam composed of any number of *elastic* beam segments and distributed masses can be easily determined by means of the standard finite element method.

2. Property matrices of an elastic-and-rigid-combined beam element

It is well known that the motions of a *rigid* body in a plane may be represented by the translational displacements of its c.g. G in the x and y directions (u_{Gx} and u_{Gy}) together with the rotational angle about the (z) axis passing through G (θ_{Gz}). Therefore, the assemblage of one *rigid* beam segment and the two adjacent *elastic* beam elements, as shown in Fig. 1, can be replaced by an equivalent three-node *elastic-and-rigid-combined* (or *elastic-rigid-elastic*) beam element shown in Fig. 2.

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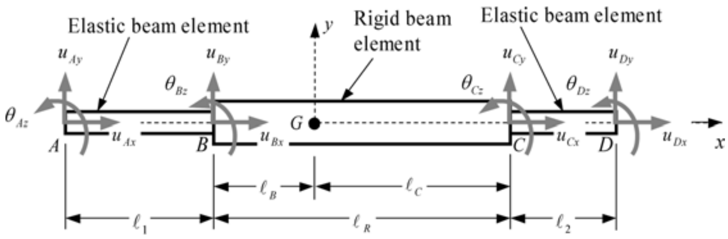

Fig. 1 A two-node rigid beam segment BC connected with two elastic beam segments, AB and CD, at its two ends B and C, respectively.

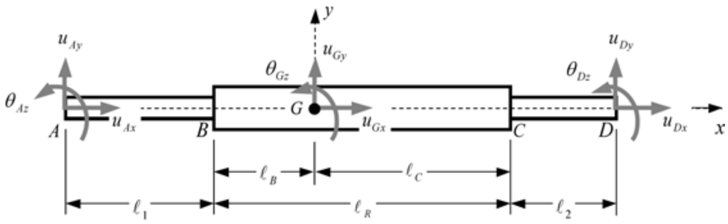

Fig. 2 A rigid beam segment BC connected with two elastic beam segments, AB and CD, at its two ends B and C, as shown in Fig. 1, can be replaced by a three-node elastic-and-rigid-combined beam element

If the nodal displacements in the x and y directions and the rotational angle about the axis parallel to z-axis for each node shown in Fig. 1 are denoted by u_{ix}, u_{iy} and θ_{iz} with $i = A$, B, C and D, then the nodal displacements (including the rotational angle) of the *rigid* beam segment at the left end B, u_{Bx}, u_{By} and θ_{Bz}, and those at c.g. G, u_{Gx}, u_{Gy} and θ_{Gz}, have the following relationships

$$\{\delta_B\} = \begin{Bmatrix} u_{Gx} \\ u_{Gy} - \ell_B \theta_{Gz} \\ \theta_{Gz} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\ell_B \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{Gx} \\ u_{Gy} \\ \theta_{Gz} \end{Bmatrix} = [R_B]\{\delta_G\} \tag{1}$$

$$\{\delta_B\} = [u_{Bx} \;\; u_{By} \;\; \theta_{Bz}]^T, \quad \{\delta_G\} = [u_{Gx} \;\; u_{Gy} \;\; \theta_{Gz}]^T, \quad [R_B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\ell_B \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

Similarly, the nodal displacements (including the rotational angle) of the *rigid* beam segment at the right end C and c.g. G have the following relationships

$$\{\delta_C\} = \begin{Bmatrix} u_{Gx} \\ u_{Gy} + \ell_C \theta_{Gz} \\ \theta_{Gz} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \ell_C \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{Gx} \\ u_{Gy} \\ \theta_{Gz} \end{Bmatrix} = [R_C]\{\delta_G\} \tag{3}$$

$$\{\delta_C\} = [u_{Cx} \;\; u_{Cy} \;\; \theta_{Cz}]^T, \quad [R_C] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \ell_C \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

For conveniences, the stiffness and mass matrices of the *elastic* beam element AB are respectively denoted by

$$[k_e^{AB}] = \begin{bmatrix} [k]_{AA} & [k]_{AB} \\ [k]_{BA} & [k]_{BB} \end{bmatrix}_{6\times 6}, \quad [m_e^{AB}] = \begin{bmatrix} [m]_{AA} & [m]_{AB} \\ [m]_{BA} & [m]_{BB} \end{bmatrix}_{6\times 6} \tag{5}$$

where $[k]_i$ and $[m]_i$ with $i = AA$, AB, BA and BB are 3×3 sub-matrices.

Based on Eq. (5), the nodal forces for the elastic beam element AB are given by

$$\{F_A\} = [k]_{AA}\{\delta_A\} + [k]_{AB}\{\delta_B\} + [m]_{AA}\{\ddot{\delta}_A\} + [m]_{AB}\{\ddot{\delta}_B\} \tag{6a}$$

$$\{F_B\} = [k]_{BA}\{\delta_A\} + [k]_{BB}\{\delta_B\} + [m]_{BA}\{\ddot{\delta}_A\} + [m]_{BB}\{\ddot{\delta}_B\} \tag{6b}$$

$$\{\delta_A\} = [u_{Ax}\ \ u_{Ay}\ \ \theta_{Az}]^T \tag{7}$$

Substituting Eq. (1) into Eqs. (6a) and (6b) and writing the resulting expressions in matrix form, one has

$$\begin{Bmatrix} F_A \\ F_B \end{Bmatrix} = \begin{bmatrix} [k]_{AA} & [k]_{AB}[R_B] \\ [k]_{BA} & [k]_{BB}[R_B] \end{bmatrix} \begin{Bmatrix} \delta_A \\ \delta_G \end{Bmatrix} + \begin{bmatrix} [m]_{AA} & [m]_{AB}[R_B] \\ [m]_{BA} & [m]_{BB}[R_B] \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_A \\ \ddot{\delta}_G \end{Bmatrix} \tag{8}$$

Similarly, the nodal forces for the elastic beam element CD can be obtained from

$$\begin{Bmatrix} F_C \\ F_D \end{Bmatrix} = \begin{bmatrix} [k]_{CC}[R_C] & [k]_{CD} \\ [k]_{DC}[R_C] & [k]_{DD} \end{bmatrix} \begin{Bmatrix} \delta_G \\ \delta_D \end{Bmatrix} + \begin{bmatrix} [m]_{CC}[R_C] & [m]_{CD} \\ [m]_{DC}[R_C] & [m]_{DD} \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_G \\ \ddot{\delta}_D \end{Bmatrix} \tag{9}$$

From Eqs. (8) and (9) one sees that the forces at nodes A, B, C and D can be replaced by those at nodes A, G and D, i.e.

$$\begin{Bmatrix} F_A \\ F_B \\ F_C \\ F_D \end{Bmatrix} \equiv \begin{Bmatrix} F_A \\ F_G \\ F_D \end{Bmatrix} = [k_{ere}^{AGD}] \begin{Bmatrix} \delta_A \\ \delta_G \\ \delta_D \end{Bmatrix} + [m_{ere}^{AGD}] \begin{Bmatrix} \ddot{\delta}_A \\ \ddot{\delta}_G \\ \ddot{\delta}_D \end{Bmatrix} \tag{10}$$

$$[k_{ere}^{AGD}] = \begin{bmatrix} [k]_{AA} & [k]_{AB}[R_B] & 0 \\ [k]_{BA} & [k]_{BB}[R_B] + [k]_{CC}[R_C] & [k]_{CD} \\ 0 & [k]_{DC}[R_C] & [k]_{DD} \end{bmatrix}_{9\times 9} \tag{11a}$$

$$[m_{ere}^{AGD}] = \begin{bmatrix} [m]_{AA} & [m]_{AB}[R_B] & 0 \\ [m]_{BA} & [m]_{BB}[R_B] + [m]_{CC}[R_C] & [m]_{CD} \\ 0 & [m]_{DC}[R_C] & [m]_{DD} \end{bmatrix}_{9\times 9} \tag{11b}$$

The relationships given by Eqs. (10) and (11) reveal that the *hybrid* beam element shown in Fig. 1 can be replaced by a three-node *elastic-and-rigid-combined* (or *elastic-rigid-elastic*) beam element shown in Fig. 2 with its stiffness matrix $[k_{ere}^{AGD}]$ and mass matrix $[m_{ere}^{AGD}]$ defined by Eqs. (11a) and

(11b), respectively. Note that, if the inertia effects of the *rigid* beam segment must be considered, then Eq. (11b) must be replaced by

$$[m_{ere}^{AGD}] = \begin{bmatrix} [m]_{AA} & [m]_{AB}[R_B] & 0 \\ [m]_{BA} & [m]_{BB}[R_B] + [m]_{CC}[R_C] + [m_R] & [m]_{CD} \\ 0 & [m]_{DC}[R_C] & [m]_{DD} \end{bmatrix}_{9\times 9}, \quad [m_R] = \begin{bmatrix} m_R & 0 & 0 \\ 0 & m_R & 0 \\ 0 & 0 & J_R \end{bmatrix} \tag{12}$$

where m_R and J_R are respectively the mass and mass moment of inertia of the *rigid* beam segment.

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