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A new high-order response surface method for structural reliability analysis

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Abstract. In order to consider high-order effects on the actual limit state function, a new response surface method is proposed for structural reliability analysis by the use of high-order approximation concept in this study. Hermite polynomials are used to determine the highest orders of input random variables, and the sampling points for the determination of highest orders are located on Gaussian points of Gauss-Hermite integration. The cross terms between two random variables, only in case that their corresponding percent contributions to the total variation of limit state function are significant, will be added to the response surface function to improve the approximation accuracy. As a result, significant reduction in computational cost is achieved with this strategy. Due to the addition of cross terms, the additional sampling points, laid on two-dimensional Gaussian points off axis on the plane of two significant variables, are required to determine the coefficients of the approximated limit state function. All available sampling points are employed to construct the final response surface function. Then, Monte Carlo Simulation is carried out on the final approximation response surface function to estimate the failure probability. Due to the use of high order polynomial, the proposed method is more accurate than the traditional second-order or linear response surface method. It also provides much more efficient solutions than the available high-order response surface method with less loss in accuracy. The efficiency and the accuracy of the proposed method compared with those of various response surface methods available are illustrated by five numerical examples.

Keywords: structural reliability analysis; Hermite polynomial; high-order response surface; Gaussian points

1. Introduction

The assessment of failure probability is one of challenges in structural reliability discipline. Commonly, the failure probability P_f is expressed as

$$P_f = P\{g(\mathbf{X}) \le 0\} = \int \cdots \int_{g(\mathbf{X}) \le 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{1}$$

where **x** is an input random vector which model the uncertainties in load conditions, material properties, geometrical dimensions etc, $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function (PDF) of

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random vector $\mathbf{x}$, and $g(\mathbf{x})$ is the so-called limit state function or performance function. The integration domain is determined by $g(\mathbf{x}) \leq 0$ which defines the failure domain. Usually the limit state function $g(\mathbf{x})$ is a nonlinear, implicit and multidimensional function of input random vector $\mathbf{x}$, so a closed-solution to the multidimensional integration in Eq. (1) rarely exists.

Many techniques have been proposed to estimate the failure probability in the past. According to their features, they can be mainly classified into three categories: (1) moment method (Rackwitz 2001, Zhao and Ono 1999, Hasofer and Lind 1974, Rackwitz and Fiessler 1978, Wang and Grandhi 1996, Ditleven and Madsen 1996, Zhao and Ono 2001, 2004, Zhao *et al.* 2003), (2) sampling method (Melchers 1989, Au and Beck 1999, Hurtado 2007, Melcher 1994, Au and Beck 2001), and (3) surrogate method (Bucher and Bourgund 1990, Rajashekhar and Ellingwood 1993, Kim and Na 1997, Das and Zhneg 2000, Guan and Melchers 2001, Kaymaz and McMahon 2005, Lee and Kwak 2006, Gavin and Yau 2008, Hurtado and Alvarez 2001, Papadrakakis and Lagaros 2002, Deng *et al.* 2005, Dong 2006, Kaymaz 2005, Hurtado and Alvarez 2003, Li *et al.* 2006, Chen 2007). In order to alleviate the computational effort in structural reliability analysis, it is widely recognized that approximating methods should be developed to express the relationship of the input parameters (loading conditions, material properties, etc.) and output quantities (responses in term of displacements, stress, etc.) of the structure. So surrogate method has been suggested as a common way to construct these relationships. There are currently several surrogate methods, such as Response Surface Method (RSM) (Bucher and Bourgund 1990, Rajashekhar and Ellingwood 1993, Kim and Na 1997, Das and Zhneg 2000, Guan and Melchers 2001, Kaymaz and McMahon 2005, Lee and Kwak 2006, Gavin and Yau 2008), Artificial Neural Network (ANN) (Hurtado and Alvarez 2001, Papadrakakis and Lagaros 2002, Deng *et al.* 2005, Dong 2006) more recently, Kriging Method (KM) (Kaymaz 2005) and Support Vector Machine (SVM) (Hurtado and Alvarez 2003, Li *et al.* 2006, Chen 2007) for reliability analysis problems. The central idea of surrogate methods is to approximate implicit limit state functions, given by a numerical calculation code, with a relatively simple explicit function, and the failure probability of the explicit function is used to replace that of actual limit state function. In early studies of surrogate methods, the polynomial-based Response Surface Method (RSM) fitted by the least square method (LSM) is very popular. The concept of RSM stemmed from design of experiment (DOE) and was later introduced into reliability assessment of structural systems (Bucher and Bourgund 1990, Rajashekhar and Ellingwood 1993, Kim and Na 1997, Das and Zhneg 2000, Guan and Melchers 2001, Kaymaz and McMahon 2005, Lee and Kwak 2006, Gavin and Yau 2008). Polynomials and the LSM are used to fit the response surface function (RSF). Once the RSF is completely constructed, the reliability analysis method can be completed on this explicit RSF. The disadvantages of RSM have been investigated by many researchers, and they showed that the inflexibility of the RSF (Bucher and Bourgund 1990, Rajashekhar and Ellingwood 1993, Kim and Na 1997, Das and Zhneg 2000) and the selection of response surface parameters (RSP) (Guan and Melchers 2001), which are used to define the locations of experimental points in RSM, have not been solved. The application of RSM in reliability analysis are not new, but it is still developing.

In this study a new high-order response surface is proposed, and this new method includes the following aspects: (1) Gaussian Points and Gauss-Hermite integration; (2) Hermite polynomial; (3) the probabilistic uncertainty contribution of each variable to limit state function. Once the response surface function is constructed, the failure probability is evaluated by Monte Carlo Simulation. The state of art and the problems of the polynomial based RSM are given in section 2. Section 3 analyzes the exiting high-order response surface method. Section 4 describes the procedure to construct high-order response surface function in detail. The accuracy and the efficiency of the proposed method are examined by five numerical examples in section 5. Section 6 is the closure of this paper.

2. Response surface method

In this section, some polynomial-based Response Surface Methods for structural reliability evaluations are briefly reviewed. In Bucher and Bourgund (1990) suggested an adaptive interpolation scheme of selecting the sampling points to form a quadratic RSF without cross terms for structural reliability analysis. In their method, the actual limit state function $g(\mathbf{x})$ is approximated by

$$\tilde{g}(\mathbf{x}) = a + \sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n} c_i x_i^2 \tag{2}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is the input random vector, a, b_i and $c_i (i = 1, \ldots, n)$ are $2n + 1$ unknown coefficients and they should be determined by the design of experiment. In order to determine the $2n + 1$ coefficients, $2n + 1$ samples have to be calculated to fit the polynomials at least. The first $2n$ samples are taken along the coordinate axes of each random variable at $\mu_i - h\sigma_i$ and $\mu_i + h\sigma_i$ in which h is the so-called Response Surface Parameter (RSP), and one more sample locates at the mean value vector $\mathbf{x}_\mu = (\mu_1, \mu_2, \ldots, \mu_n)$. After the approximated RSF $\tilde{g}(\mathbf{x})$ is constructed, the reliability index β and the corresponding design point $\mathbf{x}_D$ can be determined from the explicit RSF by First Order Reliability Method (Zhao and Ono 1999, Hasofer and Lind 1974, Rackwitz and Fiessler 1978) (FORM). Once the design point $\mathbf{x}_D$ is obtained, an update on the location of center sampling point $\mathbf{x}_M$ is achieved by a linear interpolation

$$\mathbf{x}_M = \mathbf{x} + (\mathbf{x}_\mu - \mathbf{x}_D)\frac{g(\mathbf{x}_\mu)}{g(\mathbf{x}_\mu) - g(\mathbf{x}_D)} \tag{3}$$

This makes the location of new center sampling point closer to the actual limit state $g(\mathbf{x}) = 0$. Around the new center sampling point $\mathbf{x}_M$, a new set of sampling points is selected to construct the final RSF. Thus, $4n + 3$ limit state function calls are required in Bucher and Bourgund's RSM. In later work, Rajasheknar and Ellingwood (1993) pointed out that the approximating accuracy of RSM depends on the characteristics of limit state function and one updating cycle may be insufficient. Therefore, based on the distance between the design point and the center sampling point in each cycle they proposed a criterion to judge whether the iteration is convergent or not. They also tried to generate samples including the probability distribution information of input random variables. However, it was found that sampling in the tail regions of the random variables distribution instead of over the entire region does not lead to significant improvement in the accuracy of RSF or reliability estimates. An improvement to RSM using weighted regression has been proposed by Kaymaz and McMahon (2005), in their method the sampling points closer to the limit state $g(\mathbf{x}) = 0$ hold larger weights for unknown coefficients evaluation. It should be pointed out that the value of RSP, which was set to 3 in Bucher and Bourgund' study and was gradually reduced in the later iteration in the work of Rajasheknar and Ellingwood, has significant effect on the estimated failure probability (Guan and Melchers 2001).

An improvement to Eq. (2) is adding cross terms to RSF

$$\tilde{g}(\mathbf{x}) = a + \sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n} c_i x_i^2 + \sum_{i \neq j} d_{ij} x_i x_j \tag{4}$$

Based on Eq. (4), Lee and Kwak (2006) have integrated the RSM with moment method to take the advantages of the two methods. However, the highest order of Lee and Kwak's RSM is not beyond the second order.

In Kim and Na (1997), first-order polynomial and vector projection technique are used to construct linear RSF for structural reliability analysis. However, this work also showed that the failure probability estimated by their RSM also varied with different values of RSP. The research of this type RSM was further extended by Das and Zheng (2000) through a cumulative manner.

Due to the facts that high-order polynomial may result in ill-conditional system of equations for unknown coefficients and exhibit irregular behavior outside of the domain of samples, the utilization of high-order polynomials in RSM has received relatively little attention (Rajashekhar and Ellingwood 1993, Gavin and Yau 2008). Recently, a different kind of RSM, which adopts high-order polynomials as regression bases, has been developed by Gavin and Yau (2008), referred as high-order response surface method (HORSM). In this method, a polynomial response surface including high order items is used to approximate the actual limit state function

$$\tilde{g}(\mathbf{x}) = a + \sum_{i=1}^{n} \sum_{j=1}^{k_i} b_{ij} x_i^j + \sum_{q=1}^{t} c_q \prod_{i=1}^{n} x_i^{p_{iq}} \tag{5}$$

where b_{ij} is the coefficient for terms involving only one random variable, k_i is the highest order for ith random variable, c_q is the coefficient for mixed terms, t is the total number of mixed terms, and p_{iq} is the order of a random variable in a mixed term.

Regression using Chebyshev polynomials associated with a statistical analysis is performed one-by-one to obtain the highest polynomial order of each random variable in RSF. In this stage, parameter h_{ord} is used to bound the domain of sampling points in $[\mu - h_{ord}\sigma, \mu + h_{ord}\sigma]$. After the highest polynomial orders are determined, all the mixed terms are also fixed through the criteria proposed by Gavin and Yau. Then the regression coefficients are evaluated from additional sampling points within the domain of $[\mu - h_{reg}\sigma, \mu + h_{reg}\sigma]$. At last, the failure probability is evaluated using MCS on the high-order RSF.

3. Analysis of the available HORSM

From above description, it can be concluded that two principal factors determine the final accuracy and efficiency of RSM, one is the type of RSM, linear, quadratic or high-order, the other is the positions of sampling points. We now examine the Gavin and Yau's HORSM from the view of these two factors. Compared with linear or quadratic RSM, the HORSM has shown a better approximating ability for highly nonlinear limit state function, but it is also accompanied by significant increase of computational cost. Furthermore, two response surface parameters h_{ord} and h_{reg} controlled the sampling positions are needed to be selected, which may have certain effects on the determination of highest polynomial orders and the accuracy of final failure probability estimation.

An example with explicit limit state function (Guan and Melchers 2001) is considered to investigate the Gavin and Yau's HORSM

Table 1 The results by Gavin and Yau's HORSM

h_{ord}	k_1	k_2	k_3	$P_f(\times 10^{-4})$	Function calls for determining k_i	Total function calls
1.0	4	2	1	2.94	49	91
2.0	4	2	1	2.94	49	91
3.0	4	2	1	2.94	49	91
4.0	4	2	1	2.94	49	91
5.0	4	2	1	2.94	49	91
6.0	4	2	1	2.94	49	91
7.0	4	2	1	2.94	49	91

$$g(\mathbf{x}) = \frac{x_1^4}{40} + 2x_2^2 + x_3 + 3 \tag{6}$$

where x_1, x_2 and x_3 are standard normal variables. The results computed by Gavin and Yau's HORSM are shown in Table 1.

The reference failure probability of this problem is 3.04×10^{-4} with 10^9 Monte Carlo simulations. In order to examine the effects of parameter h_{ord}, the value of parameter h_{reg} is set to 3 in this stage. Though there are no cross terms in this example, 13 cross terms are still found for each values of h_{ord} by HORSM, which results in additional sampling computation for the determination of unknown coefficients of cross terms. The total number of function calls is 91. It is not efficient for a problem involved 3 input random variables. The same phenomenon is observed when h_{ord} takes 3 and h_{reg} changes.

4. The proposed algorithm

To over the shortcomings of current HORSM, a new HORSM is proposed in this study. Hermite polynomials are used to replace the Chebyshev polynomials in the new HORSM because the weight functions of Hermite polynomials match the normal probability density functions, which benefit for generating samples in construction of RSF. New strategies are designed to generate samples and determine the highest orders of polynomials and cross terms.

The aim of this study is to improve the efficiency of HORSM by reducing unnecessary additional sampling points. To reduce unnecessary additional sampling points, it is desirable to classify the significant and insignificant random variables because the cross terms between insignificant variables provide little contribution to the total variation of output even though the actual limit state function includes them. Only the cross terms between two significant variables are taken into account in next step approximation. So the high-order RSF in this study can be expressed as

$$\tilde{g}(\mathbf{x}) = a + \sum_{i=1}^{n}\sum_{j=1}^{k_i} b_{ij}x_i^j + \sum_{i \neq j, (i, j \in \text{sign})} c_{ij}x_i^{p_i}x_j^{p_j} \tag{7}$$

where sign represents the set of significant variables.

4.1 Overall procedure

The major steps of the proposed method are detailed as following:

1. Use the sampling method described in section 4.2 to generate the sampling points for all input random variables.
2. Determine the highest order of each random variable based on a regression model using Hermite polynomials.
3. Determine the significant variables which have important contribution to the total variation of the limit state function.
4. Locate additional sampling points on two-dimensional Gaussian integration (also named as Gaussian Points) points of two significant variables while the other variables are set to their mean values.
5. All sampling points generated in Step 1 and Step 4 are used to form system of equations, then the coefficients of polynomials are evaluated by LSM.
6. Finally, MCS is carried out on the ultimately determined high-order RSF to estimate the failure probability.

4.2 Sampling method

Sampling methods have been investigated by many researchers, and several practical techniques have been established for structural reliability analysis, such as Bucher and Bourgund's proposal (Bucher and Bourgund 1990, Rajashekhar and Ellingwood 1993, Guan and Melchers 2001, Kaymaz and McMahon 2005), factorial design scheme (Gavin and Yau 2008) and vector projection (Kim and Na 1997, Das and Zhneg 2000), etc. The sampling method considering the probabilistic distribution information of random variables was also studied in the literature (Rajashekhar and Ellingwood 1993). In this paper, we use Gaussian points and variable transformation to develop a new sampling method for HORSM in structural reliability analysis.

For the sake of convenient discussion, we firstly assume that all input random variables are mutually independent. The central sampling point is located at $\mathbf{u}_\mu = (u_{\mu_1}, u_{\mu_2}, \ldots, u_{\mu_n})$ in the standard normal space usually denoted by u-space corresponding to the mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ in the original variable space usually denoted by x-space. For a non-normal variable, it can be treated as a normal variable by marginal transformation (Ditleven and Madsen 1996)

$$u = \Phi^{-1}(F_X(x)) \tag{8}$$

The sampling points firstly are generated in u-space. The jth component of the generated sampling point is set to the Gaussian points u_{ij} ($j = 1, \ldots, m$, m is the number of Gaussian points of one-dimensional integration) for ith input random variables while the rest components are set to $u_{\mu_k}(k = 1, 2, \ldots, n,\ k \neq i)$ in u-space. So a sampling points $\mathbf{u}_{ij}$ in u-space is given by Eq. (9)

$$\mathbf{u}_{ij} = (u_{\mu_1}, u_{\mu_2}, \ldots, u_{\mu_{i-1}}, u_{ij}, u_{\mu_{i+1}}, \ldots, u_{\mu_n}), \quad i = 1, \ldots, n,\ j = 1, \ldots, m \tag{9}$$

In practice, m is always taken an odd integer. In case all input random variables obey normal distribution, the m Gaussian points are symmetrical about coordinate origin in u-space, hence the total number of sampling points according to Eq. (9) is $(m-1)n+1$. For case with non-normal distributed variables, the total number of sampling points according to Eq. (9) is $m \times n$ due to $\mathbf{u}_\mu = (u_{\mu_1}, u_{\mu_2}, \ldots, u_{\mu_n}) \neq (0, 0, \ldots, 0)$.

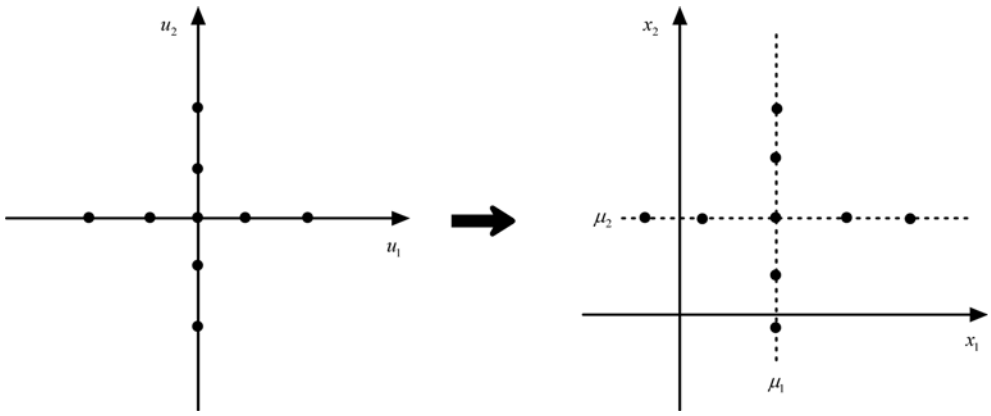

Fig. 1 Sampling points for the determination of highest order in two-dimensional case

Clearly, the sampling points are located on coordinate axis for normally distributed variables, and this sampling method can be viewed as an improvement of Bucher and Bourgund's proposal (Guan and Melchers 2001). After the sampling points are selected, they are transformed from u-space to x-space by inverse marginal transformation when the input variables are mutually independent (Ditleven and Madsen 1996)

$$\begin{aligned} x_{ij} &= F_{X_i}^{-1}(\Phi(u_{ij})), \quad i = 1,\ldots,n,\ j = 1,\ldots,m \\ \mathbf{x}_{ij} &= (\mu_1,\ldots,\mu_{i-1},x_{ij},\mu_{i+1},\ldots,\mu_n),\ i = 1,\ldots,n \end{aligned} \tag{10}$$

where $F_{X_i}^{-1}(\cdot)$ is the inverse cumulative distribution of input random variable X_i. A two-dimensional example for normally distributed variables with $m = 5$ is shown in Fig. 1.

It should be noted that if the input variables are dependent, there is a slight difference in sampling. For the problem with a complete joint PDF, the inverse Rosenblatt transformation (Rosenblatt 1952) can be used to solve the transformation from $\mathbf{u}_{ij}$ to $\mathbf{x}_{ij}$ needed for evaluating the limit state function value. However, the joint PDF required in Rosenblatt transformation are seldom available for the engineering problems with limited statistical data. Usually, the PDFs of each random variable, denoted by $f_i(x_i)(i = 1,\ldots,n)$, and the correlated matrix $\boldsymbol{\rho}$ are known for engineering problems. For this incomplete probabilistic information case (Der Kiureghian and Liu 1986, Liu and Der Kiureghian 1986), the inverse Nataf transformation can be employed to obtain the transformation from $\mathbf{u}_{ij}$ to $\mathbf{x}_{ij}$. More details about Nataf transformation can be found in (Der Kiureghian and Liu 1986, Liu and Der Kiureghian 1986).

4.3 The determination of highest polynomial orders

In order to eliminate the effects of the different physical meaning and dimensions of the basic random variables on the RSF, data scaling shown in Eq. (11) is needed before approximation

$$\mathbf{x}'_{ij} = \frac{\mathbf{x}_{ij} - \mu_i}{\sigma_i}, \quad i = 1,\ldots,n,\ j = 1,\ldots,m \tag{11}$$

The data scaling can reduce rounding error of computers and eliminate the ill-conditioned problems.

In the proposed method, the highest order for each random variables is determined in the case of neglecting the cross terms. The limit state function is first approximated (Zhao and Ono 2001, Rahman and Xu 2004) by $\tilde{g}(\mathbf{x})$ shown in Eq. (12)

$$g(\mathbf{x}) \approx \tilde{g}(\mathbf{x}) = \sum_{i=1}^{n}(g_i - g_\mu) + g_\mu \tag{12}$$

where $g_\mu = g(\mu_1, \mu_2, \ldots, \mu_n)$ is the evaluation performed at the mean vector of all input random variables, and $g_i = g(x_i) = g(\mu_1, \mu_2, \ldots, \mu_{i-1}, x_i, \mu_{i+1}, \ldots, \mu_n)$ represents that x_i is the only random variable with the other variables set equal to their mean values. The sampling points in ith coordinate axis and the Hermite polynomials are employed to construct the response surface $\tilde{g}(x_i)$ for g_i

$$g_i = g(x_i) \approx \tilde{g}(x_i) = a_0 H_0(x_i) + a_1 H_1(x_i) + a_2 H_2(x_i) + \ldots \tag{13}$$

where $a_i (i = 0, 1, 2, \ldots)$ are the regression coefficients, $H_i(\cdot)$ $(i = 0, 1, 2, \ldots)$ is ith order Hermite polynomials. The first five order Hermite polynomials are given by

$$\begin{aligned} &H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1 \\ &H_3(x) = x^3 - 3x, \quad H_4(x) = x^4 - 6x^2 + 3 \\ &H_5(x) = x^5 - 10x^3 + 15x \end{aligned} \tag{14}$$

In practice, the order of Hermite polynomial is suggested not to beyond 5. If the coefficient a_i of $H_i(\cdot)$ is less than a threshold value, then the effect of this term can be neglected. Thus, the highest term also can be determined easily. The determination of this threshold value is a trade-off between approximate accuracy and computational effort. If it is too small, more Hermite polynomials may be included in the final RSF and then the computational effort would increases. If it is too large, some polynomials would be missed and may lead to loss in approximate accuracy. Based on our experience, we suggest taking 10^{-4} as the threshold value in this paper. This procedure is performed on each variable one-by-one for determining of the valid highest order k_i in Eq. (7).

The method preserves the advantage reported by Gavin and Yau, i.e., one-dimensional approximation instead of a multi-dimensional approximation is employed due to much more computational efficiency, especially in cases involving a large number of random variables (Gavin and Yau 2008).

4.4 The determination of cross terms

It should be pointed out that the inclusion of cross terms without control will increase the computational cost and may even lead to an inaccurate estimate if the condition number of the normal equations becomes large. Once the valid highest order for each random variable are determined, three criterions should be abided to keep the number of unknown coefficients of cross terms as small as possible when adding cross terms into the response surface. The first one is that only the cross terms between two significant variables are taken into account in the next approximation. The significant variables and insignificant ones are identified through the contribution of the corresponding random variables to the total variation, which is presented in the following context. The other two criterions are developed in Gavin and Yau (2008), i.e., (1) the power p_i of a variable in a mixed term should not be larger than the highest order of the variable alone, i.e., $p_i \le k_i$; (2) the sum of orders of variables in a mixed term should not be larger than the highest order, $(p_i + p_j) \le \max(k_i, k_j)$.

The next step is to classify the significant variables and insignificant ones by their contributions to the variation of the limit state function. Based on the approximation in Eq. (12), the variance of $g(\mathbf{x})$ can be calculated by

$$\sigma_g^2 = \sum_{i=1}^{n} \sigma_{g_i}^2 \tag{15}$$

where σ_{g_i} is the standard deviation of the univariable function g_i and it can be evaluated by the Gassian-Hermite integral formulas shown in Eq. (17) (Zhao and Ono 2000)

$$\mu_{g_i} \approx \sum_{j=1}^{m} w_j g(x_{ij}) \tag{16}$$

$$\sigma_{g_i}^2 \approx \sum_{j=1}^{m} w_j (g(x_{ij}) - \mu_{g_i})^2 \tag{17}$$

where w_j is the weights of Gauss-Hermite integration and x_{ij} is the corresponding values in x-space of the Gaussian point in u-space, respectively. Because the sampling points in first stage locate on Gaussian points, there is no need to perform additional function calls to accomplish the computation in Eqs. (15)-(17). The percent contribution of each random variable to the variation of limit state function $g(\mathbf{x})$ can be evaluated by

$$P_i = \frac{\sigma_{g_i}^2}{\sigma_g^2} \times 100\% \tag{18}$$

Then the significance of each input variables can be measured by P_i.

Actually, the significant variables should contribute the main variation of the limit state function, such as 95%, which can guide the selection of them. In brief, we first sort P_i from large to small. Secondly the first few contributions of the input random variables in this sequence are added together until the total contribution of them is larger than 95%. Then the corresponding random variables are viewed as significant variables.

4.5 Final response surface approximation and assessment of failure probability

After the valid highest orders of each random variable and the valid cross terms have been determined, the type of response surface function is also fixed. If the cross terms are included in the final response surface function, additional sampling points are needed to form system of equations for estimating the unknown coefficients. The random method, which selects randomly sampling points over the defined domains of random variables, is often adapted to select the additional sampling points in some real applications. However, a major disadvantage of the random method is that it does not guarantee that the sampling points can cover the entire definition domains uniformly.

In this study, the additional sampling points are located on the Gaussian points off axis in two-dimensional plane (significant variables) while the rest of the random variables are set to their mean

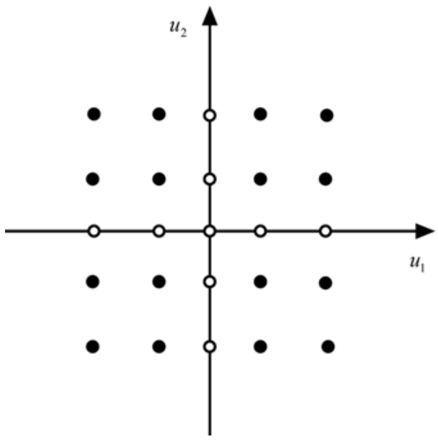

Fig. 2 Additional sampling points

Table 2 The maximum possible numbers of cross terms and those of the additional sampling points

m	The maximum possible number of cross terms	The number of additional sampling points
3	1	4
5	6	16
7	15	36

values. An example is shown in Fig. 2. The sampling points lie along axes denoted by circles have been used to obtain the RSF without cross terms in section 4.2, while the solid points are the additional samples for the determination of cross terms.

When m samples are selected for a random variable along the axis, the highest possible order of the corresponding polynomial is $m-1$. According to the three criterions for selecting cross terms, if the powers of two significant variables simultaneously reached at the highest possible value, i.e., $m-1$, the total number of cross terms of them is $(m-2)(m-1)/2$ which is less than $(m-1)^2$, the number of additional sampling points. It can be concluded that the number of additional samples generated at Gaussian points is absolutely sufficient for the least square estimators. A summary about the maximum possible numbers of the cross terms and those of the additional sampling points corresponding to different m is listed in Table 2. In practice, a part of Gaussian points which are off axis, not whole, is randomly selected as the additional sampling points.

Combining all samples generated for the determination of the highest orders of variables and the cross terms, a well-conditional system matrix for regression is conducted to determine the unknown coefficients by LSM, and the final RSF can be obtained.

At the last step, the failure probability is evaluated by Monte Carlo Simulation on the final explicit RSF. Since the limit state function is approximated by the explicit expression, the computation cost of Monte Carlo Simulation to obtain an accurate failure probability estimator can be acceptable.

5. Numerical examples

The proposed method is validated and demonstrated by five numerical examples. The first three examples are explicit mathematical functions, while the first example has no cross terms but the following two have. The last two examples have implicit limit state functions, which need to call finite element software for deterministic structural analysis. In the given examples, comparisons are made between the proposed HORSM and the various RSMs or Monte Carlo Simulation.

5.1 Example 1

The first example comes from Guan and Melchers (2001) with an explicit function

$$g(\mathbf{x}) = \frac{x_1^4}{40} + 2x_2^2 + x_3 + 3 \tag{19}$$

in which all random variables have standard normal distributions and are mutually independent.

This problem was investigated by Guan and Melchers in detail, and it is shown that the failure probability evaluations vary from 0.78×10^{-4} to 4.01×10^{-4} for different response parameters (Guan and Melchers 2001). The failure probability of this limit state function is also evaluated by the proposed HORSM described in section 3. Table 3 and Table 4 show the results for different m. From Table 3, it is observed that in case $m = 5$ or 7, the highest orders of each variables identified by the proposed method are exactly equal to the real values. The percent contributions of each variable to the total variation are also listed in Table 3. It is shown that random variable x_1 whose contribution is less than 1% can be considered as an insignificant variable. The other two variables are considered as significant variables. According to the criterions for adding cross terms, one cross term, i.e., x_2x_3, is needed to be taken into account. But in the final response surface function, the coefficients of x_2x_3 are 2.44×10^{-16} for $m = 3$, 1.28×10^{-15} for $m = 5$ and 9.89×10^{-17} for $m = 7$. It is obvious that the effect of cross term can be neglected in the final result. So the limit state function is approximated exactly with the proposed method though a cross term has been added to the response surface.

The exact solution using MCS with 10^9 simulations is 3.04×10^{-4}. When $m = 5$ or 7, the same results are obtained by MCS. When $m = 3$, the approximated response surface is

Table 3 Determination of highest order and evaluation of percent contribution of each variable

m	k_1	k_2	k_3	P_1 (%)	P_2 (%)	P_3 (%)
3	2	2	1	0. 12	88.78	11.10
5	4	2	1	0. 16	88.30	11.04
7	4	2	1	0. 16	88.30	11.04

Table 4 Failure probabilities for example 1

m	$P_f (\times 10^{-4})$	Function calls for determining k_i	Total function calls
3	3.00	7	9
5	3.21	13	15
7	3.21	19	21

$$\tilde{g}(\mathbf{x}) = 0.075x_1^2 + 2x_2^2 + x_3 + 3 \tag{20}$$

Based on Eq. (20), a comparable result, 3.00×10^{-4}, is obtained by MCS with 10^6 simulations.

The number of original limit state function calls is the same as the sample size. As seen from Table 4, Only 15 function calls are needed to approximate exactly the limit state function, but 91 function calls are required by the HORSM proposed by Gavin and Yau. So the new HORSM proposed in this study is more efficient than the old one.

In the next step, we change the standard deviations of input random variables from 1.0 to 0.8 which changes P_f from 3.04×10^{-4} to 2.1×10^{-5}. The proposed HORSM identifies the true limit state function for $m = 5$ and 7, while for $m = 3$, the response surface function is

$$\tilde{g}(\mathbf{x}) = 0.048x_1^2 + 2x_2^2 + x_3 + 3 \tag{21}$$

which yields a failure probability of 2.0×10^{-5}.

5.2 Example 2

The second example which limit state function is shown in Eq. (22) is well known in FORM, because it is not convergent when Hasofer-Lind and Rackwitz-Fiessler algorithm (Hasofer and Lind 1974, Rackwitz and Fiessler 1978) is used. Originally introduced by Wang and Grandhi (1996), the failure probability estimation was investigated by Kaymaz and McMahon (2005) using a response surface method based on weighted regression. The limit state function is

$$g(\mathbf{x}) = x_1^3 + x_1^2 x_2 + x_2^3 - 18 \tag{22}$$

where x_1 and x_2 obey normal distributions with mean values $\mu_1 = 10.0, \mu_2 = 9.9$ and standard deviation $\sigma_1 = \sigma_2 = 5.0$.

It behaves highly nonlinearity around design point as shown in Fig. 3. The results calculated by various methods are summarized in Table 5 and Fig. 3. When $m = 5$ in this example, the number of additional samples is selected twice as much as the number of the cross terms for the proposed HORSM. Since the contributions of x_1 and x_2 are all relatively large, they are both considered as

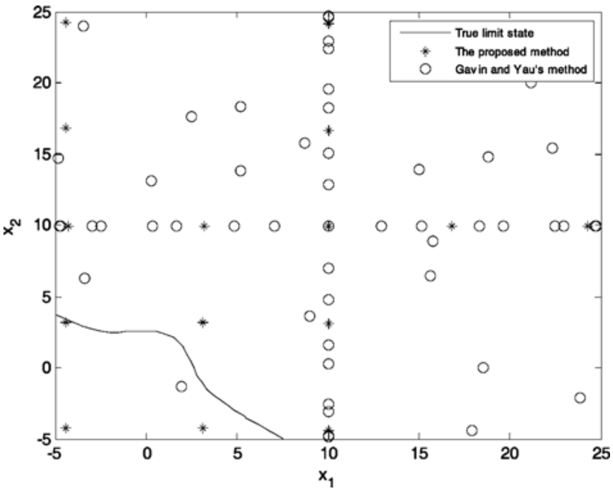

Fig. 3 True limit state and comparison of sampling points for Example 2

Table 5 Comparison of results for example 2

Method	k_1	k_2	P_1 (%)	P_2 (%)	P_f ($\times 10^{-3}$)	Function calls for determining k_i	Function calls
The proposed method (m = 5)	3	3	33.98	66.02	6.880	9	15
Gavin and Yan's method	3	3	--	--	6.880	34	54
Kaymaz and McMahon's method	2	2	--	--	5.396†	--	15
NESUSS-MCS	--	--	--	--	6.880	--	10^6

†Kaymaz and McMahon 2005

significant variables. After the highest orders are determined, three cross terms, i.e., $x_1^2x_2, x_2^2x_1$ and x_1x_2, are added to the final response surface, however only one cross term $x_1^2x_2$ has significant effect on the output because the regression coefficients of the other two cross terms are close to zero. The proposed HORSM perfectly identifies the exact form of the limit state function in this problem, so it provides an exact estimate of the failure probability. Gavin and Yau's method also gives the exact result for the present problem, but its computational cost is much higher than that of the proposed method shown in Table 5. The result of Kaymaz and McMahon' method has some discrepancy with the exact result.

Fig. 3 shows the sampling points involved in Gavin and Yau's and the proposed HORSM. They have almost the same range of sampling, but the proposed method produces more representative samples. Referenced to Gavin and Yau's HORSM with 54 function call, the proposed method with only 15 function calls yields the exact solution.

5.3 Example 3

This example is taken from Gavin and Yau (2008) to compare the performance of the proposed HORSM and Gavin and Yau's method further. The limit state function is defined as

$$g(\mathbf{x}) = 0.16(x_1-1)^3 - x_2 + 4 - 0.04\cos(x_1x_2) \tag{23}$$

where all input random variables follow standard normal distribution and are mutually independent.

Gavin and Yau's original paper contains a typographical error because the estimated failure probability of this limit state function is 3.2833×10^{-2}, obtained by a MCS with 10^6 samples in NESSUS (2005), but not 9.5×10^{-5} asserted in their work.

A comparison of results is shown in Table 6 and Fig. 4. The proposed method is quite efficient and more accurate for the determination of the highest orders of variables with 9 sampling points,

Table 6 Comparison of results for example 3

Method	k_1	k_2	P_1 (%)	P_2 (%)	P_f ($\times 10^{-2}$)	Function calls for determining k_i	Total function calls
The proposed method (m = 5)	3	1	60.57	39.43	3.3661	9	13
Gavin and Yan's method	3	2	--	--	3.2466	25	43
NESUSS-MCS	--	--	--	--	3.2833	--	10^6

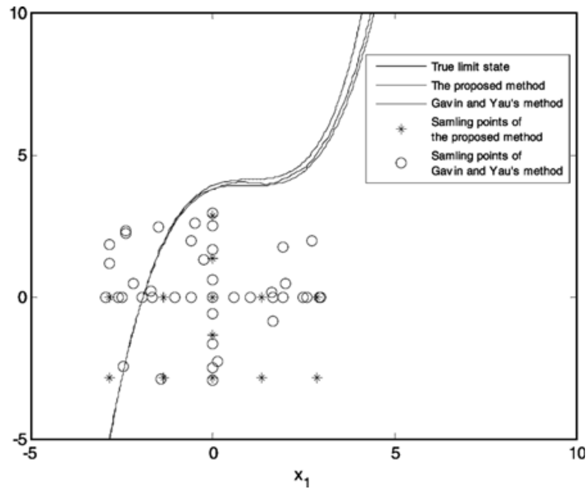

Fig. 4 Comparison of approximated limit state and sampling points for Example 3

while 25 function calls are required for Gavin and Yau's method. It can be found that the contributions of x_1 and x_2 are all relatively large, they are both considered as significant variables. Thus, two cross terms, i.e., x_1x_2 and $x_1^2x_2$, are included in the final RSF. However, the coefficient of x_1x_2 is so close to zero that the effect of x_1x_2 can be neglected. The true limit state function is approximated as

$$\tilde{g}(\mathbf{x}) = 0.16x_1^3 - 0.4812x_1^2 + 0.48x_1 - 1.0091x_2 - 0.0017x_1^2x_2 + 3.8084 \tag{24}$$

It produces an acceptable estimate value of 3.3661×10^{-2} for the failure probability estimation. While Gavin and Yau's method identifies three cross terms that lead to an increase in the computational effort. As shown in Table 6, the total function calls of Gavin and Yau's method is thrice that of the proposed method in this study, but it achieves little increase on the accuracy of failure probability. So the proposed HORSM is significantly more efficient than the existing one.

5.4 Example 4

This example is a truss structure (Kim and Na 1997, Lee and Kwak 2006) that has an implicit limit state function evaluated by finite element method. This structure has 23 members, as shown in Fig. 5. The statistical parameters of input random variables are summarized in Table 7, in which E represents Young's modulus and A is the cross sectional area. The limit state function is defined as the displacement at the center point (point C in Fig. 5) of the truss structure not beyond 11.0 cm

$$g(\mathbf{x}) = 11.0 - D(\mathbf{x}) \tag{25}$$

The proposed HORSM is applied to this problem. There are significant diversity in the ranges of input variables, so the data scaling method in Eq. (11) is adapted. Table 8 gives the results of the highest order identification for $m = 3$ and $m = 5$. Table 9 shows the estimated contribution of each variable to the total variation of the limit state function, and the failure probabilities estimated by several methods are listed in Table 10. From Table 9, it can be seen that the total contribution of random variable x_2, x_4, x_5 and x_{10} is less than 4%, so they are ranked as insignificant variables and

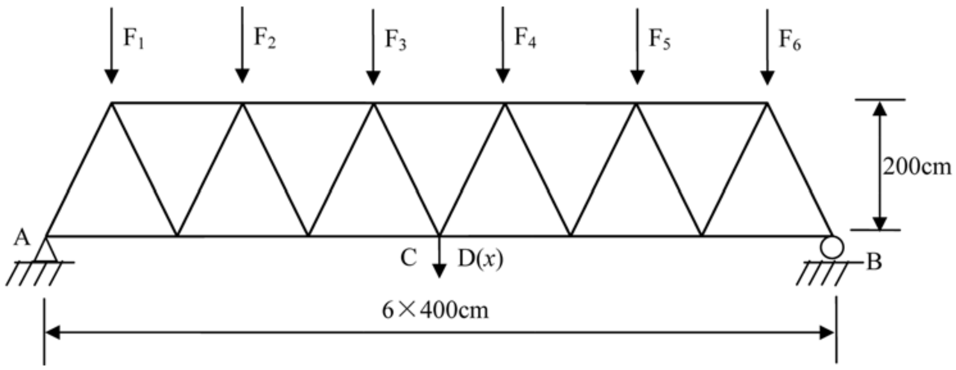

Fig. 5 Truss structure

Table 7 The statistical parameters of input random variables

Random variable	Description	Distribution type	Mean	Standard deviation
x_1	E of horizontal member	Log-normal	2100000(kg/cm^2)	210000
x_2	A of horizontal member	Log-normal	20(cm^2)	2
x_3	E of diagonal member	Log-normal	210000(kg/cm^2)	21000
x_4	A of diagonal member	Log-normal	10(cm^2)	1
x_5	Load F_1	Type-I-Largest	5000(kg)	750
x_6	Load F_2	Type-I-Largest	5000(kg)	750
x_7	Load F_3	Type-I-Largest	5000(kg)	750
x_8	Load F_4	Type-I-Largest	5000(kg)	750
x_9	Load F_5	Type-I-Largest	5000(kg)	750
x_{10}	Load F_6	Type-I-Largest	5000(kg)	750

Table 8 Determination of highest order of each variable in Example 4

Method	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}
The proposed method (m = 3)	2	2	2	2	1	1	1	1	1	1
The proposed method (m = 5)	4	4	4	4	1	1	1	1	1	1

Table 9 Evaluation of percent contribution of each variable in Example 4 (%)

Method	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
The proposed method (m = 3)	37.08	1.26	37.08	1.26	0.46	3.65	7.55	7.55	3.65	0.46
The proposed method (m = 5)	37.05	1.26	37.05	1.26	0.46	3.66	7.60	7.60	3.66	0.46

the cross terms involving them are not taken into account in computational process. Integrated the computational results in Tables 8 and 10, 9 cross terms are considered for the case of $m = 3$, and 30 cross terms for the case of $m = 5$. The size of additional samples is selected as the same of the number of cross terms. For this problem, the reference solution can be found in Lee and Kwak's study (2006), i.e., $P_f = 8.33 \times 10^{-3}$ listed in Table 10 (Lee and Kwak 2006). The results calculated by the proposed method show a good agreement with the reference one. Lee and Kwak's (2006)

Table 10 Comparison of failure probabilities for Example 4

Method	m	$P_f(\times 10^{-3})$	Function calls for determining k_i	Total function calls
The proposed method (m = 3)	3	8.39	21	30
The proposed method (m = 5)	5	8.28	41	71
Lee and Kwak's method	--	8.80‡	--	45
Kim and Na's method	--	7.47†‡	--	--
Bucher and Bourgund's RSM (convergent)	--	9.52	--	89
MCS	--	8.33‡	--	10^5

‡Lee and Kwak 2006
†‡Kim and Na 1997

and Kim and Na's (1997) methods also produce comparable estimations for this problem, but they show larger deviation from the exact one than that of the proposed HORSM. The total number of function calls for the proposed method with $m = 3$ is 30, and the total number of function calls of Lee and Kwak's method is 45. The Bucher and Bourgund's quadratic response surface with convergent strategy (Guan and Melchers 2001) (RSP takes 3) is also applied to this problem, and the proposed HORSM exhibits more accurate and efficient than this convergent quadratic response surface.

5.5 Example 5

In order to demonstrate the accuracy and efficiency of the proposed method further, a three-bay five-storey frame structure, as shown in Fig. 6, with an implicit limit state function is employed as the last example in this paper. This reliability problem has been studied by Liu and Der Kiureghian (1991), Bucher and Bourgand (1990) and Guan and Melchers (2001), which involved correlated

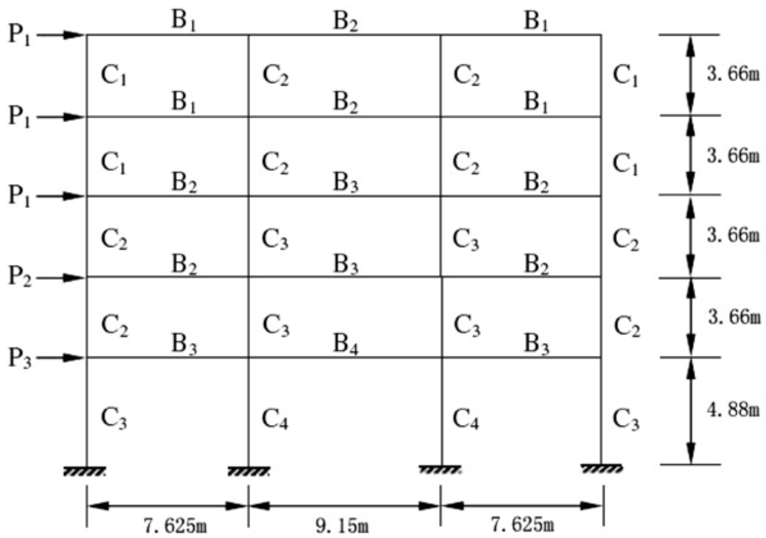

Fig. 6 Three bay five storey frame structure

Table 11 Frame element properties

Element	Moduli of elasticity	Moments of inertia	Cross-sectional area
B_1	E_4	I_{10}	A_{18}
B_2	E_4	I_{11}	A_{19}
B_3	E_4	I_{12}	A_{20}
B_4	E_4	I_{13}	A_{21}
C_1	E_5	I_6	A_{14}
C_2	E_5	I_7	A_{15}
C_3	E_5	I_8	A_{16}
C_4	E_5	I_9	A_{17}

Table 12 Statistical parameters of frame structure

Random variable	Distribution type	Mean	Standard deviation
P_1	Gumbel max	133.454	40.04
P_2	Gumbel max	88.97	35.59
P_3	Gumbel max	71.175	28.47
E_4	Normal	2.173752×10^{7}	1.9152×10^{6}
E_5	Normal	2.379636×10^{7}	1.9152×10^{6}
I_6	Normal	0.813443×10^{-2}	1.08344×10^{-3}
I_7	Normal	1.150936×10^{-2}	1.298048×10^{-3}
I_8	Normal	2.137452×10^{-2}	2.59609×10^{-3}
I_9	Normal	2.596095×10^{-2}	3.028778×10^{-3}
I_{10}	Normal	1.081706×10^{-2}	2.596095×10^{-3}
I_{11}	Normal	1.410545×10^{-2}	3.46146×10^{-3}
I_{12}	Normal	2.327852×10^{-2}	5.624873×10^{-3}
I_{13}	Normal	2.596095×10^{-2}	6.490238×10^{-3}
A_{14}	Normal	0.312564	0.055815
A_{15}	Normal	0.3721	0.07442
A_{16}	Normal	0.50606	0.093025
A_{17}	Normal	0.55815	0.11163
A_{18}	Normal	0.253028	0.093025
A_{19}	Normal	0.29116825	0.1023275
A_{20}	Normal	0.37303	0.1209325
A_{21}	Normal	0.4186	0.1395375

random variables.

There are 21 input random variables: (a) three lateral loadings denoted by P_i $(i=1,2,3)$, (b) two moduli of elasticity denoted by E_4 and E_5, (c) eight moments of inertia denoted by I_j $(j=6,7,\ldots,13)$, and (d) eight cross-sectional areas denoted by $A_k (k=14,15,\ldots,21)$. The statistical parameters and the structural data are summarized in Table 11 and Table 12, respectively. All lateral loadings are assumed to be correlated by $\rho=0.95$, and two moduli of elasticity are correlated by $\rho=0.9$. The cross-sectional areas and moment of inertias of each beam column elements are highly correlated by $\rho=0.95$, and other correlations between cross-sectional areas and moment of inertias are correlated as

$$\rho_{A_iA_j} = \rho_{I_iI_j} = \rho_{I_iA_j} = 0.13 \tag{26}$$

All other random variables are assumed to be uncorrelated.

The limit state function is defined as the top floor horizontal displacement $u_{\mathbf{x}}$ not exceeding 0.061 m

$$g(\mathbf{x}) = 0.061 - u_{\mathbf{x}} \tag{27}$$

The deterministic analysis of the frame structure is computed using ANSYS Parametric Design Language (APDL) (2007), so it is very easy to rewrite ADPL file for repeating the deterministic structural analysis.

Since only marginal probability density functions and their correlation matrix of the input variables are available, the Nataf transformation is suitable for this problem. The proposed HORSM identified the same significant variables for three values of m, that is, P_1, E_4, I_8, I_{11} and I_{12}. Table 13 shows the percent contributions of five significant variables. These five significant variables contribute about 96% uncertainty in the variation of limit state function, so they can reflect the main uncertainty source.

Nataf transformation based Monte Carlo Simulation (NBMCS) (Chang *et al.* 1993) is employed to yield a reference estimate of failure probability with a sampling size $N = 5 \times 10^5$. The failure probabilities from the proposed HORSM are compared with these of NBMCS and Bucher and Bourgund's RSM in Table 14. The Bucher and Bourgund's RSM in the fifth row of Table 14 is updated only once, which requires fewer computations of the limit state function than the proposed HORSM. In fact, if Bucher and Bourgund's response surface is updated until the convergence (Guan and Melchers 2001) is reached, the function calls is 221 and the failure probability is 3.95×10^{-4}. Table 14 clearly indicates that the proposed HORSM with $m = 7$ yields the most

Table 13 Evaluation of percent contribution of significant variable in Example 5 (%)

Method	P_{P_1}	P_{E_4}	P_{I_8}	$P_{I_{11}}$	$P_{I_{12}}$
The proposed method ($m = 3$)	83.48	4.41	1.13	3.76	3.15
The proposed method ($m = 5$)	83.41	4.41	1.13	3.79	3.17
The proposed method ($m = 7$)	83.42	4.41	1.13	3.79	3.17

Table 14 Comparison of failure probabilities for Example 5

Method	m	$P_f\,(\times 10^{-4})$	Function calls for determining k_i	Total function calls
The proposed method ($m = 3$)	3	2.65	43	63
The proposed method ($m = 5$)	5	3.80	85	129
The proposed method ($m = 7$)	7	4.68	127	171
Bucher and Bourgund's RSM	--	5.00‡‡	--	87
Bucher and Bourgund's RSM (convergent)	--	3.95	--	221
NBMCS	--	4.67	--	5×10^5

‡‡Bucher and Bourguna 1990

accurate estimate than other methods and the other values of m can also yield a comparable estimate to the convergent Bucher and Bourgund's response surface with a small number of samples.

6. Conclusions

This study devotes to improve the accuracy and the efficiency of response surface method for structural reliability analysis. A new high-order response surface method (HORSM) is proposed for this purpose. The proposed method employs Hermite polynomials and the one-dimensional Gaussian points as sampling points to determine the highest order of each variables. This method needs only $(m-1)n+1$ or $m \times n$ function calls for the determination of highest orders (n is the number of random variables, and m, which is an odd number, is the sampling size for one variables). Then, the significant random variables, which have important contributions to the output variation, are screened out by a method designed using Gauss-Hermite integration. Only the cross terms between two significant random variables are considered in the final RSF. Additional sampling points are located at two-dimensional Gaussian points. Combining two parts of sampling points, a least square method based on Hermite polynomials is performed to obtain the unknown coefficients of high-order RSF. Finally, a Monte Carlo Simulation is carried out on the high-order response surface to estimate the failure probability. As demonstrated by the numerical examples, the proposed method shows more efficient than Gavin and Yau's HORSM. Since the proposed HORSM can capture the nonlinearity of the true limit state function and has no iteration compared with other second-order or linear RSM available now, it is therefore more accurate and efficient.

The proposed method could be further improved by an iterative strategy and including the cross items between three or above significant variables. However, it would involve some other problems such as the increase in computational cost, multiple design points etc, which needs further investigations.

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