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Non-linear vibration and stability analysis of an axially moving rotor in sub-critical transporting speed range

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Abstract. Parametric and forced non-linear vibrations of an axially moving rotor both in non-resonance and near-resonance cases have been investigated analytically in this paper. The axial speed is assumed to involve a mean value along with small harmonic fluctuations. Hamilton's principle is employed for this gyroscopic system to derive three coupled non-linear equations of motion. Longitudinal inertia is neglected under the quasi-static stretch assumption and two integro-partial-differential equations are obtained. With introducing a complex variable, the equations of motion is presented in the form of a single, complex equation. The method of multiple scales is applied directly to the resulting equation and the approximate closed-form solution is obtained. Stability boundaries for the steady-state response are formulated and the frequency-response curves are drawn. A number of case studies are considered and the numerical simulations are presented to highlight the effects of system parameters on the linear and non-linear natural frequencies, mode shapes, limit cycles and the frequency-response curves of the system.

Keywords: non-linear vibrations; multiple-scale method; axially moving rotor.

1. Introduction

Many technological devices such as drill strings and cardan shafts can be modeled as an axially moving rotor.

The literature on the vibration and stability of axially moving systems is quite extensive. Wickert (1992) investigated the non-linear vibration of an axially moving tensioned beam in the sub- and super-critical transport speed ranges. Stylianou and Tabarrok (1994) investigated the effects of the system parameters on the stability boundaries of an axially moving beam via finite element method. Oz *et al*. (1998) employed a perturbation technique to develop an outer solution, and then examined the transition behavior of the system from a string model to a beam one. Pellicano and Zirilli (1998) did not consider the flexural stiffness and carried out an investigation on the non-linear vibration of

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an axially moving beam. Pakdemirli and Ozkaya (1998) used the matched asymptotic expansion to solve the boundary layer problem of an axially moving beam undergoing a constant axial speed. Oz and Pakdemirli (1999) used the multiple-scale method to obtain the approximate analytic solution and stability boundaries of an axially moving beam. The super-critical dynamics of an axially moving string supported additionally by an elastic foundation was studied by Parker (1999). Ozkaya and Pakdemirli (2000) investigated the boundary layer problem of an axially moving beam, in which the flexural stiffness was assumed as a small value. Chung *et al.* (2001) employed the Galerkin method to study the vibrations of an axially moving string. Non-linear parametric vibration and stability of an axially moving visco-elastic Rayleigh beam were investigated by Ghayesh and Balar (2008). The non-linear vibration of an axially moving visco-elastic string guided by a visco-elastic foundation was investigated by Ghayesh (2008). Two dynamic models of axially moving beams were considered by Chen and Yang (2005) to investigate the differences in the dynamical behavior of these two models. Galerkin's technique was employed by Zhang and Chen (2005) to investigate the chaotic motion and bifurcation of a visco-elastic moving string. Chen and Zhao (2005) investigated the non-linear vibration of an axially moving beam under a low transport speed. Chen *et al.* (2005) analyzed the transverse vibrations of an axially moving string via modified finite difference method. Zhang (2008) used the Galerkin method to analyze the bifurcation and chaos of an axially moving visco-elastic string. Chen (2006) investigated the coupled planar vibration of axially moving string via energetics. Chen and Yang (2006) used the multiple-scale method to solve the equation of motion of an axially moving beam constrained by hybrid supports. Shin *et al.* (2006) used the Galerkin method to analyze the dynamical response of an axially moving membrane considering both in-plane and out-of-plane vibrations. The stability characteristics of an axially accelerating string, additionally supported by an elastic foundation were investigated by Ghayesh (2009).

In the present study, the non-linear vibration and stability of an axially moving rotor is investigated. The axial velocity of the rotor is assumed to involve a mean value along with small harmonic variations. First, the parametric vibration of the system, undergoing the time-dependent velocity is addressed and the linear and non-linear natural frequencies along with complex eigenfunctions of the system are obtained via the method of multiple scales. Second, the forced vibration and the stability of the system are investigated analytically. Numerical results are included to show the effects of system parameters such as the mean velocity, the amplitude of speed fluctuation, the detuning parameter and the flexural stiffness on the linear and non-linear natural frequencies, mode shapes, frequency-response curves and amplitudes of the rotor system.

2. Free vibration[‡]

2.1 Equations of motion

A continuous rotor which is supported simply is shown in Fig. 1. This system is moving axially with the time-dependent speed $v^*(t)$, including a mean velocity along with small harmonic fluctuations.

[‡]Since the axial speed is time-dependent, "Parametric vibration" may be found as a better title for this section. The current title is chosen in a way to make clear contrast between this section and Section 3.

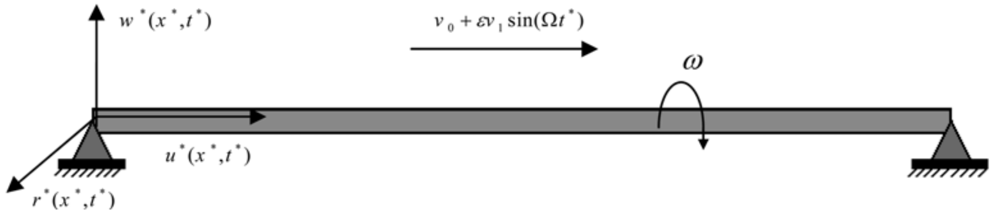

Fig. 1 An axially moving rotor under a time-variant axial velocity

Under Euler-Bernoulli beam theory assumption and the constant rotating speed of the shaft, the kinetic energy of the system is given by

$$\begin{aligned} T_t = {} & \frac{1}{2}\rho A\int_0^{\ell}\left\{\left[\frac{\partial u^*}{\partial t^*}+v^*\left(1+\frac{\partial u^*}{\partial x^*}\right)\right]^2+\left(\frac{\partial w^*}{\partial t^*}+v^*\frac{\partial w^*}{\partial x^*}\right)^2+\left(\frac{\partial r^*}{\partial t^*}+v^*\frac{\partial r^*}{\partial x^*}\right)^2\right\}dx \\ & -\frac{1}{2}\omega I_p\int_0^{\ell}\left\{\frac{\partial w^*}{\partial x^*}\left(\frac{\partial^2 r^*}{\partial t^*\partial x^*}+v^*\frac{\partial^2 r^*}{\partial x^{*2}}\right)-\frac{\partial r^*}{\partial x^*}\left(\frac{\partial^2 w^*}{\partial t^*\partial x^*}+v^*\frac{\partial^2 w^*}{\partial x^{*2}}\right)\right\}dx+\frac{1}{2}\int_0^{\ell}I_p\omega^2 dx \end{aligned} \tag{1}$$

where ρA, $u^*, v^*, w^*, r^*, \omega$ and I_p are the constant mass per unit length, longitudinal displacement, harmonic axial speed, the first and the second transverse displacements, rotating speed of the shaft, and polar moment of inertia of the rotor per unit length, respectively.

The expression for potential energy, related to the material properties of the rotor can be written as

$$U = \int_0^{\ell}\left[pe_{xx}+\frac{1}{2}EAe_{xx}^2+\frac{1}{2}EI\left(\frac{\partial^2 w^*}{\partial x^{*2}}\right)^2+\frac{1}{2}EI\left(\frac{\partial^2 r^*}{\partial x^{*2}}\right)^2\right]dx \tag{2}$$

in which, p, EA and EI are respectively the pretension, axial stiffness and flexural rigidity of the rotor.

The non-linear strain in Eq. (2) may be expressed as

$$e_{xx} = \frac{\partial u^*}{\partial x^*}+\frac{1}{2}\left(\frac{\partial w^*}{\partial x^*}\right)^2+\frac{1}{2}\left(\frac{\partial r^*}{\partial x^*}\right)^2 \tag{3}$$

Using Eqs. (1)-(3), Hamilton's principle leads to

$$\begin{aligned} & \left(\frac{\partial^2 w}{\partial t^2}+v^2\frac{\partial^2 w}{\partial x^2}+\frac{dv}{dt}\frac{\partial w}{\partial x}+2v\frac{\partial^2 w}{\partial x\partial t}\right)-\omega_p^2\left(\frac{\partial^3 r}{\partial t\partial x^2}+v\frac{\partial^3 r}{\partial x^3}\right)-\frac{\partial^2 w}{\partial x^2} \\ & -v_1^2\left(\frac{3}{2}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial x^2}+\frac{\partial w}{\partial x}\frac{\partial^2 r}{\partial x^2}\frac{\partial r}{\partial x}+\frac{\partial^2 w}{\partial x^2}\frac{\partial u}{\partial x}+\frac{1}{2}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial r}{\partial x}\right)^2\right)+v_f^2\frac{\partial^4 w}{\partial x^4} = 0 \end{aligned} \tag{4}$$

$$\begin{aligned} & \left(\frac{\partial^2 r}{\partial t^2}+v^2\frac{\partial^2 r}{\partial x^2}+\frac{dv}{dt}\frac{\partial r}{\partial x}+2v\frac{\partial^2 r}{\partial x\partial t}\right)+\omega_p^2\left(\frac{\partial^3 w}{\partial t\partial x^2}+v\frac{\partial^3 w}{\partial x^3}\right)-\frac{\partial^2 r}{\partial x^2} \\ & -v_1^2\left(\frac{3}{2}\frac{\partial^2 r}{\partial x^2}\left(\frac{\partial r}{\partial x}\right)^2+\frac{\partial r}{\partial x}\frac{\partial^2 u}{\partial x^2}+\frac{\partial r}{\partial x}\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial x}+\frac{\partial^2 r}{\partial x^2}\frac{\partial u}{\partial x}+\frac{1}{2}\frac{\partial^2 r}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2\right)+v_f^2\frac{\partial^4 r}{\partial x^4} = 0 \end{aligned} \tag{5}$$

where the following dimensionless quantities have been used

$$v = \frac{v^*}{\sqrt{\frac{p}{\rho A}}}, \quad v_1 = \sqrt{\frac{EA}{\rho}}, \quad v_f = \sqrt{\frac{EA}{PL^2}}, \quad \omega_p^2 = \frac{I_p \omega}{\rho \sqrt{\rho A p}}$$

$$u = \frac{u^*}{L}, \quad w = \frac{w^*}{L}, \quad r = \frac{r^*}{L}, \quad t = t^* \sqrt{\frac{p}{\rho A L^2}} \tag{6}$$

Under the quasi-static stretch assumption (Wickert 1992, Oz and Pakdemirli 1999), one has

$$e_{xx} = \frac{1}{2}\int_0^1 \left(\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial x}\right)^2 \right) dx \tag{7}$$

Using Eqs. (4), (5) and (7), one gets

$$\begin{gathered} \left(\frac{\partial^2 w}{\partial t^2} + v^2 \frac{\partial^2 w}{\partial x^2} + \frac{dv}{dt}\frac{\partial w}{\partial x} + 2v \frac{\partial^2 w}{\partial x \partial t} \right) - \omega_p^2 \left(\frac{\partial^3 r}{\partial t \partial x^2} + v \frac{\partial^3 r}{\partial x^3} \right) - \frac{\partial^2 w}{\partial x^2} + v_f^2 \frac{\partial^4 w}{\partial x^4} = \\ \frac{1}{2} v_1^2 \frac{\partial^2 w}{\partial x^2} \int_0^1 \left(\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial x}\right)^2 \right) dx \end{gathered} \tag{8}$$

$$\begin{gathered} \left(\frac{\partial^2 r}{\partial t^2} + v^2 \frac{\partial^2 r}{\partial x^2} + \frac{dv}{dt}\frac{\partial r}{\partial x} + 2v \frac{\partial^2 r}{\partial x \partial t} \right) + \omega_p^2 \left(\frac{\partial^3 w}{\partial t \partial x^2} + v \frac{\partial^3 w}{\partial x^3} \right) - \frac{\partial^2 r}{\partial x^2} + v_f^2 \frac{\partial^4 r}{\partial x^4} = \\ \frac{1}{2} v_1^2 \frac{\partial^2 r}{\partial x^2} \int_0^1 \left(\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial x}\right)^2 \right) dx \end{gathered} \tag{9}$$

Introducing the complex variable $z = w + ir$, Eqs. (8) and (9) may be rewritten in the following form

$$\begin{gathered} \frac{\partial^2 z}{\partial t^2} + (v^2 - 1)\frac{\partial^2 z}{\partial x^2} + \frac{dv}{dt}\frac{\partial z}{\partial x} + 2v \frac{\partial^2 z}{\partial x \partial t} + i\omega_p^2 \left(\frac{\partial^3 z}{\partial t \partial x^2} + v \frac{\partial^3 z}{\partial x^3} \right) + v_f^2 \frac{\partial^4 z}{\partial x^4} = \\ \frac{1}{2} v_1^2 \frac{\partial^2 z}{\partial x^2} \int_0^1 \left(\frac{\partial \bar{z}}{\partial x}\frac{\partial z}{\partial x} \right) dx \end{gathered} \tag{10}$$

Supposing the transformation $z = \sqrt{\varepsilon} z^*$, making the non-linear terms in Eq. (10) weak, gives[§]

$$\begin{gathered} \frac{\partial^2 z}{\partial t^2} + (v^2 - 1)\frac{\partial^2 z}{\partial x^2} + \frac{\partial v}{\partial t}\frac{\partial z}{\partial x} + 2v \frac{\partial^2 z}{\partial x \partial t} + i\omega_p^2 \left(\frac{\partial^3 z}{\partial t \partial x^2} + v \frac{\partial^3 z}{\partial x^3} \right) + v_f^2 \frac{\partial^4 z}{\partial x^4} = \\ \frac{1}{2} \varepsilon v_1^2 \frac{\partial^2 z}{\partial x^2} \int_0^1 \left(\frac{\partial \bar{z}}{\partial x}\frac{\partial z}{\partial x} \right) dx \end{gathered} \tag{11}$$

[§]The sign * has been omitted.

2.2 Multiple-scale analysis

The method of multiple scales is employed in this section to obtain the solution of the system equations. In the method of multiple scales, the solution is generally assumed in the following form (Nayfeh 1981, 1993, Kevorkian and Cole 1981)

$$z(x,t;\varepsilon) = \sum_{i=0}^{1} \varepsilon^i z_i(x, T_0, T_1) + O(\varepsilon^2) \tag{12}$$

The axial velocity is assumed to include a mean value, along with small harmonic variations, i.e.

$$v = v_0 + \varepsilon v_1 \sin\Omega t \tag{13}$$

where v_0, v_1 and Ω are the mean velocity, the amplitude of harmonic fluctuations and the frequency of fluctuations, respectively.

Substitution of Eqs. (12) and (13) into Eq. (11) leads to the following equations of order one and epsilon

$$O(\varepsilon^0):\ \frac{\partial^2 z_0}{\partial T_0^2} + (v_0^2 - 1)\frac{\partial^2 z_0}{\partial x^2} + 2v_0 \frac{\partial^2 z_0}{\partial T_0 \partial x} + i\omega_p^2\left(\frac{\partial^3 z_0}{\partial x^2 \partial T_0} + v_0 \frac{\partial^3 z_0}{\partial x^3}\right) + v_f^2 \frac{\partial^4 z_0}{\partial x^4} = 0 \tag{14}$$

$$\begin{aligned} O(\varepsilon):\ & \frac{\partial^2 z_1}{\partial T_0^2} + (v_0^2 - 1)\frac{\partial^2 z_1}{\partial x^2} + 2v_0 \frac{\partial^2 z_1}{\partial T_0 \partial x} + i\omega_p^2\left(\frac{\partial^3 z_1}{\partial x^2 \partial T_0} + v_0 \frac{\partial^3 z_1}{\partial x^3}\right) + v_f^2 \frac{\partial^4 z_1}{\partial x^4} = \\ & -2\frac{\partial^2 z_0}{\partial T_0 \partial T_1} - 2v_0 v_1 \sin\Omega t \times \frac{\partial^2 z_0}{\partial x^2} - 2v_0 \frac{\partial^2 z_0}{\partial T_1 \partial x} - v_1 \Omega \cos\Omega t \times \frac{\partial z_0}{\partial x} \\ & -2v_1 \sin\Omega t \times \frac{\partial^2 z_0}{\partial T_0 \partial x} - i\omega_p^2 \frac{\partial^3 z_0}{\partial x^2 \partial T_1} + \frac{1}{2} v_1^2 \frac{\partial^2 z_0}{\partial x^2} \int_0^1 \left(\frac{\partial \bar{z}_0}{\partial x}\frac{\partial z_0}{\partial x}\right) dx \end{aligned} \tag{15}$$

The solution of the equation order one (Eq. (14)) may be assumed in the following complex form

$$z_0(x,t) = A_n(T_1)\exp(i\omega_n T_0) z_n(x) \tag{16}$$

Inserting Eq. (16) into Eq. (14), for a simply-supported system, the spatial functions $z_n(x)$ must satisfy the following equations and boundary conditions

$$v_f^2 \frac{\partial^4 z_n}{\partial x^4} + i\omega_p^2 v_0 \frac{\partial^3 z_n}{\partial x^3} + (v_0^2 - 1 - \omega_p^2 \omega_n)\frac{\partial^2 z_n}{\partial x^2} + 2iv_0\omega_n \frac{\partial z_n}{\partial x} - \omega_n^2 z_n = 0 \tag{17}$$

$$z_n(0) = 0,\ z_n(1) = 0,\ \frac{\partial^2 z_n}{\partial x^2}(0) = 0,\ \frac{\partial^2 z_n}{\partial x^2}(1) = 0 \tag{18}$$

The solution of Eq. (17) can be assumed as

$$z_n(x) = c_{1n}[e^{i\theta_{1n}x} + c_{2n}e^{i\theta_{2n}x} + c_{3n}e^{i\theta_{3n}x} + c_{4n}e^{i\theta_{4n}x}] \tag{19}$$

Substituting Eq. (19) into the boundary conditions (Eq. (18)) and making subsequent determinant of coefficients equal to zero yields

$$
\begin{aligned}
&(\theta_{1n}^2-\theta_{2n}^2)(\theta_{3n}^2-\theta_{4n}^2)[e^{i(\theta_{1n}-\theta_{2n})}+e^{i(\theta_{3n}-\theta_{4n})}] \\
&+(\theta_{2n}^2-\theta_{4n}^2)(\theta_{3n}^2-\theta_{1n}^2)[e^{i(\theta_{1n}-\theta_{3n})}+e^{i(\theta_{2n}-\theta_{4n})}] \\
&+(\theta_{1n}^2-\theta_{4n}^2)(\theta_{2n}^2-\theta_{3n}^2)[e^{i(\theta_{2n}-\theta_{3n})}+e^{i(\theta_{1n}-\theta_{4n})}]=0
\end{aligned}
\tag{20}
$$

Eqs. (17) and (20) should be solved together numerically to obtain the linear natural frequencies of the system.

So, the mode shape equation of the system can be obtained as (from Eqs. (18) and (19))

$$
\begin{aligned}
z_n(x)=c_{1n}\Bigg\{&\left(-1+\frac{(\theta_{4n}^2-\theta_{3n}^2)(\theta_{4n}^2-\theta_{1n}^2)(e^{i\theta_{3n}}-e^{i\theta_{1n}})-(\theta_{4n}^2-\theta_{1n}^2)(\theta_{4n}^2-\theta_{2n}^2)(e^{i\theta_{2n}}-e^{i\theta_{1n}})}{(\theta_{4n}^2-\theta_{2n}^2)(\theta_{4n}^2-\theta_{3n}^2)(e^{i\theta_{3n}}-e^{i\theta_{2n}})}\right)e^{i\theta_{4n}x} \\
&-\frac{(\theta_{4n}^2-\theta_{1n}^2)(e^{i\theta_{2n}}-e^{i\theta_{1n}})}{(\theta_{4n}^2-\theta_{3n}^2)(e^{i\theta_{2n}}-e^{i\theta_{3n}})}e^{i\theta_{3n}x}+\frac{(\theta_{4n}^2-\theta_{1n}^2)(e^{i\theta_{3n}}-e^{i\theta_{1n}})}{(\theta_{4n}^2-\theta_{2n}^2)(e^{i\theta_{2n}}-e^{i\theta_{3n}})}e^{i\theta_{2n}x}+e^{i\theta_{1n}x}\Bigg\}
\end{aligned}
\tag{21}
$$

Substitution of Eq. (16) into (15) gives

$$
\begin{aligned}
O(\varepsilon):\ &\frac{\partial^2 z_1}{\partial T_0^2}+(v_0^2-1)\frac{\partial^2 z_1}{\partial x^2}+2v_0\frac{\partial^2 z_1}{\partial T_0\partial x}+i\omega_p^2\left(\frac{\partial^3 z_1}{\partial x^2\partial T_0}+v_0\frac{\partial^3 z_1}{\partial x^3}\right)+v_f^2\frac{\partial^4 z_1}{\partial x^4}= \\
&\exp(i\omega_n T_0)\left[-2i\omega_n A_n' z_n(x)-2v_0A_n'z_n'(x)-i\omega_p^2A_n'z_n''(x)\right] \\
&+A_n\exp(i(\Omega+\omega_n)T_0)\left[iv_1v_0z_n''(x)-\frac{1}{2}v_1\Omega z_n'(x)-v_1\omega_n z_n'(x)\right] \\
&+A_n\exp(i(\omega_n-\Omega)T_0)\left[-iv_1v_0z_n''(x)-\frac{1}{2}v_1\Omega z_n'(x)+v_1\omega_1 z_n'(x)\right] \\
&+A_n^2\bar{A}_n\exp(i\omega_n T_0)\left[\frac{1}{2}v_1^2z_n''(x)\int_0^1 z_n'(x)\bar{z}_n'(x)dx\right]
\end{aligned}
\tag{22}
$$

2.3 Solvability condition

In this section, two possible cases ($\Omega\neq 0$ and Ω near to zero) are considered and the non-linear natural frequencies and the limit cycles of the system are obtained.

When the frequency of the axial velocity is away from zero, the displacement, z_1, can be assumed as (Oz and Pakdemirli 1999)

$$z_1(x,T_0,T_1)=\varphi_n(x,T_1)\exp(i\omega_n T_0) \tag{23}$$

then Eq. (22) becomes

$$
\begin{aligned}
&\frac{\partial^2 z_1}{\partial T_0^2}+(v_0^2-1)\frac{\partial^2 z_1}{\partial x^2}+2v_0\frac{\partial^2 z_1}{\partial T_0\partial x}+i\omega_p^2\left(\frac{\partial^3 z_1}{\partial x^2\partial T_0}+v_0\frac{\partial^3 z_1}{\partial x^3}\right)+v_f^2\frac{\partial^4 z_1}{\partial x^4}= \\
&\exp(i\omega_n T_0)\left[-2A_n'i\omega_n z_n(x)-i\omega_p^2A_n'z_n''(x)-2v_0A_n'z_n'(x)\right] \\
&+A_n^2\bar{A}_n\exp(i\omega_n T_0)\left[\frac{1}{2}v_1^2z_n''(x)\int_0^1 z_n'(x)\bar{z}_n'(x)dx\right]
\end{aligned}
\tag{24}
$$

Substituting Eq. (23) into (24) results in

$$v_f^{\ 2}\frac{\partial^4\varphi_n(x,T_1)}{\partial x^4}+i\omega_p^2 v_0\frac{\partial^3\varphi_n(x,T_1)}{\partial x^3}+(v_0^2-1-\omega_p^2\omega_n)\frac{\partial^2\varphi_n(x,T_1)}{\partial x^2}+2iv_0\omega_n\frac{\partial\varphi_n(x,T_1)}{\partial x}$$
$$-\omega_n^{\ 2}\varphi_n(x,T_1)=-2A_n' i\omega_n z_n(x)-i\omega_p^2 A_n' z_n''(x)-2v_0A_n' z_n'(x)+\frac{1}{2}A_n^2\bar{A}_n v_1^2 z_n''(x)\int_0^1 z_n'(x)\bar{z}_n'(x)dx \tag{25}$$

Applying the solvability condition (Nayfeh 1981) to the right-hand side of Eq. (25), one gets

$$A_n' + k_1 A_n^2\bar{A}_n = 0 \tag{26}$$

where

$$k_1=\frac{-\frac{1}{2}v_1^2\int_0^1\bar{z}_n z_n''dx\int_0^1 z_n'\bar{z}_n'dx}{2i\omega_n\int_0^1 z_n\bar{z}_n dx+i\omega_p^2\int_0^1 z_n''\bar{z}_n dx+2v_0\int_0^1 z_n'\bar{z}_n dx} \tag{27}$$

Expressing A_n in a polar form of

$$A_n = \frac{1}{2}a_n\exp(i\alpha_n) \tag{28}$$

and substituting it into Eq. (26), the modulation equations can be obtained as

$$a_n' + \frac{1}{4}k_{1r}a_n^3 = 0$$
$$a_n a_n' + \frac{1}{4}k_{1i}a_n^3 = 0 \tag{29}$$

Numerical verifications show that $k_{1r} = 0$. Then, solving Eq. (29) gives

$$a_n = a_{0n}$$
$$\alpha_n = -\frac{1}{4}k_{1i}(a_{0n})^2 T_1 + \alpha_{0n} \tag{30}$$

where a_{0n}, α_{0n} are constant values. Then, the non-linear natural frequencies of the system can be obtained as

$$(\omega_n)_{nl} = -\frac{1}{4}\varepsilon k_{1i}a_{0n}^2 + \omega_n \tag{31}$$

When the frequency of the axial velocity approaches zero, a detuning parameter, σ, is introduced such that

$$\Omega = \varepsilon\sigma \tag{32}$$

where $\varepsilon << 1$.

Substituting Eq. (32) into (22) results in

$$
\begin{aligned}
O(\varepsilon):\ & \frac{\partial^2 z_1}{\partial T_0^2}+(v_0^2-1)\frac{\partial^2 z_1}{\partial x^2}+2v_0\frac{\partial^2 z_1}{\partial T_0\partial x}+i\omega_p^2\left(\frac{\partial^3 z_1}{\partial x^2\partial T_0}+v_0\frac{\partial^3 z_1}{\partial x^3}\right)+v_f^2\frac{\partial^4 z_1}{\partial x^4}= \\
& \exp(i\omega_n T_0)\left[-2A_n' i\omega_n z_n(x)-i\omega_p^2 A_n' z_n''(x)-2v_0 A_n' z_n'(x)\right] \\
& +A_n\exp(i\omega_n T_0)\left[iv_1 v_0 z_n''(x)-\frac{1}{2}v_1\Omega z_n'(x)-v_1\omega_n z_n'(x)\right]\exp(i\sigma T_1) \\
& +A_n\exp(i\omega_n T_0)\left[-iv_1 v_0 z_n''(x)-\frac{1}{2}v_1\Omega z_n'(x)+v_1\omega_1 z_n'(x)\right]\exp(-i\sigma T_1) \\
& +A_n^2\overline{A}_n\exp(i\omega_n T_0)\left[\frac{1}{2}v_1^2 z_n''(x)\int_0^1 z_n'(x)\overline{z}_n'(x)dx\right]
\end{aligned} \tag{33}
$$

From applying the solvability condition to Eq.(33), the following equation is obtained

$$A_n'+k_1A_n^2\overline{A}_n+k_2\exp(i\sigma T_1)A_n+k_3\exp(-i\sigma T_1)A_n=0 \tag{34}$$

in which

$$k_2=\frac{v_1\left(\frac{1}{2}\Omega+\omega_n\right)\int_0^1 z_n'\overline{z}_n dx-iv_0v_1\int_0^1 z_n''\overline{z}_n dx}{2i\omega_n\int_0^1 z_n\overline{z}_n dx+i\omega_p^2\int_0^1 z_n''\overline{z}_n dx+2v_0\int_0^1 z_n'\overline{z}_n dx} \tag{35}$$

$$k_3=\frac{v_1\left(\frac{1}{2}\Omega-\omega_n\right)\int_0^1 z_n'\overline{z}_n dx+iv_0v_1\int_0^1 z_n''\overline{z}_n dx}{2i\omega_n\int_0^1 z_n\overline{z}_n dx+i\omega_p^2\int_0^1 z_n''\overline{z}_n dx+2v_0\int_0^1 z_n'\overline{z}_n dx} \tag{36}$$

Numerical verifications show that the imaginary parts of k_2 and k_3 are equal to zero.

Substituting Eq. (28) into (34) and separating the real and imaginary parts, the following equations are obtained

$$a_n' = 0 \tag{37}$$

$$a_n' + \frac{1}{4}k_{1i}a_n^2+(k_{2r}-k_{3r})\sin\sigma T_1 = 0 \tag{38}$$

After solving and simplifying Eqs. (37) and (38), one gets

$$(\omega_n)_{nl} = \varepsilon\left(-\frac{1}{4}k_{1i}a_{0n}^2+(k_{3r}-k_{2r})\sin\sigma T_1\right)+\omega_n \tag{39}$$

3. Forced vibration

3.1 Equations of motion

Suppose a rotor with magnitude and direction of the eccentricity of the center of gravity constant with respect to x. Therefore, the virtual work of this eccentricity takes the form**

$$\delta w_e = \rho A e \omega^2 dx(\cos(\omega t)\delta w + \sin(\omega t)\delta r) \tag{40}$$

Using Hamilton's principle for Eqs. (1)-(3), (7) and (40), the non-dimensional equations of motion for the system take the form

$$\begin{gathered}\left(\frac{\partial^2 w}{\partial t^2} + v^2\frac{\partial^2 w}{\partial x^2} + \frac{dv}{dt}\frac{\partial w}{\partial x} + 2v\frac{\partial^2 w}{\partial x \partial t}\right) - \omega I_p^*\left(\frac{\partial^3 r}{\partial t \partial x^2} + v\frac{\partial^3 r}{\partial x^3}\right) - \frac{\partial^2 w}{\partial x^2} + {v_f}^2\frac{\partial^4 w}{\partial x^4} = \\ \frac{1}{2}{v_1}^2\frac{\partial^2 w}{\partial x^2}\int_0^1\left((\frac{\partial w}{\partial x})^2 + (\frac{\partial r}{\partial x})^2\right)dx + \varepsilon^{3/2}\omega_e\omega^2\cos(\omega t)\end{gathered} \tag{41}$$

$$\begin{gathered}\left(\frac{\partial^2 r}{\partial t^2} + v^2\frac{\partial^2 r}{\partial x^2} + \frac{dv}{dt}\frac{\partial r}{\partial x} + 2v\frac{\partial^2 r}{\partial x \partial t}\right) + \omega I_p^*\left(\frac{\partial^3 w}{\partial t \partial x^2} + v\frac{\partial^3 w}{\partial x^3}\right) - \frac{\partial^2 r}{\partial x^2} + {v_f}^2\frac{\partial^4 r}{\partial x^4} = \\ \frac{1}{2}{v_1}^2\frac{\partial^2 r}{\partial x^2}\int_0^1\left((\frac{\partial w}{\partial x})^2 + (\frac{\partial r}{\partial x})^2\right)dx + \varepsilon^{3/2}\omega_e\omega^2\sin(\omega t)\end{gathered} \tag{42}$$

in which the following new parameters are introduced

$$I_p^* = \frac{I_p}{L\sqrt{\rho A p}}, \quad \omega_e = \frac{\rho A e L}{\varepsilon^{3/2}} \tag{43}$$

Using Eqs. (7), (41) and (42), and writing $z = w + ir$, one gets

$$\begin{gathered}\frac{\partial^2 z}{\partial t^2} + (v^2 - 1)\frac{\partial^2 z}{\partial x^2} + \frac{\partial v}{\partial t}\frac{\partial z}{\partial x} + 2v\frac{\partial^2 z}{\partial x \partial t} + i\omega I_p^*\left(\frac{\partial^3 z}{\partial t \partial x^2} + v\frac{\partial^3 z}{\partial x^3}\right) + {v_f}^2\frac{\partial^4 z}{\partial x^4} = \\ \frac{1}{2}{v_1}^2\frac{\partial^2 z}{\partial x^2}\int_0^1\left(\frac{\partial \bar{z}}{\partial x}\frac{\partial z}{\partial x}\right)dx + \varepsilon^{3/2}\omega_e\omega^2\exp(i\omega t)\end{gathered} \tag{44}$$

Introducing the transformation $z = \sqrt{\varepsilon z}^*$, one gets‡†

$$\begin{gathered}\frac{\partial^2 z}{\partial t^2} + (v^2 - 1)\frac{\partial^2 z}{\partial x^2} + \frac{\partial v}{\partial t}\frac{\partial z}{\partial x} + 2v\frac{\partial^2 z}{\partial x \partial t} + i\omega I_p^*\left(\frac{\partial^3 z}{\partial t \partial x^2} + v\frac{\partial^3 z}{\partial x^3}\right) + {v_f}^2\frac{\partial^4 z}{\partial x^4} = \\ \frac{1}{2}\varepsilon{v_1}^2\frac{\partial^2 z}{\partial x^2}\int_0^1\left(\frac{\partial \bar{z}}{\partial x}\frac{\partial z}{\partial x}\right)dx + \varepsilon\omega_e\omega^2\exp(i\omega t)\end{gathered} \tag{45}$$

**The effect of axial motion on the force due to eccentricity has been neglected.

‡†The sign * has been omitted.

3.2 Method of multiple scales for the primary resonance case

Substituting Eqs. (12) and (13) into Eq. (45), the equations of orders one and epsilon for the primary resonance case ($\omega = \omega_n + \varepsilon\mu$) take the form

$$O(\varepsilon^0):\ \frac{\partial^2 z_0}{\partial T_0^2}+2v_0\frac{\partial^2 z_0}{\partial T_0\partial x}+(v_0^2-1)\frac{\partial^2 z_0}{\partial x^2}+v_f^2\frac{\partial^4 z_0}{\partial x^4}+i\omega_n I_p^*\left(\frac{\partial^3 z_0}{\partial x^2\partial T_0}+v_0\frac{\partial^3 z_0}{\partial x^3}\right)=0 \tag{46}$$

$$\begin{aligned}O(\varepsilon):\ &\frac{\partial^2 z_1}{\partial T_0^2}+2v_0\frac{\partial^2 z_1}{\partial T_0\partial x}+(v_0^2-1)\frac{\partial^2 z_1}{\partial x^2}+v_f^2\frac{\partial^4 z_1}{\partial x^4}+i\omega_n I_p^*\left(\frac{\partial^3 z_1}{\partial x^2\partial T_0}+v_0\frac{\partial^3 z_1}{\partial x^3}\right)=\\&-2\frac{\partial^2 z_0}{\partial T_0\partial T_1}-2v_0v_1\sin\Omega t\times\frac{\partial^2 z_0}{\partial x^2}-2v_0\frac{\partial^2 z_0}{\partial T_1\partial x}-v_1\Omega\cos\Omega t\times\frac{\partial z_0}{\partial x}\\&-2v_1\sin\Omega t\times\frac{\partial^2 z_0}{\partial T_0\partial x}+\frac{1}{2}v_1^2\frac{\partial^2 z_0}{\partial x^2}\int_0^1\left(\frac{\partial \bar{z}_0}{\partial x}\frac{\partial z_0}{\partial x}\right)dx\\&-i\omega_n I_p^*\left(\frac{\partial^3 z_0}{\partial T_1\partial x^2}+v_1\sin\Omega t\times\frac{\partial^3 z_0}{\partial x^3}\right)-i\mu I_p^*\left(\frac{\partial^3 z_0}{\partial T_0\partial x^2}+v_0\frac{\partial^3 z_0}{\partial x^3}\right)\\&+\omega_e\omega_n^2\exp(i\mu T_1)\exp(i\omega_n T_1)\end{aligned} \tag{47}$$

In the above equations, μ is a detuning parameter showing the deviation of the rotating frequency from the natural frequency of the system.

Similar to the procedure followed in Section 2.2 (Eqs. (16)-(21)), the mode shapes and the natural frequencies of the forced system can be obtained via Eq. (46).

Substitution of Eq. (16) into Eq. (47) leads to

$$\begin{aligned}O(\varepsilon):\ &\frac{\partial^2 z_1}{\partial T_0^2}+2v_0\frac{\partial^2 z_1}{\partial T_0\partial x}+(v_0^2-1)\frac{\partial^2 z_1}{\partial x^2}+v_f^2\frac{\partial^4 z_1}{\partial x^4}+i\omega_n I_p^*\left(\frac{\partial^3 z_1}{\partial x^2\partial T_0}+v_0\frac{\partial^3 z_1}{\partial x^3}\right)=\\&\exp(i\omega_n T_0)\Big[-2A_n' i\omega_n z_n(x)-2v_0A_n'z_n'(x)+\frac{1}{2}v_1^2A_n^2\bar{A}_n z_n''(x)\int_0^1 z_n'(x)\bar{z}_n'(x)dx\\&-i\omega_n I_p^*A_n'z_n''(x)+\omega_e\omega_n^2\exp(i\mu T_1)-i\mu I_p^*A_n\left(i\omega_n z_n''(x)+v_0z_n'''(x)\right)\Big]\\&+A_n\exp(i(\Omega+\omega_n)T_0)\left[iv_1v_0z_n''(x)-\frac{1}{2}v_1\Omega z_n'(x)-v_1\omega_n z_n'(x)-\frac{1}{2}\omega_n I_p^*v_1z_n'''(x)\right]\\&+A_n\exp(i(\omega_n-\Omega)T_0)\left[-iv_1v_0z_n''(x)-\frac{1}{2}v_1\Omega z_n'(x)+v_1\omega_1 z_n'(x)+\frac{1}{2}\omega_n I_p^*v_1z_n'''(x)\right]\end{aligned} \tag{48}$$

3.3 Solvability condition and stability

When the axial speed frequency is away from zero, the solvability condition (Nayfeh 1981) of Eq. (48) leads to

$$A_n' + \eta_1 A_n^2\bar{A}_n + \eta_2\mu A_n + \frac{1}{2}\eta_3 e^{i\mu T_1} = 0 \tag{49}$$

where

$$\eta_1 = \frac{-\frac{1}{2}v_1^2\int_0^1 \bar{z}_n z_n''\,dx \int_0^1 z_n'(x)\bar{z}_n'(x)\,dx}{2i\omega_n\int_0^1 z_n\bar{z}_n\,dx + 2v_0\int_0^1 z_n'(x)\bar{z}_n(x)\,dx + i\omega_n I_p^*\int_0^1 \bar{z}_n z_n''\,dx} \tag{50}$$

$$\eta_2 = \frac{-I_p^*(\omega_n\int_0^1 \bar{z}_n z_n''\,dx - iv_0\int_0^1 \bar{z}_n z_n'''\,dx)}{2i\omega_n\int_0^1 z_n\bar{z}_n\,dx + 2v_0\int_0^1 z_n'(x)\bar{z}_n(x)\,dx + i\omega_n I_p^*\int_0^1 \bar{z}_n z_n''\,dx} \tag{51}$$

$$\eta_3 = \frac{-2\omega_e\omega_n^2\int_0^1 \bar{z}_n(x)\,dx}{2i\omega_n\int_0^1 z_n\bar{z}_n\,dx + 2v_0\int_0^1 z_n'(x)\bar{z}_n(x)\,dx + i\omega_n I_p^*\int_0^1 \bar{z}_n z_n''\,dx} \tag{52}$$

By expressing

$$\begin{aligned} A_n &= \frac{1}{2}a_n\exp(i\beta_n) \\ \eta_j &= \eta_{jr} + i\eta_{ji}, \qquad j = 1,2,3 \end{aligned} \tag{53}$$

Eq. (49) takes the form

$$a_n' + \frac{1}{4}\eta_{1r}a_n^3 + \eta_{2r}\mu a_n + \eta_{3r}\cos\gamma_n - \eta_{3i}\sin\gamma_n = 0 \tag{54}$$

$$(\mu - \gamma_n')a_n + \frac{1}{4}\eta_{1i}a_n^3 + \eta_{2i}\mu a_n + \eta_{3r}\sin\gamma_n + \eta_{3i}\cos\gamma_n = 0 \tag{55}$$

where

$$\gamma_n = \mu T_1 - \beta_n \tag{56}$$

For the stationary response, γ_n' and a_n' are equal to zero. Therefore, eliminating γ_n from Eqs. (54) and (55) leads to the relationship between detuning parameter, μ, and a_n, i.e.

$$\mu_{1,2} = \frac{-B \mp \sqrt{B^2 - 4AC}}{2A} \tag{57}$$

where

$$\begin{aligned} A &= (\eta_{2r}^2 + (1 + \eta_{2i})^2)a_n^2 \\ B &= \frac{1}{4}(\eta_{1r}\eta_{2r} + (1 + \eta_{2i})\eta_{1i})a_n^4 \\ C &= \frac{1}{16}(\eta_{1r}^2 + \eta_{1i}^2)a_n^4 - (\eta_{3r}^2 + \eta_{3i}^2) \end{aligned} \tag{58}$$

In what follows, the local stability conditions of the steady-state response will be obtained. Considering the modulation equations (Eqs. (54) and (55)), the Jacobian matrix can be constructed as

$$J = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{59}$$

where

$$
\begin{aligned}
A_{11} &= -\frac{3}{4}\eta_{1r}a_n^2 - \eta_{2r}\mu \\
A_{11} &= -(1+\eta_{2i})\mu a_n - \frac{1}{4}\eta_{1i}a_n^3 \\
A_{21} &= \frac{1}{2}\eta_{1i}a_n + \frac{1}{a_n}\left(\mu(1+\eta_{2i}) + \frac{1}{4}\eta_{1i}a_n^2\right) \\
A_{22} &= -\frac{1}{4}\eta_{1r}a_n^2 - \eta_{2r}\mu
\end{aligned}
\tag{60}
$$

Calculating the eigenvalues of the above matrix (Eq. (59)) results in

$$\lambda^2 - (A_{11} + A_{22})\lambda + (A_{11}A_{22} - A_{12}A_{21}) = 0 \tag{61}$$

Using the Routh-Hurwitz criterion, the stability condition can be obtained as

$$
\begin{aligned}
&A_{11} + A_{22} < 0, \quad \text{and} \\
&A_{11}A_{22} > A_{12}A_{21}
\end{aligned}
\tag{62}
$$

The stability conditions for the first and second detuning parameters, μ_1 and μ_2, are attainable from substituting Eq. (57) into (61).

4. Numerical results

In this section, some numerical results are presented to highlight the effects of system parameters on the linear and non-linear natural frequencies, complex mode shapes, frequency-response curves and amplitudes of the system.

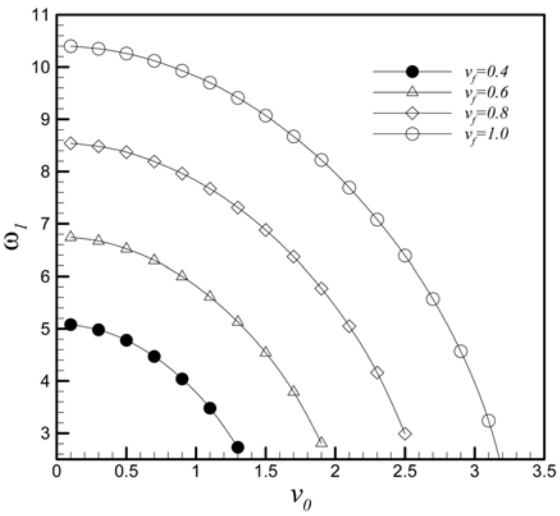

Fig. 2 The first natural frequency of the system as a function of the mean velocity for a selection of v_f; $\omega_p = 0.1$

With increasing the mean velocity, the first linear natural frequency of the system decreases, while the increasing flexural stiffness of the system increases the first natural frequency of the system, as seen in Fig. 2.

Figs. 3 and 4 display the influence of the mean velocity, v_0, on the real and imaginary components of the first complex eigenfunction of the system, $\mathrm{Re}(z_1)$ and $\mathrm{Im}(z_1)$, respectively, indicating a travelling wave-form of the oscillation. The increasing mean velocity increases the amplitude of the vibration of each point of the rotor.

As seen in Fig. 5, any increase in the angular velocity of the rotor, ω_p, increases the first linear

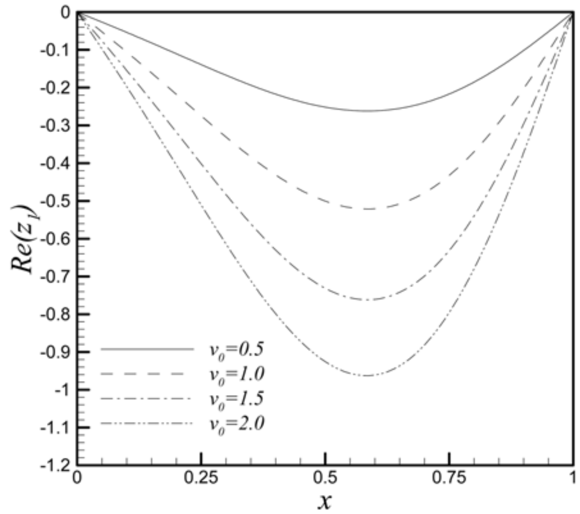

Fig. 3 The real component of the first complex eigenfunction of the system; $v_f = 1$, $\omega_p = 0.1$

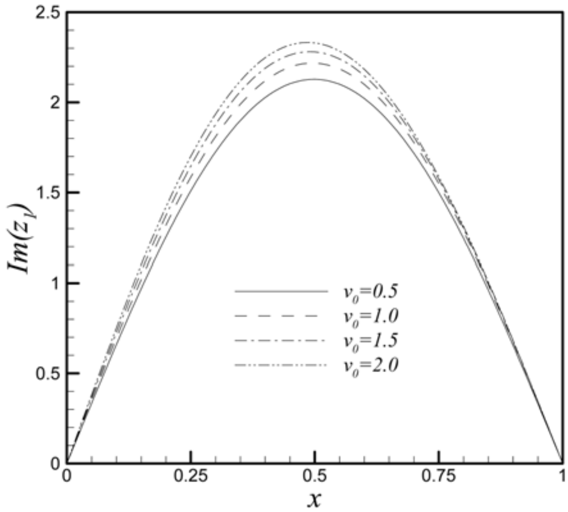

Fig. 4 The imaginary component of the first complex eigenfunction of the system; $v_f = 1$, $\omega_p = 0.1$

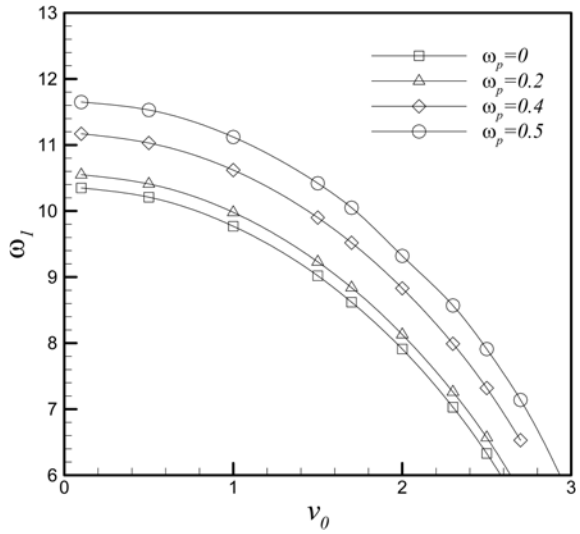

Fig. 5 The first natural frequency of the system versus mean velocity for various ω_p; $v_f = 0.1$

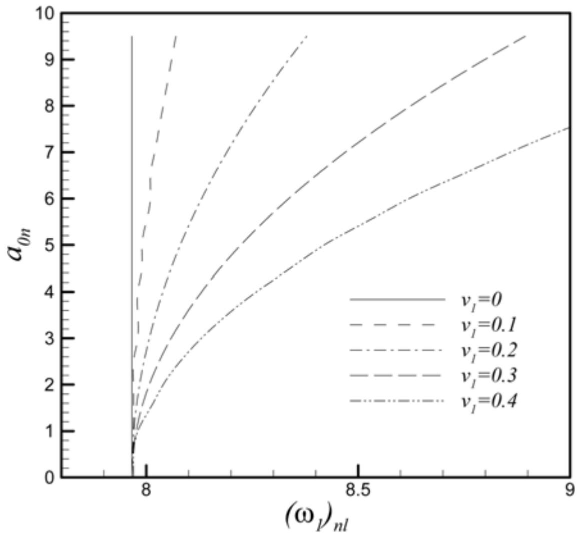

Fig. 6 The first non-linear natural frequency versus a_{01} for various v_1; $\omega_p = 0.1$, $v_0 = 2$, $v_f = 1$, $\varepsilon = 0.05$

natural frequency of the system.

The first non-linear natural frequency of the system is illustrated in Fig. 6 for various values of the speed fluctuation's amplitude, v_1. For $v_1 = 0$, the non-linear natural frequency is independent of a_{0n}; in fact, the system is linear in this case. By increasing v_1, the system displays more non-linearity.

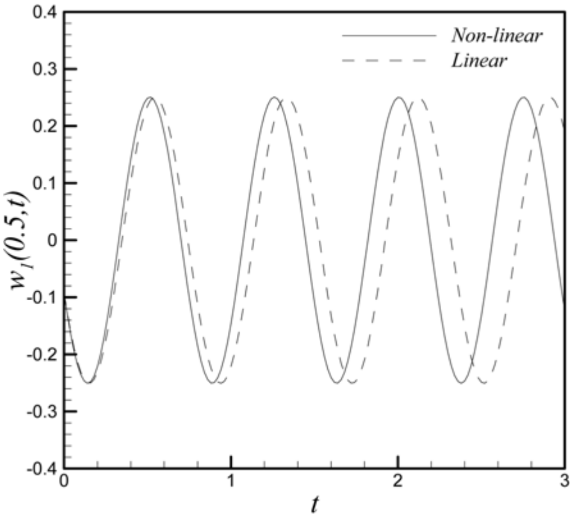

Fig. 7 The linear and non-linear time traces of the system in the w direction :line: non-linear vibration, dashed line: linear vibration; $\omega_p = 0.1$, $v_0 = 2$, $v_f = 1$, $v_1 = 1$, $\varepsilon = 0.05$

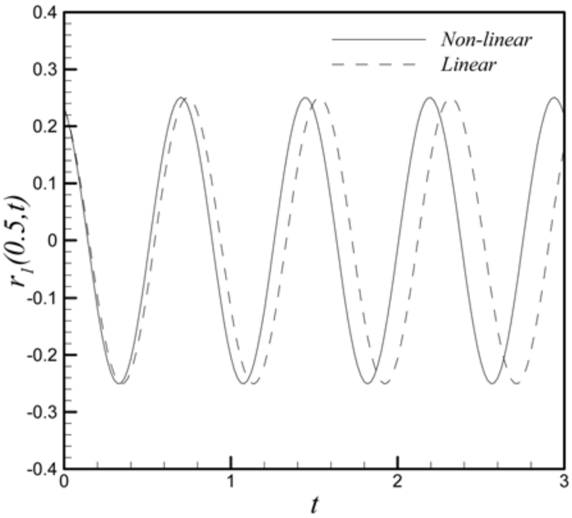

Fig. 8 The linear and non-linear time traces of the system in the r direction :line: non-linear vibration, dashed line: linear vibration; $\omega_p = 0.1$, $v_0 = 2$, $v_f = 1$, $v_1 = 1$, $\varepsilon = 0.05$

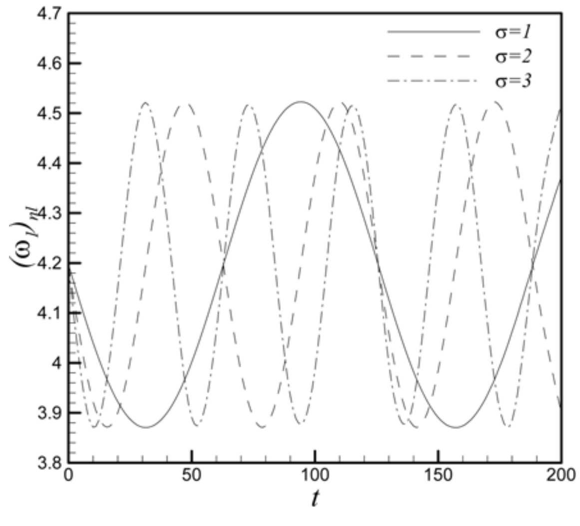

Fig. 9 The first time-dependent non-linear natural frequency for a selection of detuning parameters when the velocity frequency is near to zero; $v_1 = 1$, $v_0 = 3$, $v_f = 1$, $\omega_p = 0.1$; $\varepsilon = 0.05$

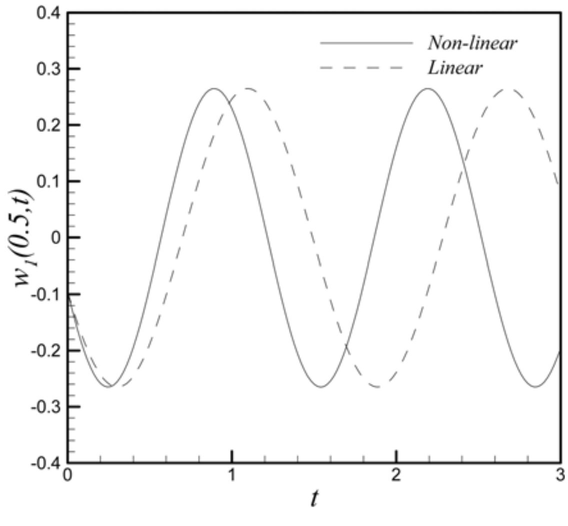

Fig. 10 The linear and non-linear time traces of the system in the w direction for the case where the velocity frequency is close to zero :line: non-linear vibration, dashed line: linear vibration; $\omega_p = 0.1$, $v_0 = 3$, $v_f = 1$, $v_1 = 1$; $\sigma = 1$, $\varepsilon = 0.05$

The linear and non-linear time traces in both the transverse directions (w and r) for the first mode are shown in Figs. 7 and 8, respectively. It is helpful to say that the linear and non-linear limit cycles are obtained using w_1 and $(w_1)_{nl}$, respectively.

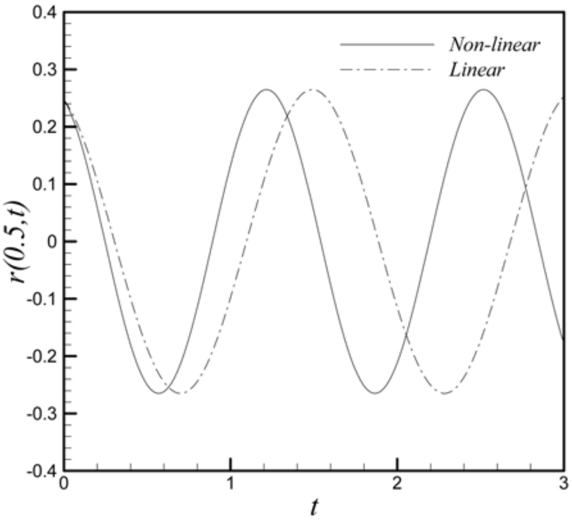

Fig. 11 The linear and non-linear time traces of the system in the r direction for the case where the velocity frequency is close to zero :line: non-linear vibration, dashed line: linear vibration; $\omega_p = 0.1$, $v_0 = 3$, $v_f = 1$, $v_1 = 1$; $\sigma = 1$, $\varepsilon = 0.05$

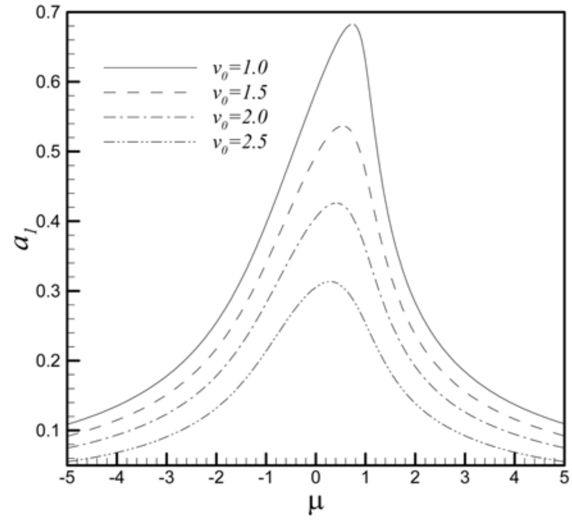

Fig. 12 Frequency-response curve of the system in the first mode for a selection of mean axial speed; $v_f = 1$, $v_1 = 1$; $I_p^* = 0.0015$; $\omega_e = 0.1$

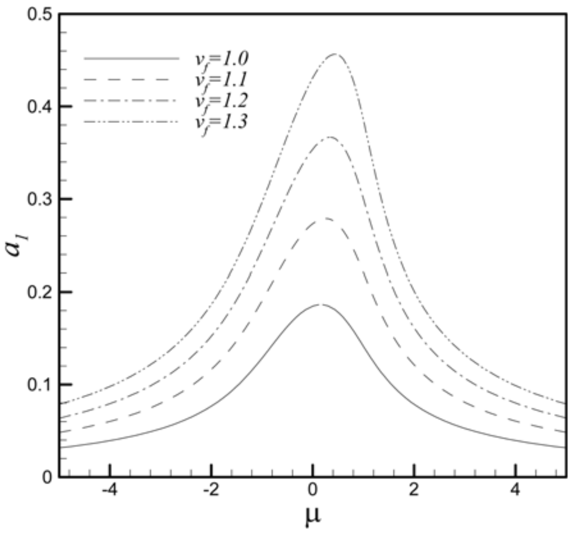

Fig. 13 Frequency-response curve of the system in the first mode for a selection of flexural rigidity of the shaft; $v_0 = 3$, $v_1 = 1$; $I_p^* = 0.0015$; $\omega_e = 0.1$

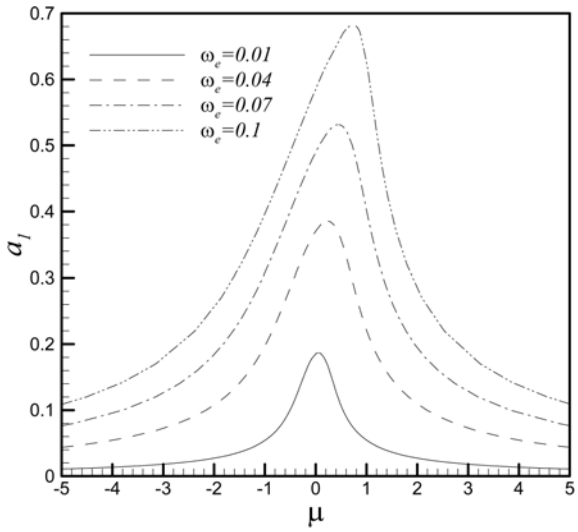

Fig. 14 Frequency-response curve of the system in the first mode for a selection of ω_e; $v_f = 1$, $v_0 = 1$, $v_1 = 1$; $I_p^* = 0.0015$

Fig. 9 shows the dependency of the non-linear natural frequency of the system on time for a selection of detuning parameters, in the case at which the velocity frequency is near to zero. As seen in Fig. 9, the first non-linear natural frequency varies periodically with time. Moreover, the extermum values of the non-linear natural frequency seem to be independent of the detuning parameter chosen, but the time-period of changes in the non-linear natural frequency increases by decreasing the detuning parameter.

The linear and non-linear time traces of the two transverse displacements, w and r, are illustrated in Figs. 10 and 11, when the speed frequency is near to zero. It can be concluded that the time-period of the linear response is larger than the non-linear one.

As seen in Fig. 12, by increasing the mean value of the axial speed, the amplitude of the response decreases. Moreover, the distance between the peak of the response and line $\mu = 0$ gets smaller, as the mean axial speed is increased; the hardening type non-linearity decreases.

Figs. 13 and 14 show that any increase in either the flexural rigidity or the eccentricity increases the amplitude of the response and makes the peak of the response to tend to right more; the hardening type of non-linearity increases.

5. Conclusions

The aim of the study described in this paper was to investigate the parametric and forced non-linear vibration and stability of an axially moving rotor both in the non-resonant and near-resonant cases. The system was traveling axially under a time-dependent velocity. Hamilton's principle was employed to derive the equations of motion. Then, the method of multiple scales was applied directly to the equations of motion and linear and non-linear natural frequencies along with mode shapes and frequency-response characteristics were obtained. Any increase in the mean velocity decreases the linear and non-linear natural frequencies, and the amplitude of the frequency-response curve, whereas increases both the real and imaginary components of the first complex eigenfunctions of the system. The linear natural frequencies of the system are increased by increasing either the flexural rigidity or angular velocity of the rotor. The frequency of vibrations of the non-linear system is larger than the linear one. Increasing either the flexural rigidity or the eccentricity, increases the amplitude of the response of the frequency-response curve and makes the peak of the response to tend to right more.

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