Structural Engineering and Mechanics, Vol. 33, No. 5 (2009) 573-588 573
DOI: http://dx.doi.org/10.12989/sem.2009.33.5.573

An assumed-stress hybrid element for modeling of plates with shear deformations on elastic foundation

Kutlu Darılmaz†

Department of Civil Engineering, Istanbul Technical University 34469, Maslak, Istanbul, Turkey

(Received April 20, 2009, Accepted September 24, 2009)

Abstract. In this paper a four-node hybrid stress element is proposed for analysing arbitrarily shaped plates on a two parameter elastic foundation. The element is developed by combining a hybrid plate stress element and a soil element. The formulation is based on Hellinger-Reissner variational principle in which both inter element compatible boundary displacement and equilibrated stress fields for the plate as well as the foundation are chosen separately. This formulation also allows a low order polynomial interpolation functions. Numerical examples are presented to show that the validity and efficiency of the present element for the plate analysis resting on an elastic foundation. In these examples the effect of soil depth, interaction between closed plates on soil parameters, comparison with Winkler hypothesis is investigated.

Keywords: plate; elastic foundation; assumed stress hybrid formulation; finite element.

1. Introduction

Plates resting on elastic foundations have wide application in modern engineering and pose great technical problems in structural design. As a result, numerous researches involving the calculation and analysis approach for plates on elastic foundation have been presented.

As is known to all, the Winkler model of elastic foundation is the most preliminary in which the vertical displacement is assumed to be proportional to the contact pressure at an arbitrary point, Winkler (1867), Abdalla and Ibrahim (2006). The main disadvantages of this model are the discontinuity in the soil displacement between the soil under the structure and that outside the structure and the necessity of determining the modulus of subgrade reaction k. In order to perform a better model than the Winkler hypothesis Pasternak (1954), Vlasov and Leontev (1966) presented a two-parameter model elastic foundation and analysed beams and slabs on it. Vlasov, in his model, introduced a parameter γ to characterize the vertical displacement distribution in the elastic foundation. Vallabhan and Das (1988) determined the γ parameter as a function of the characteristic of the structure and the foundation using an iterative procedure and named this model as modified Vlasov model. They emphasized that the parameters depend on the properties of the soil and the structure as well as the type and magnitude of the loading and the depth of the soil.

Çelik and Saygun proposed an iterative method to analyze the plates on a two parameter elastic foundation (1999), Ozgan and Daloğlu (2008) studied the effect of transverse shear strains on plates

† Associate Professor, Ph.D., E-mail: kdarilmaz@ins.itu.edu.tr

resting on elastic foundation and presented a four-noded quadrilateral and an eight-noded quadrilateral plate bending element based on Mindlin plate theory which are adopted for the analysis of thin and thick plates resting on elastic foundation using modified Vlasov model. Eratli and Aköz (1997), using the Gateaux differential, developed the mixed element formulation for the thick plates on elastic foundation.

Nogami and Lam (1987) developed a two parameter model for slabs on elastic foundations where the foundation layer is divided into a number of horizontal layers. Ayvaz and Oğuzhan (2008) studied the free vibration analysis of plates resting on elastic foundations using modified Vlasov model. Güler and Celep (1995) studied static and dynamic responses of a thin circular plate on a tensionless elastic foundation.

Vallabhan and Das (1988) developed an iterative procedure by minimizing the total potential energy to obtain a mode shape parameter where the elastic constants of the beam and the mode of loading are used as a function in addition to the thickness of the compressible layer and the elastic constants of the foundation. They also determined the elastic bedding and shear parameter coefficients. Wang, Tham and Cheung (2005) reviews the state-of-the-art of interaction action between structures and supporting soil media.

In this study an efficient assumed stress hybrid finite element formulation is presented for analysing arbitrarily shaped plates on a two parameter elastic foundation. The formulation is based on Hellinger-Reissner variational principle in which both displacements, internal stress and subgrade reaction fields are chosen seperately. This allows a low order polynomial interpolation functions and avoids shear and foundation locking which are typically exhibited by alternative models based on displacement based formulation (Gendy and Saleeb 1999). A number of numerical examples are given to show that the validity and efficiency of the present element.

2. Governing equations and the expressions for plates on an elastic foundation

Total potential energy of the plate-soil system can be written as

$$\Pi = \frac{1}{2}\int_{\Omega}\left(\frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}, 2\frac{\partial^2 w}{\partial x \partial y}\right)[D]\left(\frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}, 2\frac{\partial^2 w}{\partial x \partial y}\right)^T dxdy$$
$$+\frac{1}{2}\int_{\Omega}(Cw^2 + 2C_t(\nabla w)^2)dxdy - \int_{\Omega} qwdxdy \tag{1}$$

C and $2C_t$ in the above equation are the soil parameters and can be defined as

$$C = \int_0^H E_s \frac{(1-\nu_s)}{(1+\nu_s)(1-2\nu_s)}\left(\frac{\partial \phi}{\partial z}\right)^2 dz \tag{2}$$

$$2C_t = \int_0^H G_s \phi^2 dz \tag{3}$$

The derivations of Eqs. (2) and (3) are given in Appendix.

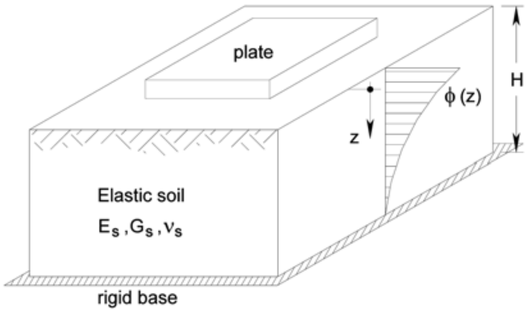
The vertical displacement of the soil is assumed

$$w_z = w(x,y)\phi(z) \tag{4}$$

Where $w(x, y)$ is the deflection of the soil surface and $\phi(z)$ is the function which defines the variation of vertical displacement in the vertical direction. The boundary conditions of mode shape function $\phi(z)$ are

$$\phi(z=0)=1,\ \phi(z=H)=0$$

where H is the height of compressible soil.



The field equation in the plate domain is

$$D\nabla^4 w - 2C_t\nabla^2 w + Cw = q \tag{5}$$

where D is the flexural rigidity of plate, q is the external load on the plate, ∇^4 is the biharmonic and ∇^2 is the Laplace operator.

Outside the plate domain

$$-2C_t\nabla^2 w + Cw = 0 \tag{6}$$

The mode shape function can be expressed as (See Appendix)

$$\phi(z) = \frac{\sinh\left[\gamma\left(1-\frac{z}{H}\right)\right]}{\sinh\gamma} \tag{7}$$

where γ denotes the mode shape parameter. The derivation of Eq. (7) is given in Appendix.

The soil parameters, C is the bedding coefficient and $2C_t$ is the shear coefficient parameter, can be obtained as follows

$$C = \int_0^H E_s\frac{(1-\nu_s)}{(1+\nu_s)(1-2\nu_s)}\left(\frac{\partial\phi}{\partial z}\right)^2 dz = E_s\frac{(1-\nu_s)}{(1+\nu_s)(1-2\nu_s)H}\frac{\gamma(\sinh 2\gamma + 2\gamma)}{4\sinh^2\gamma} \tag{8}$$

$$2C_t = \int_0^H G_s\phi^2 dz = G_s\frac{H}{\gamma}\frac{(\sinh 2\gamma - 2\gamma)}{4\sinh^2\gamma} \tag{9}$$

where E_s, G_s, ν_s are the elastic constants of soil.

The mode shape parameter γ yields as follows

$$\gamma^2 = H^2\frac{(1-2\nu_s)}{2(1-\nu_s)}\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]dxdy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w^2 dxdy} \tag{10}$$

It should be noted that γ depends on the deflection shape of the plate-soil system and needs an iterative solution for obtaining.

3. Finite element formulation

The Hellinger-Reissner variational principle in which stresses and displacements are assumed independently is used. The Hellinger-Reissner functional of linear elasticity allows displacements and stresses to be varied separately. This establishes the master fields. Two slave strain fields appear, one coming from displacements and one from stresses, (Pian and Chen 1982, Washizu 1982).

The general form of Hellinger-Reissner functional can be written as

$$\Pi_{RH} = \int_V \underset{\sim}{\sigma}^T \underset{\sim}{D}\underset{\sim}{u} dV - \frac{1}{2}\int_V \underset{\sim}{\sigma}^T \underset{\sim}{S}\underset{\sim}{\sigma} dV \tag{11}$$

Where $\underset{\sim}{\sigma}$ is the stress vector, $\underset{\sim}{S}$ is the compliance matrix relating strains, $\underset{\sim}{\varepsilon}$, to stress ($\underset{\sim}{\varepsilon} = \underset{\sim}{S}\underset{\sim}{\sigma}$), $\underset{\sim}{D}$ is the differential operator matrix corresponding to the linear strain-displacement relations ($\underset{\sim}{\varepsilon} = \underset{\sim}{D}\underset{\sim}{u}$) and V is the volume of structure.

The approximation for stress and displacements can be incorporated in the functional. The stress field is described in the interior of the element as

$$\underset{\sim}{\sigma} = \underset{\sim}{P}\underset{\sim}{\beta} \tag{12}$$

and a compatible displacement field is described by

$$\underset{\sim}{u} = \underset{\sim}{N}\underset{\sim}{q} \tag{13}$$

where P and N are matrices of stress and displacement interpolation functions, respectively, and β and q are the unknown stress and nodal displacement parameters, respectively. Intra-element equilibrating stresses and compatible displacements are independently interpolated. Since stresses are independent from element to element, the stress parameters are eliminated at the element level and a conventional stiffness matrix results. This leaves only the nodal displacement parameters to be assembled into the global system of equations.

Substituting the stress and displacement approximations Eq. (12), Eq. (13) in the functional Eq. (11)

$$\Pi_{RH} = \underset{\sim}{\beta}^T \underset{\sim}{G}\underset{\sim}{q} - \frac{1}{2}\underset{\sim}{\beta}\underset{\sim}{H}\underset{\sim}{\beta} \tag{14}$$

where

$$\underset{\sim}{H} = \int_V \underset{\sim}{P}^T \underset{\sim}{S}\underset{\sim}{P} dV \tag{15}$$

$$\underset{\sim}{G} = \int \underset{\sim}{P}^T \underset{\sim}{D}\underset{\sim}{N} dV \tag{16}$$

Now imposing stationary conditions on the functional with respect to the stress parameters β gives

$$\underset{\sim}{\beta} = \underset{\sim}{H}^{-1}\underset{\sim}{G}\underset{\sim}{q} \tag{17}$$

Substitution of β in Eq. (14), the functional reduces to

$$\Pi_{RH} = \frac{1}{2}\underset{\sim}{q}^{T}\underset{\sim}{G}^{T}\underset{\sim}{H}^{-1}\underset{\sim}{G}\underset{\sim}{q} = \frac{1}{2}\underset{\sim}{q}^{T}\underset{\sim}{K}\underset{\sim}{q} \tag{18}$$

where

$$\underset{\sim}{K} = \underset{\sim}{G}^{T}\underset{\sim}{H}^{-1}\underset{\sim}{G} \tag{19}$$

is recognized as a stiffness matrix.

For a plate element on a two parameter elastic foundation the functional can be written as

$$\Pi_{RH} = \left[\int_V\left(\underset{\sim}{\sigma}^{T}\underset{\sim}{D}\underset{\sim}{u} - \frac{1}{2}\underset{\sim}{\sigma}\underset{\sim}{S}\underset{\sim}{\sigma}\right)dV\right] - \left[\int_A \underset{\sim}{F}_1\underset{\sim}{w} - \frac{1}{2}\underset{\sim}{F}_1\underset{\sim}{D}_{s1}^{-1}\underset{\sim}{F}_1 dA\right] - \left[\int_A \underset{\sim}{F}_2\underset{\sim}{w} - \frac{1}{2}\underset{\sim}{F}_2\underset{\sim}{D}_{s2}^{-1}\underset{\sim}{F}_2 dA\right] - W \tag{20}$$

The first term in the above equation is the internal strain energy for the plate element, the second and third terms are foundation effect, including bedding and shear effects respectively, the term W denotes the work of external forces.

Differential Eq. (6) is mathematically equal in form to the differential equation of a plate which exhibits only shear behavior ($Gh' = 2C_t$) and having a bedding coefficient C. By using this analogy, an assumed stress soil finite element is developed, combined with an assumed stress plate element and used in the plate domain. Outside the plate domain is idealized by using soil finite elements.

3.1 Soil finite element

While using assumed stress hybrid finite element, the Hellinger-Reissner, two field variational principal in which stresses and displacements are assumed independently is utilized. For a typical soil element the Hellinger-Reissner functional can be written as

$$\Pi_{RH} = \left[\int_A \underset{\sim}{F}_1\underset{\sim}{w} - \frac{1}{2}\underset{\sim}{F}_1\underset{\sim}{D}_{s1}^{-1}\underset{\sim}{F}_1 dA\right] - \left[\int_A \underset{\sim}{F}_2\underset{\sim}{w} - \frac{1}{2}\underset{\sim}{F}_2\underset{\sim}{D}_{s2}^{-1}\underset{\sim}{F}_2 dA\right] \tag{21}$$

Where F_1 and F_2 are subgrade forces, D_{s1} and D_{s2} are the differential operator matrices related to bedding and shear effects, respectively.

The nodal unknowns of the soil element are shown in Fig. 1.

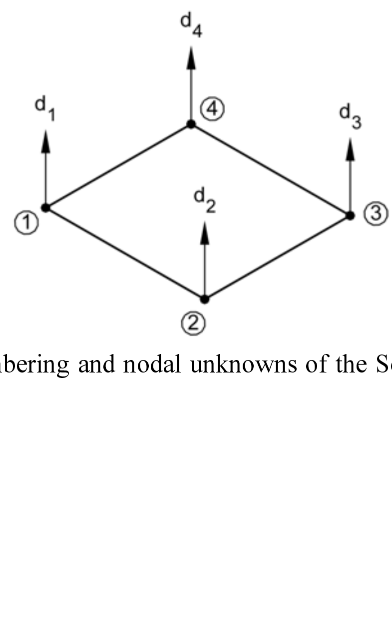


Fig. 1 Node numbering and nodal unknowns of the Soil finite element

The stress field is described in the interior of the element as

$$\underset{\sim}{F}_{s1} = \underset{\sim}{P}_{s1}\underset{\sim}{\beta}_{s1} \tag{22}$$

$$\underset{\sim}{F}_{s2} = \underset{\sim}{P}_{s2}\underset{\sim}{\beta}_{s2} \tag{23}$$

where $\{\beta_s\}$ is the unknown stress parameter. Stress interpolation functions are chosen as

$$\underset{\sim}{P}_{s1} = [1 \;\; x \;\; y \;\; xy] \tag{24}$$

$$\underset{\sim}{P}_{s2} = \begin{bmatrix} 1 & x & y & 0 & 0 \\ 0 & -y & 0 & 1 & x \end{bmatrix} \tag{25}$$

Since stresses are independent from element to element, the stress parameters are eliminated at the element level and this leaves only the nodal displacement parameters like displacement based elements.

Differential operator matrices for bedding and shear effect parts are as follows

$$D_{s1} = C \tag{26}$$

$$D_{s2} = \begin{bmatrix} C_t & 0 \\ 0 & C_t \end{bmatrix} \tag{27}$$

Compatible displacement field is described by

$$\{w\} = [N]\{d\} \tag{28}$$

where $[N]$ is the shape function matrix.

Substituting the stress and displacement approximations in the functional and imposing stationary conditions on the functional with respect to the stress parameters the bedding and shear effect part of the soil element can be obtained as

$$[C] = [G_{s1}]^T[H_{s1}]^{-1}[G_{s1}] \tag{29}$$

$$[C_t] = [G_{s2}]^T[H_{s2}]^{-1}[G_{s2}] \tag{30}$$

where

$$[H_{s1}] = \int_A \underset{\sim}{P}_{s1}^T \underset{\sim}{D}_{s1}^{-1} \underset{\sim}{P}_{s1} dA \;\; ; \;\; [H_{s2}] = \int_A \underset{\sim}{P}_{s2}^T \underset{\sim}{D}_{s2}^{-1} \underset{\sim}{P}_{s2} dA \tag{31}$$

$$[G_{s1}] = \int \underset{\sim}{P}_{s1}^T \underset{\sim}{D}_{s1} \underset{\sim}{N} dA \;\; ; \;\; [G_{s2}] = \int \underset{\sim}{P}_{s2}^T \underset{\sim}{D}_{s2} \underset{\sim}{N} dA \tag{32}$$

3.2 Plate element

The plate element, is taken from a previous study presented by the author, Darılmaz (2005), and corresponds to the Mindlin/Reissner plate theory. Only the assumed stress field which satisfies the equilibrium conditions for the plate part is given here.

$$
\begin{aligned}
M_x &= \beta_1 + \beta_4 y + \beta_6 x + \beta_8 xy \\
M_y &= \beta_2 + \beta_5 x + \beta_7 y + \beta_9 xy \\
M_{xy} &= \beta_3 + \beta_{10} x + \beta_{11} y + \beta_{12} x^2/2 + \beta_{13} y^2/2 \\
Q_x &= \beta_6 + \beta_{11} + \beta_8 y + \beta_{13} y \\
Q_y &= \beta_7 + \beta_{10} + \beta_9 x + \beta_{12} x
\end{aligned}
\tag{33}
$$

The nodal displacements for the plate are chosen as

$$\{d\}_{plate} = \{w_1\ \theta_{x1}\ \theta_{y1}\ w_2\ \theta_{x2}\ \theta_{y2}\ w_3\ \theta_{x3}\ \theta_{y3}\ w_4\ \theta_{x4}\ \theta_{y4}\} \tag{34}$$

Finaly the equilibrium equation of a plate on a two parameter elastic foundation can be written as

$$[K]\{d\} + [C]\{d\} + [C_t]\{d\} = \{P\} \tag{35}$$

Where $[K]$ is the stiffness matrix of the plate element, $[C]$ and $[C_t]$ is the elastic bedding and shear effect matrices of soil element, $\{P\}$ is the applied equivalent load vector of the system.

4. Computation of mode shape parameter

The mode shape parameter γ can be obtained iteratively by using Eq. (10). The terms used in Eq. (10) can be obtained as given in Eq. (36) and Eq. (37).

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w^2 dxdy = \sum_n \frac{1}{C}[d]^T[C][d] \tag{36}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]dxdy = \sum_n \frac{1}{2C_t}[d]^T[C_t][d] \tag{37}$$

Where n denotes number of elements.

5. Numerical examples

Some numerical examples have been used for assessing the accuracy of the element. The results obtained are compared with other researchers' solutions.

Example 1

A rectangular plate on an elastic foundation subjected to uniformly distributed load and concentrated load is analyzed compared with those of Vallabhan *et al.* (1991); Çelik and Saygun (1999); Ozgan and Daloğlu (2008).

The properties of the plate-soil system are as follows. The modulus of elasticity of the subsoil is $E_s = 68950$ kN/m^2, Poisson ratio of the subsoil is $\nu_s = 0.25$, the modulus of elasticity of the plate is $E_p = 20\ 685\ 000$ kN/m^2, Poisson ratio of the plate equals to $\nu_p = 0.20$, the thickness of the plate is considered as $h = 0.1524$ m, and plate dimensions are 9.144×12.192 m. The uniformly distributed load on the plate is 23.94 kN/m^2 and concentrated load is 133.34 kN. The calculations are performed for four depth of the soil, H = 3.048, 6.096, 9.144 and 15.240 m.

Results for uniformly distributed load and concentrated load are given in Table 1 and Table 2. The results for the presented element are in a good agreement with the reference results. As it can be seen from tables, the parameter C decreases as the depth of the soil, H, increases while the parameter $2C_t$ increases with H. The values for displacements, bending moments get closer to each other for both cases as the depth of the soil increases. An increase in the soil depth does not affect the results after certain value of H.

The deformed of the system for two different loads are given Fig. 2.

Table 1 Results for uniformly distributed load

H (m)	Reference	C (kN/m^3)	C_t (kN/m)	γ	$w_{\max}$ (cm)	M_x (kNm/m)
3.048	Çelik, Saygun	27192	13413	0.5766	0.0853	0.0445
	Vallabhan *et al.*	27206	13452	0.5724	0.0872	0.0529
	Ozgan, Daloğlu	27208	13421	0.5750	0.0876	0.0465
	Present study	27207	13423	0.574	0.0865	0.0553
6.096	Çelik, Saygun	13757	25205	0.9194	0.1526	0.2880
	Vallabhan *et al.*	13757	25141	0.9297	0.1524	0.3113
	Ozgan, Daloğlu	13744	25307	0.9010	0.1541	0.2546
	Present study	13751	25249	0.9113	0.1522	0.2902
9.144	Çelik, Saygun	9377	35293	1.2064	0.1893	0.4109
	Vallabhan *et al.*	9430	34753	1.2644	0.1890	0.4224
	Ozgan, Daloğlu	9339	35681	1.1640	0.1917	0.3296
	Present study	9409	34999	1.2382	0.1883	0.4057
15.24	Çelik, Saygun	5954	52332	1.6193	0.2212	0.4671
	Vallabhan *et al.*	6366	47366	1.9419	0.2070	0.4892
	Ozgan, Daloğlu	5928	51374	1.5850	0.2247	0.3228
	Present study	6045	51187	1.6923	0.2176	0.4591

Table 2 Results for concentrated load

H (m)	Reference	C (kN/m^3)	C_t (kN/m)	γ	$w_{\max}$ (cm)	M_x (kNm/m)
3.048	Çelik, Saygun	31898	9456	1.9478	0.0818	15.047
	Vallabhan *et al.*	31610	9565	1.9018	0.0480	12.544
	Present study	31709	9534	1.9217	0.0866	15.665
6.096	Çelik, Saygun	24256	11798	3.5249	0.0845	14.563
	Vallabhan *et al.*	23918	11959	3.4737	0.0975	12.544
	Present study	24124	11865	3.5039	0.0907	15.341
9.144	Çelik, Saygun	23737	12017	5.2434	0.0846	14.510
	Vallabhan *et al.*	23376	12193	5.1669	0.0975	12.544
	Present study	23531	12124	5.1973	0.0909	15.418
15.24	Çelik, Saygun	23710	12030	8.7369	0.0846	14.510
	Vallabhan *et al.*	23350	12205	8.6079	0.0975	12.544
	Present study	23516	12130	8.6627	0.0909	15.394

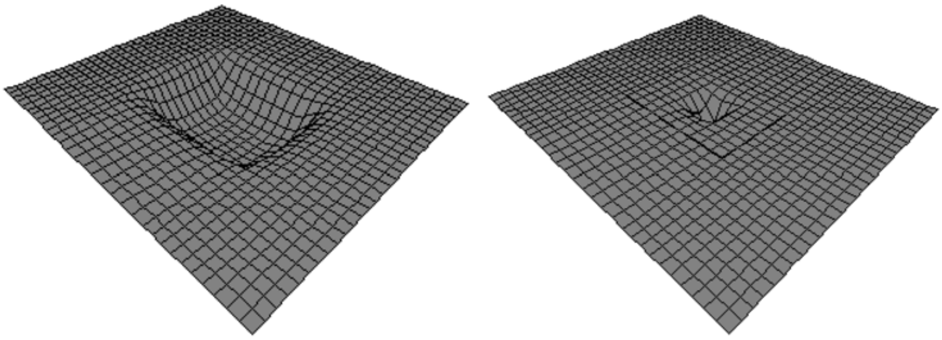


Fig. 2 Deformed shape of the system a) uniform load b) concentrated load

Example 2

The interaction between closed plates can change not only the soil coefficients but also the internal forces which depend on the deformed shape of the plate-soil system. In order to numerically demonstrate this, two similar plates on an elastic soil is analyzed. The geometric properties of the system is given in Fig. 3, dimensions and load values of the columns are given in Table 3.

The properties of the plate-soil system are as follows. The modulus of elasticity of the subsoil is $E_s = 80000$ kN/m^2, Poisson ratio of the subsoil is $\nu_s = 0.25$, the modulus of elasticity of the plate is $E_p = 2 \times 10^7$ kN/m^2, Poisson ratio of the plate is $\nu_p = 0.16$, the thickness of the plate is taken as $h = 0.6$ m. The depth of the soil is $H = 5$ m.

Results at $y = 0$ are compared with the solutions given by Çelik and Saygun (1999) and are found

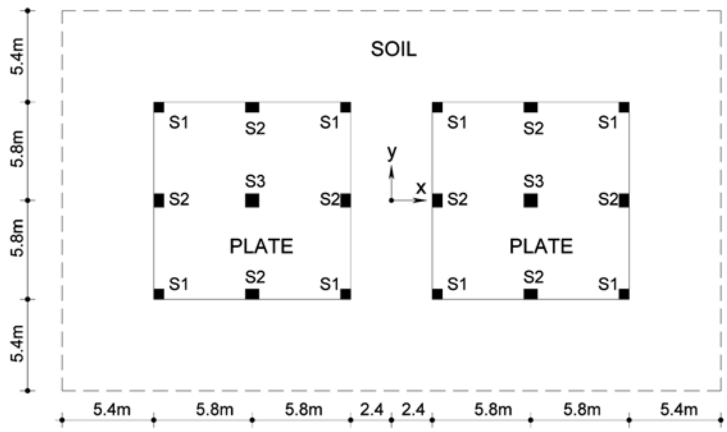


Fig. 3 Two plates on an elastic foundation

Table 3 Dimensions and load values of the columns

	$b \times h$	Load Value (kN)
S1	0.60 m×0.60 m	1200
S2	0.80 m×0.60 m	2000
S3	0.80 m×0.80 m	3200

to be in a good agreement, Fig. 4. Vertical deflection of the system at $y = 5.8$ and $x = 2.4$ are also given in Fig. 5 and Fig. 6, respectively.

The calculated soil parameters are as $C = 19657$ kN/m^3, $C_t = 23167$ kN/m. The obtained mode shape parameter $\gamma = 1.071$ is closed to previos study result $\gamma = 1.066$, Çelik and Saygun (1999). The plate is also analyzed by assuming Winkler hypothesis with obtained bedding coefficient C. It can be observed that the settlements are obtained higher in Winkler soil solutions since Winkler model idealized the soil medium by a number of mutually independent spring elements and two parameter

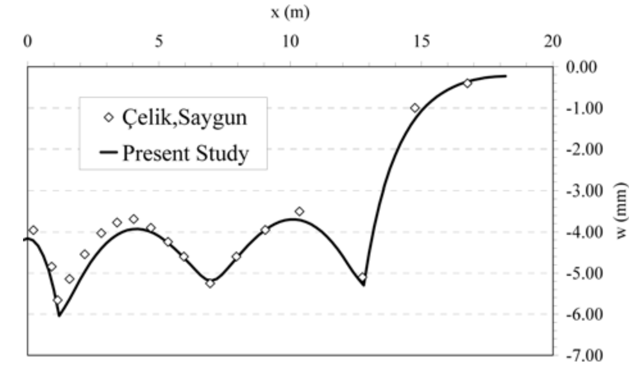


Fig. 4 Vertical Deflection of the system at $y = 0$

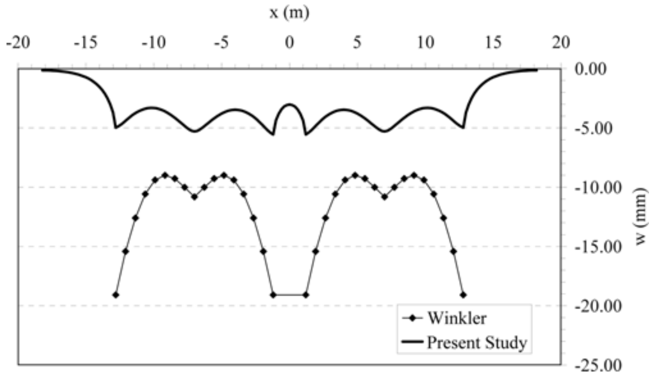


Fig. 5 Vertical Deflection of the system at $y = 5.8$ m

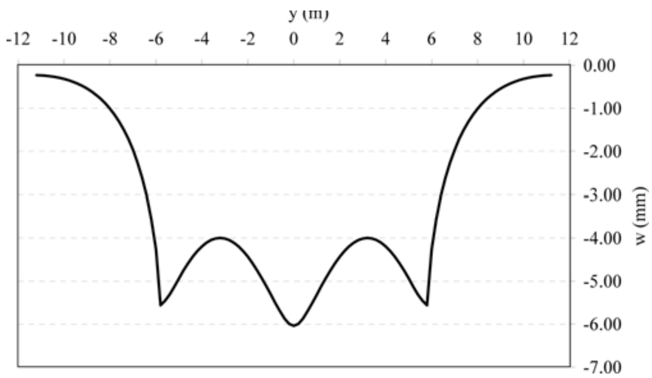


Fig. 6 Vertical Deflection of the system at $x = 2.4$ m

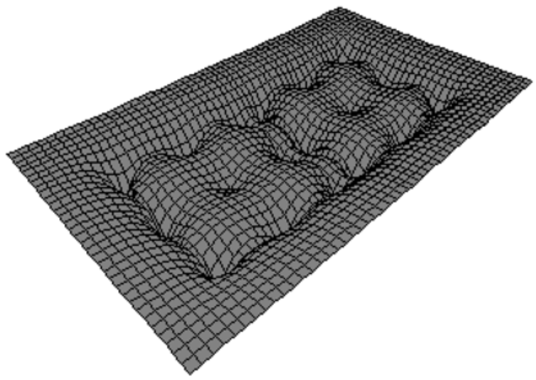


Fig. 7 Deformed shape of the system

models permits interaction among the springs. Deformed shape of the entire system is given in Fig. 7.

Example 3

In this example a circular foundation which is also investigated by Vallabhan and Das (1991), Saygun and Çelik (2003) is solved, Fig. 8(a). The properties of the plate-soil system are as follows. Radius of the plate is R = 3.05 m, modulus of elasticity of the plate is E_p = 22700000 kN/m^2, Poisson's ratio of the plate ν_p = 0.2, thickness of the plate h = 0.24 m, depth of the soil foundation H = 3.05 m, modulus of elasticity of the soil E_s = 22700 kN/m^2, Poisson's ratio of the soil ν_s = 0.2. The plate is solved for two diferent load cases, subjected to a uniformly load 26.3 kN/m^2 and 16 kN/m edge load, Fig. 8(b), Fig. 8(c).

The mode shape parameter γ obtained and given in Table 4 comparatively with the results of reference solutions. As can be seen, the results are in good agreement. The calculated soil parameters are as C = 8382 kN/m^3, C_t = 4326 kN/m for uniform load case, C = 8977 kN/m^3, C_t = 3666 kN/m for uniform edge load case.

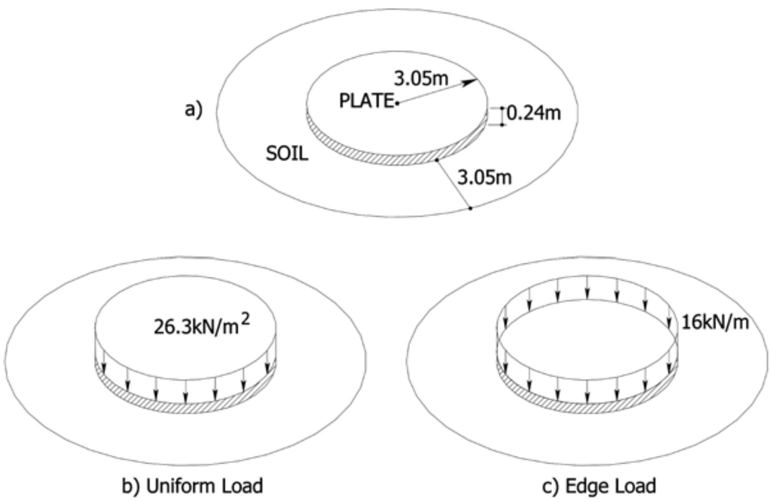


Fig. 8 Circular foundation and loads

Table 4 Mode shape parameter values for uniform and edge loading

	γ (Uniform Load)	γ (Edge Load)
Vallabhan and Das (1991)	0.915	1.49
Saygun and Çelik (2003)	0.915	1.52
Present Study	0.918	1.55

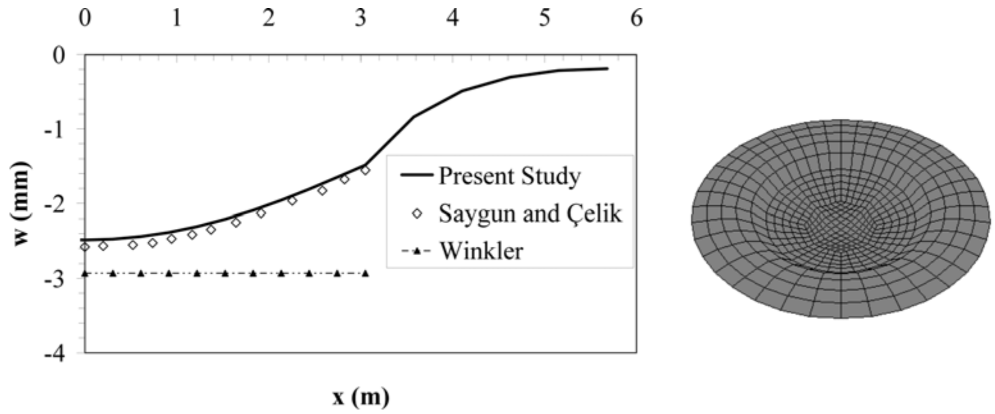


Fig. 9 Uniform load

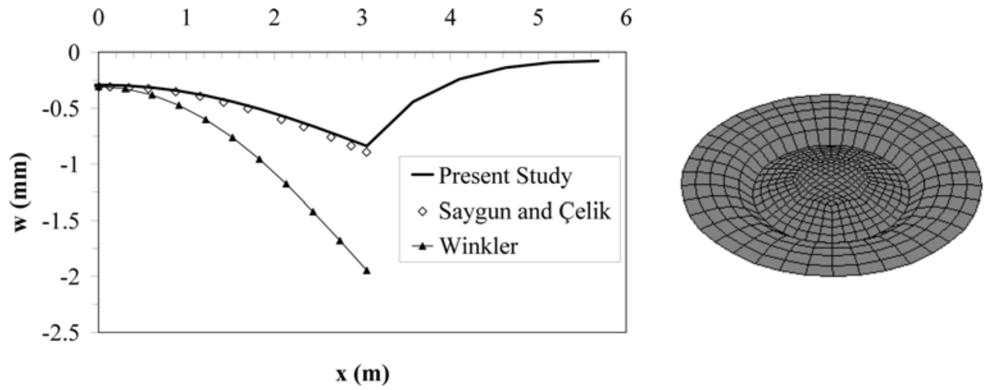


Fig. 10 Edge load

The deformed shapes for both load cases are given in Fig. 9 and Fig. 10. The results are found to be in good agreement with previous studies.

Results compared with the solutions given by Saygun and Çelik (2003) and are found to be in a good agreement. The plate is also analyzed by assuming Winkler hypothesis with obtained bedding coefficient C. It can be observed that the settlements are obtained higher in Winkler soil solutions.

Example 4

In this example an arbitrarily shaped shaped foundation is solved. The geometrical properties of the plate-soil system is given in Fig. 11. Modulus of elasticity of the plate is E_p = 20000000 kN/m^2,

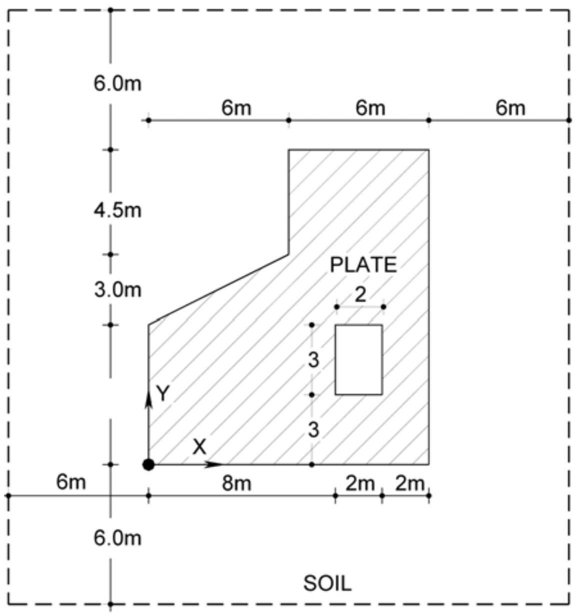
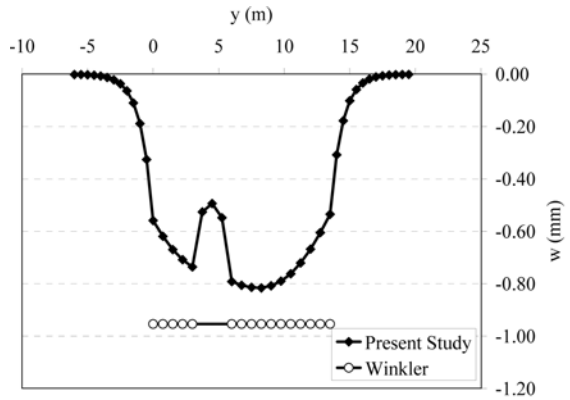


Fig. 11 Arbitrarily shaped foundation



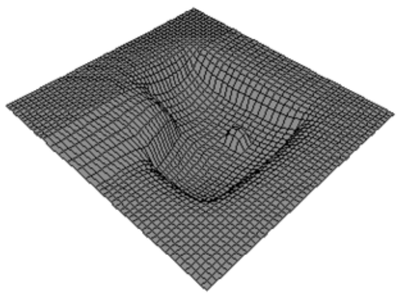


Fig. 12 Displacements at $x = 9$ m and deformed shape of arbitrarily shaped foundation

Poisson's ratio of the plate ν_p = 0.2, thickness of the plate h = 0.2 m, depth of the soil foundation H = 10 m, modulus of elasticity of the soil E_s = 70000 kN/m^2, Poisson's ratio of the soil ν_s = 0.25. The plate is subjected to a uniformly load 25 kN/m^2.

The calculated shape and soil parameters are as $\gamma = 6.2417$, $C = 26217$ kN/m^3, $C_t = 11213$ kN/m.

The deformations at $x = 9$ m for both two-parameter and Winkler hypothesis are given in Fig. 12. Again it can be observed that the settlements are obtained higher in Winkler soil solutions.

6. Conclusions

In this study an assumed stress hybrid four-node finite element is presented for modeling of plates with shear deformations on elastic foundation. The formulation is based on Hellinger-Reissner variational principle which leads to lower order polynomial interpolation functions. The plate domain is idealized with superposition of a hybrid plate and a hybrid soil element where out of the plate domain is idealized with only soil elements. Numerical examples with different loading types and soil depths are solved. It is observed that the parameter C decreases as the depth of the soil, H, increases while the parameter $2C_t$ increases with H. An increase in the soil depth does not affect the results after certain value of H. Interaction between closed plates is also investigated and showed that this type of interaction can change not only the soil coefficients but also the internal forces which depend on the deformed shape of the plate-soil system.

A comparison is made between two parameter elastic foundation hypothesis and Winkler hypothesis. It is observed that the settlements are obtained higher in Winkler soil solutions since this hypothesis ignores interaction among the soil springs which can be taken into account in two parameter elastic foundation hypothesis.

The results obtained are compared with other researchers' solutions and are found to be in good agreement.

Acknowledgements

The author gratefully acknowledge the excellent support provided by Professor Dr. Engin Orakdöğen.

References

Abdallaa, J.A. and Ibrahimb, A.M. (2008), "Development of a discrete Reissner-Mindlin element on Winkler foundation", *Finite Elem. Anal. Des.*, **42**, 740-748.

Ayvaz, Y. and Oğuzhan, C.B. (2008), "Free vibration analysis of plates resting on elastic foundations using modified Vlasov model", *Struct. Eng. Mech.*, **28**(6), 635-658.

Çelik, M. and Saygun, A. (1999), "A Method for the analysis of plates on a two-parameter foundation", *Int. J. Solids Struct*., **36**, 2891-915.

Çelik, M. and Omurtag, M.H. (2005), "Determination of the Vlasov foundation parameters quadratic variation of elasticity modulus using FE analysis", *Struct. Eng. Mech*., **19**(6), 619-637.

Darılmaz (2005), "An assumed-stress finite element for static and free vibration analysis of Reissner-Mindlin plates", *Struct. Eng. Mech.*, **19**(2), 199-215.

Eratli, N. and Aköz, A.Y. (1997), "The mixed finite element formulation for the thick plates on elastic foundations", *Comput. Struct*., **65**(4), 515-529.

Gendy, A.S. and Saleeb, A.F. (1999) "Effective modeling of beams with shear deformations on elastic foundation", *Struct. Eng. Mech.*, **8**(6), 607-622.

Güler, K. and Celep, Z. (1995), "Static and dynamic responses of a circular plate on a tensionless elastic foundation", *J. Sound Vib.*, **183**(2), 185-195.

Nogami, T. and Lam, Y.C. (1987), "Two parameter layer model for analysis of slabs on elastic foundations", *J. Eng. Mech.*, **113**(9), 1279-1291.

Ozgan, K. and Daloğlu, A.T. (2008), "Effect of transverse shear strains on plates resting on elastic foundation using modified Vlasov model", *Thin Wall. Struct.,* **46**, 1236-1250.

Pasternak, P.L. (1954), "On a new method of analysis of an elastic foundation by means of two foundation constants", *Gos. Izd. Lip. po Strait i Arkh*. Moscow, (in Russian).

Pian, T.H.H. and Chen, D.P. (1982), "Alternative ways for formulation of hybrid stress elements", *Int. J. Numer.*

Meth. Eng., **18**, 1679-1684.
Saygun, A. and Çelik, M. (2003), "Analysis of circular plates on two - parameter elastic foundation", *Struct. Eng. Mech.*, **15**(2), 249-267.
Vlasov, V.Z. and Leont'ev, U.N. (1966), "Beams, plates and shells on elastic foundation", *Israel Program for Scientific Translation*, Jerusalem.
Vallabhan, C.V.G. and Das, Y.C. (1988), "A parametric study of beams on elastic foundations", *J. Eng. Mech. Div.*, **114**(12), 2072-2082.
Vallabhan, C.V.G., Straughan, W.T. and Das, Y.C. (1991), "Refined model for analysis of plates on elastic foundation", *J. Eng. Mech.,* **117**(12), 2830-2844.
Wang, Y.H., Tham, L.G. and Cheung, Y.K. (2005), "Beams and plates on elastic foundations: a review", *Prog. Struct. Eng. Mater.*, **7**(4), 174-182.
Washizu, K. (1982), *Variational Method in Elasticity and Plasticity*, Pergamon Press, Oxford 3rd. Edn.
Winkler, E. (1867), Die Lehre von der Elastizität und Festigkeit. Dominicus, Prag.

Notation

E_p : modulus of elasticity of the plate
E_s : elasticity of the subsoil
h : thickness of the plate
H : depth of the soil
ν_s : Poisson ratio of the subsoil is
ν_p : Poisson ratio of the plate equals to
γ : mode shape parameter
$\{d\}$: nodal displacements
$[K]$: stiffness matrix of the plate
$[C]$: bedding matrices of soil element,
$[C_t]$: shear effect matrices of soil element
$[N]$: shape function matrix
$\{P\}$: load vector
$\tilde{\sigma}$: stress vector
$\tilde{\varepsilon}$: strain vector
$\tilde{D}$: differential operator matrix
$\tilde{S}$: compliance matrix
β,β_s : stress parameters
$\{w\}$: displacement

Appendix

Derivation of the formulation

Using variational principle and minimizing the total potential energy of Eq. (1) by taking variations in w and ϕ yields

$$\delta\Pi = \int_{\Omega}(D\nabla^4 w - 2C_t\nabla^2 w + Cw - q)\delta w dxdy - \int_0^H \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E_s \frac{(1-\nu_s)}{(1+\nu_s)(1-2\nu_s)} w^2 dxdy \frac{\partial^2\phi}{\partial z^2} \right)\delta\phi dz$$

$$+ \int_0^H \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_s \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right] dxdy\phi \right)\delta\phi dz + \text{boundary conditions}=0 \tag{A.1}$$

The terms in the parentheses and boundary conditions must be equal to zero since δw and $\delta\phi$ are not equal to zero.

So the field equation in the domain of the plate can be written as

$$D\nabla^4 w - 2C_t\nabla^2 w + Cw = q \tag{A.2}$$

The second and third expressions in (A.1) is the field equation for the deformation pattern of the soil in the vertical direction.

$$-\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E_s \frac{(1-\nu_s)}{(1+\nu_s)(1-2\nu_s)} w^2 dxdy \frac{\partial^2\phi}{\partial z^2} + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_s \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right] dxdy\phi = 0 \tag{A.3}$$

By using the boundary conditions $\phi(z = 0) = 1$, $\phi(z = H) = 0$, solution of Eq. (A.3) with the given boundary conditions yields mode shape function

$$\phi(z) = \frac{\sinh\left[\gamma\left(1-\frac{z}{H}\right)\right]}{\sinh\gamma} \tag{A.4}$$

The soil parameters, C is the bedding coefficient and $2C_t$ is the shear coefficient parameter, can be obtained by using mode shape function given in Eq. (A.4)

$$C = \int_0^H E_s \frac{(1-\nu_s)}{(1+\nu_s)(1-2\nu_s)} \left(\frac{\partial\phi}{\partial z}\right)^2 dz = E_s \frac{(1-\nu_s)}{(1+\nu_s)(1-2\nu_s)} \frac{\gamma}{H} \frac{(\sinh 2\gamma + 2\gamma)}{4\sinh^2\gamma} \tag{A.5}$$

$$2C_t = \int_0^H G_s\phi^2 dz = G_s \frac{H}{\gamma} \frac{(\sinh 2\gamma - 2\gamma)}{4\sinh^2\gamma} \tag{A.6}$$