Structural Engineering and Mechanics, Vol. 33, No. 4 (2009) 485-502

DOI: http://dx.doi.org/10.12989/sem.2009.33.4.485

New eight node serendipity quadrilateral plate bending element for thin and moderately thick plates using Integrated Force Method

H.R. Dhananjaya†

Department of Civil Engineering, Manipal Institute of Technology, Manipal – 576 104, India
(Visiting faculty to Malaya University, Malaysia)

P.C. Pandey‡

Department of Civil Engineering, Indian Institute of Science, Bangalore – 560012, India

J. Nagabhushanam‡†

Department of Aerospace Engineering, Indian Institute of Science, Bangalore – 560012, India

(Received April 21, 2009, Accepted September 13, 2009)

Abstract. A new 8-node serendipity quadrilateral plate bending element (MQP8) based on the Mindlin-Reissner theory for the analysis of thin and moderately thick plate bending problems using Integrated Force Method is presented in this paper. The performance of this new element (MQP8) is studied for accuracy and convergence by analyzing many standard benchmark plate bending problems. This new element MQP8 performs excellent in both thin and moderately thick plate bending situations. And also this element is free from spurious/zero energy modes and free from shear locking problem.

Keywords: Mindlin-Reissner theory; plate bending element; integrated force method; displacement fields; stress-resultant fields.

1. Introduction

Considerable amount of research work has been carried out over the past few decades by research engineers and scientists on the Mindlin-Reissner theory based plate bending elements. These elements consider C_0 continuity and avoid C_1 continuity which is rather difficult to adopt for higher order finite elements. Most of the plate bending elements developed so far use the displacement based finite element method (Choi and Park 1999, Choi *et al.* 2002, Kim and Choi 2005, Kanber and Bozkurt 2006, Ozgan and Daloglu 2007) and a very few are hybrid/mixed elements (Pian and

† Professor, Corresponding author, E-mail: djaya_hr@yahoo.com, djaya.hr@manipal.edu
‡ Professor, E-mail: pcpandey@civil.iisc.ernet.in
‡† Emeritus Professor, E-mail: naga@aero.iisc.ernet.in

Chen 1982, Dimitris *et al.* 1984, Chen and Cheung 1987, Darılmaz 2005, Darılmaz and Kumbasar 2006). In this paper Integrated Force Method, has been used to develop the Mindlin-Reissner theory based 8-node quadrilateral plate bending element which considers C_0 continuity and effect of shear deformation.

During pre-computer era, both the force method and the displacement method were popular tools for analyzing civil, mechanical and aerospace engineering structures. The popularity of the force method can be attributed to its ability to determine accurate estimates for forces in the structures. During the formulative period of structural analysis by matrix methods, earnest research was directed to automate both force and displacement methods. Automation of displacement method of analysis was successful as there were no hurdles like selection of redundant forces in statically indeterminate and continuum structures. However computer automation of the force method of analysis was not successful. Redundant force selection of statically indeterminate and continuum structures was the main cause of failure. It acted as the dominating road block in the path of the force method automation. The effort in computer-assisted generation of compatibility conditions in the process of automation of the force method was partially successful (Robinson and Haggenmacher 1971, Kaneko *et al.* 1983), as it was not extended for continuum structures.

A new novel matrix formulation of the classical force method of analysis termed "Integrated Force Method (IFM)" has been developed (Patnaik 1973) for analyzing civil, mechanical and aerospace engineering structures. In this method, all independent/internal forces are treated as unknown variables which are computed by simultaneously imposing equations of equilibrium and compatibility conditions. Unlike classical force method of analysis, the IFM is independent of redundants and the basic determinate structure. It requires explicit generation of compatibility conditions for skeletal as well as continuum structures. The advantages of IFM compare to displacement-based finite element method are reported in the reference (Patnaik *et al.* 1991).

The IFM integrates the system of equilibrium equations and the global compatibility conditions in a fashion paralleling approaches in continuum mechanics (example, the Beltrami-Michell formulation of elasticity (Love 1944). A variational energy formulation for IFM has been established by Patnaik (1986). The stationary condition of the functional yields, the equilibrium equations, compatibility and natural boundary conditions.

Generation of compatibility conditions for elasticity and discrete models have been reported by Patnaik *et al.* (2000). Nagabhushanam and Patnaik (1990) have developed a general purpose program to generate compatibility matrix for the IFM. Automatic generation of sparse and banded compatibility matrix for the Integrated Force Method has been reported by Nagabhushanam and Srinivas (1991). IFM has been successfully implemented for analyzing, plane stress Nagabhushanam and Srinivas (1991), two/three dimensional problems (Kaljevic *et al.* 1996, Kaljevic *et al.* 1996), dynamics (Patnaik and Yadagiri 1976), optimization (Patnaik *et al.* 1986) and non-linear problems (Krishnam Raju and Nagabhushanam 2000). A 4-node rectangular plate bending element based on the Kirchhoff theory has been formulated using the IFM (Patnaik *et al.* 1991) The element considers a transverse displacement and two rotations as degrees of freedom at each node. The performance of this element was compared with those obtained by force method (Przemieniecki 1968, Robinson 1973). Dhananjaya *et al.* (2007), developed a 4-node bilinear plate bending element based on the Mindlin-Reissner theory using IFM. The results of this element were compared with those of similar displacement-based 4-node quadrilateral plate bending elements available literature.

In this paper, a new 8-node serendipity quadrilateral plate bending element (MQP8) has been presented by assuming suitable stress-resultants and displacement fields for analysis of thin and moderately thick plate bending problems using Integrated Force Method. Mindlin-Reissner theory has been employed in the formulation which accounts the effect of shear deformation. Many standard plate bending benchmark problems are analyzed to test the accuracy and convergence of the element presented. The results obtained by this element are compared with those of similar displacement-based 8-node quadrilateral plate bending elements available in the literature. Results are also compared with the exact solutions. Numerical results indicate that proposed element MQP8 is free from spurious/zero energy modes and shear locking problem. The proposed element MQP8 has produced, in general, excellent results in the numerical problems considered.

2. Formulation of element equilibrium and flexibility matrices

For the sake of completeness, the basic theory of the Integrated Force Method has been given in the appendix A. In this section brief formulation on the development of equilibrium and flexibility matrices of plate bending element is described. The Mindlin-Reissner theory has been employed in the formulation. In the Mindlin-Reissner theory, a line that is straight and normal to mid-surface of the un-deformed plate remain straight but not necessarily normal to the mid-surface of the deformed plate. This leads to the following definition of the displacement components u, v, w in the **x**, **y**, **z** Cartesian coordinates system

$$u = -z\theta_x(x,y); \quad v = -z\theta_y(x,y); \quad w = w(x,y) \tag{1}$$

where

- x, y are coordinates in the reference mid-surface
- z is the coordinate through the thickness of the plate t with $-t/2 \le z \le t/2$
- w is the transverse (lateral) displacement
- θ_x, θ_y represent the rotations of the normal in **x-z** and **y-z** planes respectively

Engineering strains for the Mindlin-Reissner plate theory can be written as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = -\begin{Bmatrix} z\dfrac{\partial \theta_x}{\partial x} \\ z\dfrac{\partial \theta_y}{\partial y} \\ z\left(\dfrac{\partial \theta_x}{\partial y} + \dfrac{\partial \theta_y}{\partial x}\right) \\ \theta_y - \dfrac{\partial w}{\partial y} \\ \theta_x - \dfrac{\partial w}{\partial x} \end{Bmatrix} \tag{2}$$

The stress-strain relations for an isotropic two-dimensional plate material is given by

$$\{\sigma\} = [C_{con}]\{\varepsilon\} \tag{3}$$

where $\{\sigma\} = [\sigma_x \;\; \sigma_y \;\; \tau_{xy} \;\; \tau_{yz} \;\; \tau_{zx}]^T$ = vector of stress components

$\{\varepsilon\} = [\varepsilon_x \;\; \varepsilon_y \;\; \gamma_{xy} \;\; \gamma_{yz} \;\; \gamma_{zx}]^T$ = vector of strain components

$$[C_{con}] = \text{constitutive matrix} = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}$$

E = Young's modulus;
ν = Poisson's ratio

The stress-resultants for plates can be written as

$$\begin{aligned} M_x &= \int_{-t/2}^{t/2} z\sigma_x dz \\ M_y &= \int_{-t/2}^{t/2} z\sigma_y dz \\ M_{xy} &= \int_{-t/2}^{t/2} z\tau_{xy} dz \\ Q_y &= \int_{-t/2}^{t/2} \tau_{yz} dz \\ Q_x &= \int_{-t/2}^{t/2} \tau_{xz} dz \end{aligned} \tag{4}$$

Eqs. (2), (3) and (4) yield the moment-curvature relations as

$$\{M\} = [C_1]\{k\} \tag{5}$$

Where $\{M\}$ = vector of stress-resultants

$= [M_x \;\; M_y \;\; M_{xy} \;\; Q_y \;\; Q_x]^T$

$[C_1]$ = matrix relating stress-resultants to curvatures

$\{k\}$ = vector of curvatures

$$= \left[\frac{\partial\theta_x}{\partial x} \;\; \frac{\partial\theta_y}{\partial y} \;\; \frac{\partial\theta_x}{\partial y}+\frac{\partial\theta_y}{\partial x} \;\; \theta_y-\frac{\partial\theta_w}{\partial y} \;\; \theta_x-\frac{\partial w}{\partial x}\right]^T$$

From the Eq. (5), the curvature-moment relations can be written as

$$\{k\} = [C_1]^{-1}\{M\} = [H]\{M\} \tag{6}$$

where $[H] = [C_1]^{-1}$

= matrix relating curvatures to stress-resultants

The matrix $[H]$ for the Mindlin-Reissner plate with Reissner's shear correction factor (Reissner

1945) of 5/6 can be written as

$$[H] = \frac{1}{D_1}\begin{bmatrix} 1 & -\nu & 0 & 0 & 0 \\ -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & \dfrac{t^2(1+\nu)}{5} & 0 \\ 0 & 0 & 0 & 0 & \dfrac{t^2(1+\nu)}{5} \end{bmatrix} \tag{7}$$

where $D_1 = Et^3/12$

The strain energy U_p of the elastic plate in bending and shear is written as

$$U_p = \iint \frac{1}{2}\left[M_x\frac{\partial\theta_x}{\partial x} + M_y\frac{\partial\theta_y}{\partial y} + M_{xy}\left(\frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x}\right) + Q_y\left(\theta_y - \frac{\partial\theta_w}{\partial y}\right) + Q_x\left(\theta_x - \frac{\partial w}{\partial x}\right)\right]dxdy \tag{8}$$

The vectors $\{M\}$ and $\{k\}$ for a discrete plate bending element can be expressed in matrix notations in terms of assumed stress-resultants and displacement fields respectively as

$$\{M\} = [\psi]\{F_e\} \tag{9}$$

$$\{k\} = [D_{op}][\phi_1]\{\alpha\} = [D_{op}][\phi]\{X_e\} \tag{10}$$

where $[\psi]$ = matrix of polynomial terms for stress-resultant fields
$\{F_e\}$ = vector of force components of the discrete element
$[\phi_1]$ = matrix of polynomial terms for displacement fields
$[\phi_1] = [\phi_1][A]^{-1}$
$[A]$ = matrix formed by substituting the coordinates of the element nodes into the polynomial of displacement fields
$\{\alpha\}$ = coefficients of the displacement field polynomial
$\{X_e\}$ = vector of displacements of the discrete element

$$[D_{op}] = \text{differential operator matrix} = \begin{bmatrix} 0 & \dfrac{\partial}{\partial x} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial y} \\ 0 & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \\ -\dfrac{\partial}{\partial y} & 0 & 1 \\ -\dfrac{\partial}{\partial x} & 1 & 0 \end{bmatrix}$$

Substituting Eqs. (9) and (10) into the Eq. (8), the strain energy for the discrete element can be expressed as

$$U_p = \frac{1}{2}\{X_e\}^T[B_e]\{F_e\} \tag{11}$$

where $[B_e]$ represents the element equilibrium matrix and is given by

$$[B_e] = \iint[\phi]^T[D_{op}]^T[\psi]dxdy \tag{12}$$

The complementary strain energy for the elastic plate in bending and shear is expressed as

$$U_c = \iint\frac{1}{2}\frac{1}{D_1}\left[M_x^2 + M_y^2 - 2\,\nu M_x M_y + 2(1+\nu)M_{xy}^2 + Q_y^2\frac{2t^2(1+\nu)}{5} + Q_x^2\frac{2t^2(1+\nu)}{5}\right]dxdy$$

Using the Eq. (7), the complementary strain energy for the discrete element is written as

$$U_c = \frac{1}{2}\{F_e\}^T[G_e]\{F_e\} \tag{13}$$

where $[G_e]$ represents the element flexibility matrix and is given by

$$[G_e] = \iint[\psi]^T[H][\psi]dxdy \tag{14}$$

The Eqs. (12) and (14) are used to obtain element equilibrium matrix $[B_e]$ and element flexibility matrix $[G_e]$ respectively. These element matrices $[B_e]$ and $[G_e]$ of all elements are assembled to obtain the global equilibrium matrix $[B]$ and global flexibility matrix $[G]$ of the structure and they are used to setup the IFM governing equation to analyze the plate problems by IFM.

2.1 Displacement and stress-resultant fields

A typical 8-node quadrilateral plate bending element is shown in the Fig. 1. Three degrees of freedom namely a transverse displacement w and two rotations θ_x, θ_y are considered at each node of the proposed 8-node serendipity quadrilateral plate bending element MQP8.

The assumed polynomials for displacement fields in Integrated Force Method should satisfy the convergence requirements. Assumed displacement fields for w, θ_x and θ_y in terms of generalized displacement parameters α_1 α_2 … α_{24} are given in the Eq. (15) for this proposed 8-node quadrilateral element.

$$\begin{aligned}
w &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^2 y + \alpha_8 xy^2 \\
\theta_x &= \alpha_9 + \alpha_{10} x + \alpha_{11} y + \alpha_{12} x^2 + \alpha_{13} xy + \alpha_{14} y^2 + \alpha_{15} x^2 y + \alpha_{16} xy^2 \\
\theta_y &= \alpha_{17} + \alpha_{18} x + \alpha_{19} y + \alpha_{20} x^2 + \alpha_{21} xy + \alpha_{22} y^2 + \alpha_{23} x^2 y + \alpha_{24} xy^2
\end{aligned} \tag{15}$$

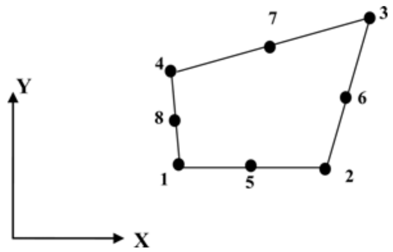


Fig. 1 A typical 8-node quadrilateral plate bending element

In the Integrated Force Method, the assumed stress-resultant fields must satisfy the equilibrium equations. Eq. (16) shows the assumed stress-resultant fields for this proposed 8-node quadrilateral plate bending element. in terms of polynomials with independent generalized force parameters $F_1\ F_2\ \ldots\ F_{21}$. The stress-resultant fields for the shear forces Q_y and Q_x are obtained by considering moment equilibrium equations of the element.

$$\begin{aligned}
M_x &= F_1 + F_2x + F_3y + F_4x^2 + F_5xy + F_6y^2 + F_7x^2y + F_8xy^2 \\
M_y &= F_9 + F_{10}x + F_{11}y + F_{12}x^2 + F_{13}xy + F_{14}y^2 + F_{15}x^2y + F_{16}xy^2 \\
M_{xy} &= F_{17} + F_{18}x + F_{19}y + F_{20}x^2 + F_{21}y^2 \\
Q_y &= (F_{11} + F_{18}) + (F_{13} + 2F_{20})x + 2F_{14}y + 2F_{16}xy + F_{15}x^2 \\
Q_x &= (F_2 + F_{19}) + 2F_4x + (F_5 + 2F_{21})y + 2F_7xy + F_8y^2
\end{aligned} \tag{16}$$

The displacement and stress-resultant fields given in the above Eqs. (15) and (16) are used in the Eqs. (12) and (14) to obtain element equilibrium and flexibility matrices for this proposed element MQP8.

3. Numerical tests and results

To validate the proposed element MQP8, the numerical tests for convergence, spurious/zero energy modes and shear locking are considered.

A square/rectangular plate with simply supported/clamped boundary conditions, the Morley's plate problem and the Razzaque's plate problem are considered to study the performance of the proposed element MQP8. These numerical/example problems are analyzed to estimate moments and deflections for various mesh sizes using the proposed element MQP8 via Integrated Force Method. The results of the proposed element MQP8 are compared for accuracy and convergence with those of existing 8-node displacement based quadrilateral plate bending elements available in the literature (Spilker 1982). The results of the proposed element MQP8 are also compared with the exact solutions (Timoshenko and Krieger 1959, Jane *et al.* 2000). The details of example problems considered here are given below.

1. A square thin plate ($t/L = 0.01$) with simply supported/clamped boundary conditions subjected to central point load. The parameters of the problem are : L = 100, B = 100, t = 1, E = 10^7, ν = 0.3, P = 400 (Spilker 1982)
2. A rectangular thin plate ($t/B = 0.01$) with aspect ratio 2 for simply supported/clamped boundary conditions subjected to uniform load. The parameters of the problem are : L = 200, B = 100, t = 1, E = 10^7, ν = 0.3, q = 10 (Spilker 1982)
3. The Morley's plate problem (Fig. 2). Parameters of the problems are: L = 100, B = 100, t = 1, E = 1092000.0, ν = 0.3 and q = 1, inclination of the plate θ = 30°, w = 0 on all boundaries (Morley 1963)
4. The Razzaque's plate problem (Fig. 3). Parameters of the problems are: L = 100, B = 100, t = 1, E = 1092000.0, ν = 0.3 and q = 1, inclination of the plate θ = 60° (Razzaque 1973)

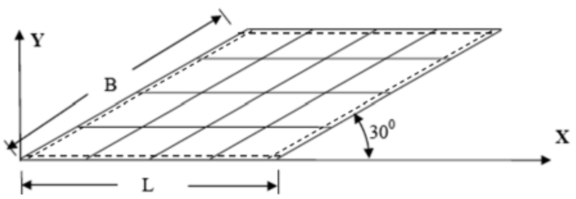


Fig. 2 Morley's plate : L = 100, B = 100, t = 1, E = 1092000.0, ν = 0.3 and q = 1, inclination of the plate θ 30°, w = 0 on all sides of the plate

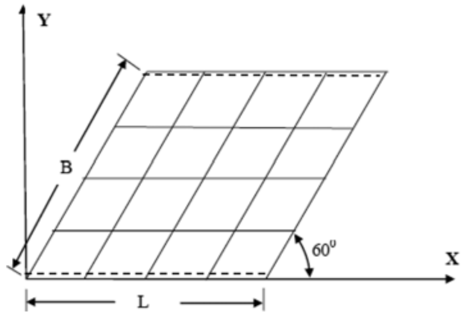


Fig. 3 Razzaque's plate L = 100, B = 100, t = 1, E = 1092000.0, ν = 0.3 and q = 1, inclination of the plate θ = 60° (Two opposite edges simply supported while other two free)

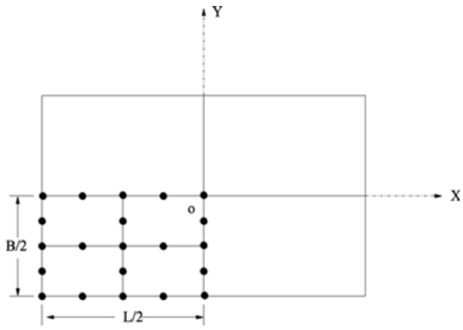


Fig. 4 A typical (2 × 2) mesh in one quadrant of the plate

Due to symmetry of the plate with respect to the shape, boundary and loading conditions in the above example problems 1 and 2, one quadrant of the plate is analyzed with appropriate boundary conditions for various mesh sizes. A typical mesh (2 × 2) considered in one quadrant of the rectangular/square plate is shown in the Fig. 4. As the force-based 8-node quadrilateral plate bending finite elements are not available in the literature, the displacement based 8-node quadrilateral plate bending finite elements are considered for the purpose of comparison of the results of the proposed element MQP8.

The displacements and moments are estimated for the above example problems using the proposed element MQP8 via IFM. The results of MQP8 are compared with those of displacement based 8-node quadrilateral plate bending elements QH1, QH2, QH3, QH4 available in the literature (Spilker 1982) for the example problems 1 and 2.

Table 1 Normalized central deflection for a simply supported square thin plate with the central point load (t/L = 0.01, Example problem 1)

Elements	QH1	QH2	QH3	QH4	MQP8
1 × 1	0.96	1.06	0.93	0.76	0.97
2 × 2	1.00	1.02	1.00	0.93	1.01
3 × 3	1.00	1.01	1.00	0.95	1.00
4 × 4	1.00	1.00	1.00	0.97	1.00

Table 2 Normalized central deflection for a clamped square plate with the central point load (t/L = 0.01, Example problem 1)

Elements	QH1	QH2	QH3	QH4	MQP8
1 × 1	1.150	1.280	1.100	0.505	1.154
2 × 2	1.000	1.040	0.980	0.810	1.002
3 × 3	1.010	1.030	1.000	0.920	1.009
4 × 4	1.010	1.020	1.010	0.960	1.009

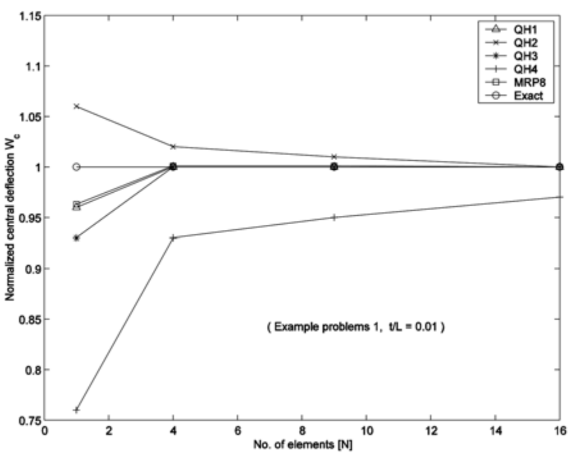


Fig. 5 Normalized central deflection for a simply supported square thin plate with central point load

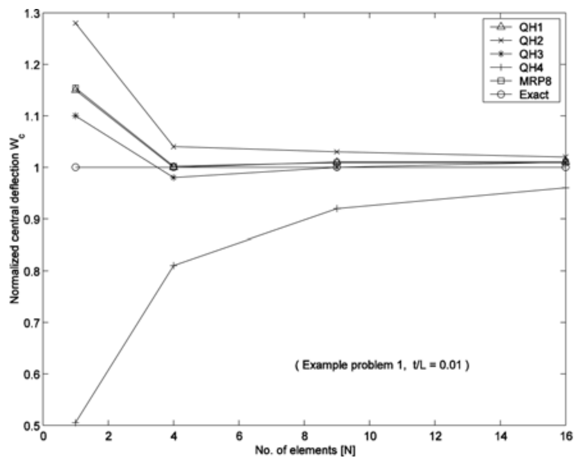


Fig. 6 Normalized central deflection for a clamped square thin plate with central point load

The central displacements and moments obtained for the example problems 1 and 2 are normalized with respect to the solution obtained from thin plate bending theory. Tables 1, 2 summarize the normalized central displacements of a square thin plate (t/L = 0.01) for the example problem 1 for various grid sizes, and the Figs. 5, 6 show the corresponding convergence trends. The normalized central displacements and moments of a rectangular thin plate (t/B = 0.01) with aspect ratios 2 for the example problem 2 are summarized in the Tables 3, 4, and the corresponding convergence trends are shown in the Figs. 7, 8. These Tables and Figs. indicate that the proposed element MQP8 is consistently performing well in estimating the central deflections and moments for thin square/rectangular plate with various boundary conditions and loadings considered and the results are fast approaching towards the exact solutions as shown in the Figs. 5-8.

Table 3 Normalized central deflection for a simply supported rectangular thin plate with uniform load (t/B = 0.01, L/B = 2, Example problem 2)

Elements	QH1	QH2	QH3	QH4	MQP8
1 × 1	0.975	0.870	0.910	0.855	0.980
2 × 2	1.000	0.990	1.010	0.980	1.010
3 × 3	1.000	1.000	1.000	0.990	1.000
4 × 4	1.000	1.000	1.000	0.995	1.000

Table 4 Normalized central moment M_x for a simply supported rectangular thin plate with uniform load (t/B = 0.01, L/B = 2, Example problem 2)

Elements	QH1	QH2	QH3	QH4	MQP8
1 × 1	0.720	1.800	1.020	1.090	0.700
2 × 2	1.040	1.550	0.970	0.980	0.980
3 × 3	1.020	1.140	0.980	0.985	0.950
4 × 4	1.010	1.090	0.990	0.990	0.990

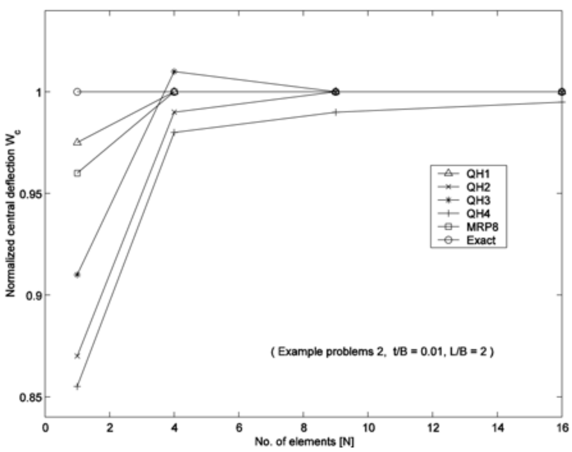


Fig. 7 Normalized central deflection for a simply supported rectangular thin plate with uniform load

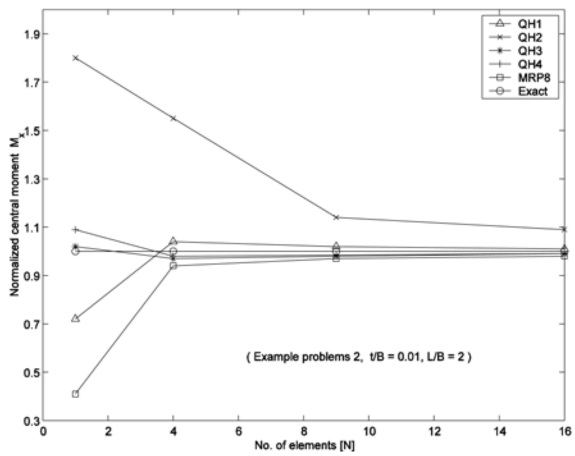


Fig. 8 Normalized central moment M_x for a simply supported rectangular thin plate with uniform load

The Morley's plate problem (example problem 3) and the Razzaque's plate problem (example problem 4) are analyzed for central deflections and moments by considering the full plate with various mesh sizes using the proposed element MQP8 via the Integrated Force Method. Central deflections and moments of the Morley's plate and the Razzaque's plate obtained by the proposed element MQP8 are plotted against no of elements in the Figs. 9-11. The exact values of central deflections and moments of the Morley's plate (finite difference solution) (Morley 1963) and the Razzaque's plate (Razzaque 1973) are also plotted in the Figs. 9-11. These Figures show that the estimated central deflections and moments of the proposed element MQP8 are fast converging to the exact solutions.

To verify the spurious/zero energy modes, regular and irregular meshes as shown in the Figs. 12(a) and 12(b) respectively are considered. The parameters of the problem are L = 100,

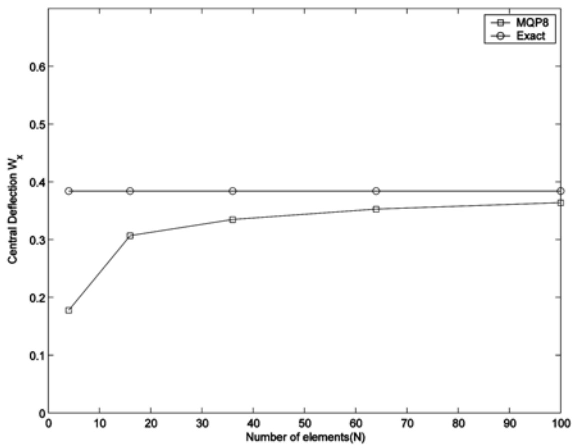


Fig. 9 Central deflection for a Morley's plate with uniform load

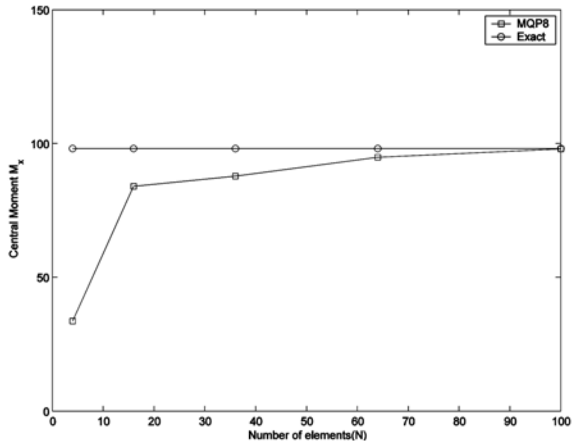


Fig. 10 Central Moment M_x for a Morley's plate with uniform load

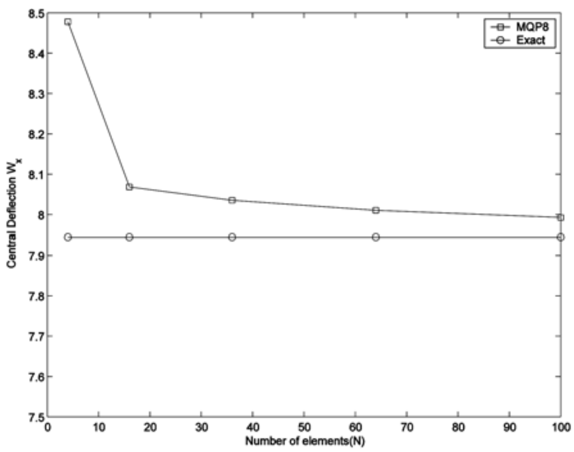
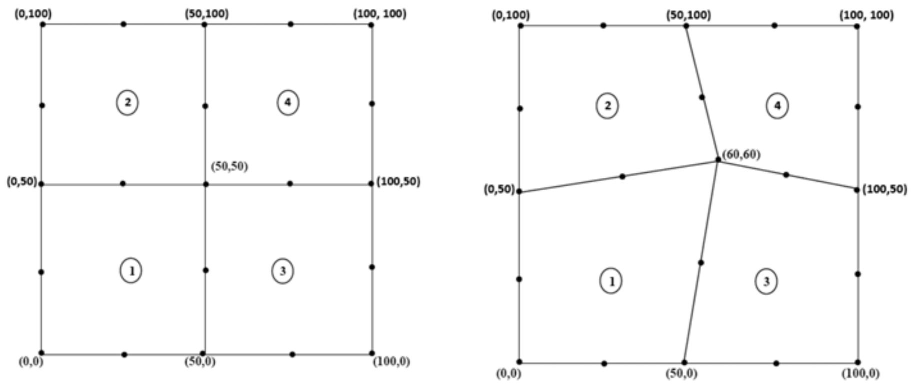


Fig. 11 Central deflection for a Razzaque's plate with uniform load



(a) (b)

Fig. 12 (a) 8-node 2 × 2 regular mesh, (b) 8-node 2 × 2 irregular mesh

Table 5 Eigen values for regular mesh (element 1 in the Fig. 14(a)), t/L = 0.1, 0.01, 0.001, 0.0001, 0.00001

t/L	Eigen values			
0.1	.3815E+10;	.3701E+10;	.2673E+10;	.1783E+10;
	.2680E+10;	.1538E+10;	.1682E+10;	.1516E+10;
	.7518E+09;	.7309E+09;	.7399E+09;	.3112E+09;
	.5770E+09;	.1594E+09;	.3179E+09;	.1371E+09;
	.2022E+08;	.1158E+07;	.3899E+07;	.6510E+07;
	.3924E+07;	-.6820E-07;	-.1031E-06;	.7443E-07;
0.01	.5525E+07;	.5673E+07;	.2310E+07;	.3034E+07;
	.3044E+07;	.1994E+07;	.2090E+07;	.2138E+07;
	.8480E+06;	.7771E+06;	.8520E+06;	.6127E+06;
	.3253E+06;	.1628E+06;	.3340E+06;	.1377E+06;
	.2275E+05;	.1232E+04;	.4268E+04;	.7027E+04;
	.4296E+04;	.8830E-10;	-.1625E-10;	-.1625E-10;
0.001	.5588E+04;	.5747E+04;	.3065E+04;	.2347E+04;
	.3063E+04;	.2266E+04;	.2182E+04;	.2149E+04;
	.8575E+03;	.7817E+03;	.8743E+03;	.6149E+03;
	.3473E+03;	.1629E+03;	.3555E+03;	.1377E+03;
	.2281E+02;	.1241E+01;	.4327E+01;	.7040E+01;
	.4359E+01;	.3658E-13;	.8672E-13;	.2055E-12;
0.0001	.5587E+01;	.5746E+01;	.3041E+01;	.2299E+01;
	.3051E+01;	.2084E+01;	.2145E+01;	.2104E+01;
	.8541E+00;	.7783E+00;	.8606E+00;	.3342E+00;
	.6145E+00;	.1629E+00;	.3430E+00;	.2281E-01;
	.1377E+00;	.1224E-02;	.4297E-02;	.7040E-02;
	.4326E-02;	.1306E-15;	-.9310E-16;	.3515E-17;
0.00001	.5587E-02;	.5746E-02;	.3041E-02;	.3051E-02;
	.2299E-02;	.2084E-02;	.2145E-02;	.2104E-02;
	.8541E-03;	.7783E-03;	.8606E-03;	.6145E-03;
	.3342E-03;	.1629E-03;	.3430E-03;	.1377E-03;
	.2281E-04;	.1224E-05;	.4297E-05;	.7040E-05;
	.4326E-05;	.2151E-19;	.1129E-18;	-.1532E-18;

$B = 100$, t = 10, 1, 0.1, 0.01, 0.001, $E = 10^7$, $\nu = 0.3$ and uniform load $q = 1$. Eigen values corresponding to element 1 of the Figs. 12(a) and 12(b) for various thicknesses-span ratios are given in the Tables 5 and 6. The results show that the proposed element MQP8 is free from spurious/zero energy modes.

The simply supported square plate with various thickness-span ratio (thin: t/L = 0.00001, 0.0001, 0.001, 0.01 and moderately thick t/L = 0.1) subjected to uniform load is analyzed using the proposed element MQP8 for the grid size 4×4 in one quadrant of the plate to estimate the central deflections and moments. The parameters of the problem considered are: $L = 50$, $B = 50$, t = 5, 0.5, 0.05, 0.005, 0.0005, $E = 200000$, $\nu = 0.3$, $q = 1$. The exact central displacements and moments are

Table 6 Eigen values for irregular mesh (element 1 in the Fig. 14(b)), *t/L* = 0.1, 0.01, 0.001, 0.0001, 0.00001

t/L	Eigen values			
0.1	.4367E+10;	.4134E+10;	.2763E+10;	.1728E+10;
	.2821E+10;	.1817E+10;	.1721E+10;	.1827E+10;
	.9234E+09;	.7344E+09;	.4587E+09;	.1940E+09;
	.8356E+09;	.6545E+09;	.4183E+09;	.1245E+09;
	.2050E+08;	.9304E+06;	.3055E+07;	.5687E+07;
	.2934E+07;	-.1459E-06;	-.4187E-07;	-.4187E-07;
0.01	.9096E+07;	.6075E+07;	.4003E+07;	.3020E+07;
	.3246E+07;	.2250E+07;	.2417E+07;	.2110E+07;
	.9966E+06;	.8004E+06;	.1228E+07;	.5489E+06;
	.1979E+06;	.7029E+06;	.4703E+06;	.1258E+06;
	.2263E+05;	.3408E+04;	.9745E+03;	.6088E+04;
	.3149E+04;	-.1195E-09;	.2334E-09;	.8574E-11;
0.001	.1449E+05;	.6643E+04;	.3880E+04;	.3009E+04;
	.3367E+04;	.2254E+04;	.2907E+04;	.2162E+04;
	.1011E+04;	.8146E+03;	.1159E+04;	.4933E+03;
	.1984E+03;	.6944E+03;	.4391E+03;	.1260E+03;
	.2312E+02;	.3517E+01;	.9755E+00;	.6215E+01;
	.3204E+01;	-.2832E-13;	.1340E-12;	-.1083E-12;
0.0001	.1449E+02;	.6643E+01;	.3880E+01;	.3009E+01;
	.3367E+01;	.2254E+01;	.2907E+01;	.2162E+01;
	.1011E+01;	.8146E+00;	.1159E+01;	.4933E+00;
	.1984E+00;	.6944E+00;	.4391E+00;	.1260E+00;
	.2312E-01;	.3517E-02;	.9755E-03;	.6215E-02;
	.3204E-02;	-.2589E-15;	-.1283E-15;	-.1418E-16;
0.00001	.1449E-01;	.6643E-02;	.3009E-02;	.3880E-02;
	.2254E-02;	.3367E-02;	.2907E-02;	.2162E-02;
	.1011E-02;	.8146E-03;	.1159E-02;	.4933E-03;
	.1984E-03;	.6944E-03;	.4391E-03;	.1260E-03;
	.2312E-04;	.3517E-05;	.9755E-06;	.6215E-05;
	.3204E-05;	.1743E-18;	-.9612E-19;	-.9612E-19;

calculated from the Kirchhoff theory (Timoshenko and Krieger 1959) and Mindlin theory (Jane *et al.* 2000) solutions for thin and moderately thick plate bending problems respectively. The results are shown in the Figs. 13 and 14. These Figures indicate that the proposed element MQP8 performs quite well for both thin and moderately thick plate bending problems.

Tables and Figures of all example problems considered here show that the proposed element MQP8 has consistently estimated the deflections and moments quite closely to exact values and are better, in general, than those of displacement based similar finite elements considered. Also the performance of the proposed element MQP8 is quite excellent for both thin and moderately thick plate bending problems.

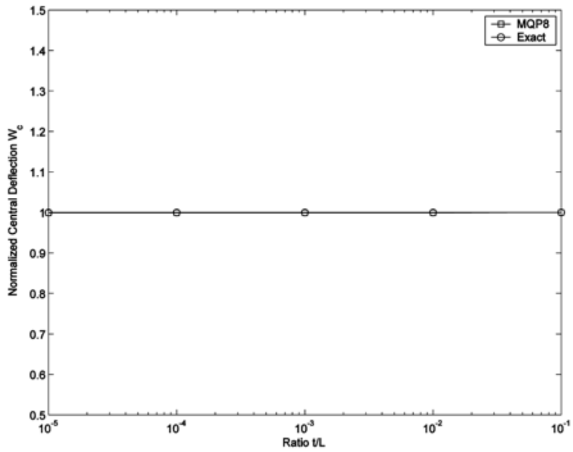


Fig. 13 Normalized central deflections for various thickness-span ratios (t/L = 0.00001, 0.0001, 0.001, 0.01 and 0.1)

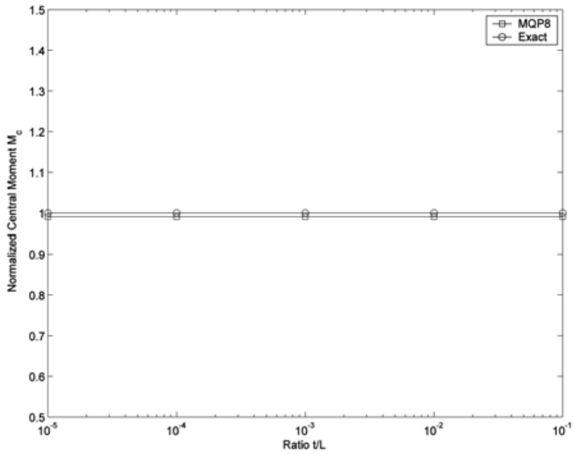


Fig. 14 Normalized central moments for various thickness-span ratios (t/L) (t/L = 0.00001, 0.0001, 0.001, 0.01 and 0.1)

4. Conclusions

The new 8 node quadrilateral plate bending element (MQP8) has been presented for the analysis of thin and moderately thick plate problems using the Integrated Force Method. The element considers three degrees of freedom namely a transverse displacement 'w' and two rotations θ_x, θ_y at each node. The Mindlin-Reissner theory has been employed in the formulation which accounts for the shear deformation. The proposed element MQP8 is free from spurious/zero energy modes.

The proposed element (MQP8) is also free from shear locking. Therefore the same element can be used to analyze thin as well as moderately thick plate bending problems. Many standard benchmark plate bending problems have been analyzed using the proposed element MQP8 via the Integrated Force Method. In all the plate bending problems considered, the proposed element MQP8 has produced, in general, excellent results. Therefore the proposed element MQP8 can be used as an alternative element to similar displacement-based 8-node quadrilateral plate bending elements considered for the analysis thin or moderately thick plate bending problems.

References

Bahattin Kanber, and Yavuz Bozkurt (2006), "Finite element analysis of elasto-plastic plate bending problems using transition rectangular plate elements", *Acta Mech. Sinica*, **22**, 355-365.

Choi, C.K. and Park, Y.M. (1999), "Quadratic NMS Mindlin-plate-bending element", *Int. J. Numer. Meth. Eng.*, **46**(8), 1273-1289.

Choi, C.K., Lee, T.Y. and Chung, K.Y. (2002), "Direct modification for nonconforming elements with drilling DOF", *Int. J. Numer. Meth. Eng.,* **55**(12), 1463-1476.

Chen, Wanji and Cheung, Y.K. (1987), "A new approach for the hybrid element method", *Int. J. Numer. Meth. Eng.*, **24**,1697-1709.

Darlmaz, K. (2005), "An assumed-stress finite element for static and free vibration analysis of Reissner-Mindlin plates", *Struct. Eng. Mech.,* **19**(2), 199-215.

Dhananjaya, H.R., Nagabhushanam, J. and Pandey, P.C. (2007), "Bilinear plate bending element for thin and moderately thick plates using Integrated Force Method", *Struct. Eng. Mech.*, **26**(1), 43-68.

Dimitris, K., Hung, L.T. and Atluri, S.N. (1984), "Mixed finite element models for plate bending analysis, A new element and its applications", *Comput. Struct.*, **19**(4), 565-581.

Jane Liu, Riggs, H.R. and Alexander Tessler (2000), "A four node shear-deformable shell element developed via explicit Kirchhoff constraints", *Int. J. Numer. Meth. Eng.*, **49**, 1065-1086.

Kaljevic, I., Patnaik, S.N. and Hopkins, D.A. (1996), "Development of finite elements for two- dimensional structural analysis using Integrated Force Method", *Comput. Struct.,* **59**(4), 691-706.

Kaljevic, I., Patnaik, S.N. and Hopkins, D.A. (1996), "Three dimensional structural analysis by Integrated Force Method", *Comput. Struct.*, **58**(5), 869-886.

Kaneko, L., Lawo, H. and Thierauf, G. (1983), "On computational procedures for the force method", *Int. J. Numer. Meth. Eng.*, **18**, 1469-1495.

Krishnam Raju, N.R.B. and Nagabhushanam, J. (2000), "Non-linear structural analysis using integrated force method", *Sadhana*, **25**(4), 353-365.

Kutlu Darlmaz and Nahit Kumbasar (2006), "An 8-node assumed stress hybrid element for analysis of shells", *Comput. Struct.*, **84**, 1990-2000.

Love, A.E.H. (1944), *A Treatise on the Mathematical Theory of Elasticity*, Dover, New York.

Morley, L.S.D. (1963), *Skew Plates and Structures*, Pergamon press, Oxford.

Nagabhushanam, J. and Patnaik, S.N. (1990), "General purpose program to generate compatibility matrix for the Integrated Force Method", *AIAA J.*, **28**, 1838-1842.

Nagabhushanam, J. and Srinivas, J. (1991), "Automatic generation of sparse and banded compatibility matrix for the Integrated Force Method, Computer Mechanics 91", *International Conference on Computing in Engineering Science, Patras, Greece*, 20-25.

Ozgan, K. and Ayse, T. Daloglu (2007), "Alternate plate finite elements for the analysis of thick plates on elastic foundations", *Struct. Eng. Mech.*, **26**(1), 69-86.

Patnaik, S.N. (1973), "An integrated force method for discrete analysis", *Int. J. Numer. Meth. Eng.*, **6**, 237-251.

Patnaik, S.N. (1986), "The variational energy formulation for the Integrated Force Method", *AIAA J.*, **24**, 129-137.

Patnaik, S.N., Berke, L. and Gallagher, R.H. (1991), "Integrated force method verses displacement method for finite element analysis", *Comput. Struct.*, **38**(4), 377-407.

Patnaik, S.N., Hopkins, D.A. and Coroneos, R. (1986), "Structural optimization with approximate sensitivities", *Comput. Struct.*, **58**, 407-418.

Patnaik, S.N., Coroneos, R.M. and Hopkins, D.A. (2000), "Compatibility conditions of structural mechanics", *Int. J. Numer. Meth. Eng.*, **47**, 685-704.

Patnaik, S.N. and Yadagiri, S. (1976), "Frequency analysis of structures by Integrated Force Method", *Comput. Meth. Appl. Mech. Eng.*, **9**, 245-265.

Pian, T.H.H. and Chen, D.P. (1982), "Alternative ways for formulation of hybrid stress elements", *Int. J. Numer. Meth. Eng.*, **19**, 1741-1752.

Przemieniecki, J.S. (1968), *Theory of Matrix Structural Analysis*, McGraw Hill, New York.

Razzaque, A. (1973), "Program for triangular plate bending element with derivative smoothing", *Int. J. Numer. Meth. Eng.*, **6**, 333-345.

Reissner, E. (1945), "The effect of transverse shear deformation on bending of plates", *J. Appl. Mech.*, **12**, A69-A77.

Robinson, J. (1973), *Integrated Theory of Finite Elements Methods*, Wiley, New York.

Robinson, J. and Haggenmacher, G.W. (1971), "Some new developments in matrix force Analysis, In Recent advances in matrix methods of structural analysis and design", *University Alabama*, 183-228.

Spilker, R.L. (1982), "Invariant 8-node hybrid-stress elements for thin and moderately thick plates", *Int. J. Numer. Meth. Eng.*, **18**, 1153-1178.

Kim, S.H. and Choi, C.K. (2005), "Modelling of plates and shells: Improvement of quadratic finite element for Mindlin plate bending", *Int. J. Numer. Meth. Eng.,* **34**(1), 197-208.

Timoshenko, S.P. and Krieger, S.W. (1959), *Theory of Plates and Shells,* Second Edition, McGraw_Hill international editions.

Notations

$[A]$	: matrix relating nodal degrees of freedom and coefficients of the polynomial
$[B]$	: global equilibrium matrix ($m \times n$)
$[B_e]$	: element equilibrium matrix ($m_e \times n_e$)
$[C]$	: compatibility matrix ($r \times n$)
$[D_{op}]$	: differential operator matrix
E	: Young's modulus
$\{F\}$	: vector of internal forces of the structure ($n \times 1$)
$\{F_e\}$	: vector of internal forces of the discrete element ($n_e \times 1$)
$[G]$	: global flexibility matrix ($n \times n$)
$[G_e]$	: element flexibility matrix ($n_e \times n_e$)
$[H]$	: matrix relating the curvatures to stress resultants
$[J]$	: deformation coefficient matrix ($m \times n$)
L, B	: Length and breadth of the plate
M_c	: central moment of the plate
$\{M\}$	: vector of stress resultants
P	: point load at the center or tip of the plate
$\{P\}$	: vector of external loads ($m \times 1$)
q	: uniform load over the plate
$[S]$	: IFM governing matrix ($n \times n$)
W_c	: central deflection of the plate
$\{X\}$	: vector of displacements of the structure ($m \times 1$)
$\{X_e\}$	: vector of displacements of the discrete element ($m_e \times 1$)
$\{k\}$	: vector of curvatures
n, m	: force and displacement degrees of freedom of the structures respectively
n_e, m_e	: element force and displacement degrees of freedom respectively
t	: thickness of the plate
$\{\alpha\}$	: generalized coordinates of the polynomial in the displacement field.
$\{\beta\}$	: vector of elastic deformations
$\{\beta_o\}$	: vector of initial deformations
ν	: Poisson's ratio
$[\phi_1]$	: matrix of polynomial terms for displacement fields
$[\psi]$	: matrix of polynomial terms for stress-resultants fields

Appendix A: Basic theory of IFM

In the Integrated Force Method of analysis, a structure idealized by finite elements is designated as "*structure(n, m)*". Where (n, m) are force and displacement degrees of freedom of the discrete model, respectively. The structure (n, m) has m equilibrium Equations and $r = (n - m)$ compatibility conditions. Equilibrium equations (EE) represent the vectorial summation of the internal forces $\{F\}$ to the external loads $\{P\}$ at the nodes of the finite element descritization. It can be written in symbolized matrix notation as

$$\text{Equilibrium Equations[EE]} : [B]\{F\} = \{P\} \tag{A.1}$$

Where $[B]$ = global equilibrium matrix
$\{F\}$ = vector of internal forces of the structure
$\{P\}$ = vector of external loads on the structure

The Compatibility Conditions (CC) are constraints on strains, and for finite element models they are also constraints on member deformations.

In IFM, St. Venant's approach has been extended for discrete mechanics to develop the compatibility conditions. Development of CC is briefly explained below:

The Deformation-Displacement Relationship(DDR) for discrete mechanics is equivalent to the strain-displacement relationship in elasticity. The DDR for discrete analysis was obtained during the development of the variational energy formulation for the IFM (Krishnam Raju and Nagabhushanam 2000)

According to work energy conservation theorem, the internal energy stored in the body in the structure is equal to the work done by the external load, that is

$$\frac{1}{2}\{F\}^T\{\beta\} = \frac{1}{2}\{P\}^T\{X\} \tag{A.2}$$

where $\{X\}$ represents nodal displacements. Eq. (A.2) can be rewritten by eliminating the load $\{P\}$ in favor of forces $\{F\}$, by using the Eq. (A.1) to obtain the following relation

$$\frac{1}{2}\{F\}^T[B]^T\{X\} = \frac{1}{2}\{F\}^T\{\beta\} \tag{A.3}$$

Eq. (A.3) can be simplified as

$$\frac{1}{2}\{F\}^T[[B]^T\{X\} - \{\beta\}] = 0 \tag{A.4}$$

Since $\{F\}$ is not a null vector, its coefficient must be equal to zero, which yields the DDR as

$$\{\beta\} = [B]^T\{X\} \tag{A.5}$$

Where $\{\beta\}$ are member deformations.

This equation represents the Deformation Displacement Relations (DDR) for the discrete structure. The elimination of m displacements from n deformations displacement relations given by the above equation yields $r = (n - m)$ compatibility conditions and the associated matrix [C].

Here while obtaining the matrix [C], no reference is made to redundant forces. Thus the concept of redundant force selection is eliminated in IFM.

It can be symbolized in matrix notations as

$$[C]\{\beta\} = 0 \tag{A.6}$$

Substituting Eq. (5) into the Eq. (A.6), we obtain

$$[C]\{\beta\} = [C][B]^T\{X\} = 0$$
$$\{X\}^T([B][C]^T) = \{0\}$$

Since the displacement vector $\{X\}$ is not a null vector, we have

$$[B][C]^T = 0 \tag{A.7}$$

where [C] is the (r x n) compatibility matrix. It is a kinematics relationship, and it is independent of design parameters, material properties and external loads. This matrix is rectangular and banded. The deformation $\{\beta\}$ in the compatibility conditions (CC) given by the Eq. (6) represents the total deformation consisting of an elastic component $\{\beta_e\}$ and the initial component $\{\beta_o\}$ as

$$\{\beta\} = \{\beta_e\} + \{\beta_0\} \tag{A.8}$$

The CC in terms of elastic deformation can be written as

$$[C]\{\beta\} = [C]\{\beta_e\} + [C]\{\beta_0\} = 0 \tag{A.9}$$

$$[C]\{\beta_e\} = \{\delta R\}$$

Where

$$\{\delta R\} = -[C]\{\beta_0\} \tag{A.10}$$

Using the flexibility characteristics, Eq. (A.6) with initial deformations can be rewritten as

$$[C][G]\{F\} = \{\delta R\} \tag{A.11}$$

Combining Eqs. (A.1) and (A.11), we lead to the IFM governing equation as

$$\begin{bmatrix} [B] \\ ([C][G]) \end{bmatrix} \{F\} = \begin{Bmatrix} P \\ \delta R \end{Bmatrix}$$

$$[S]\{F\} = \{P^*\} \tag{A.12}$$