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Stochastic free vibration analysis of smart random composite plates

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Abstract. The present study is concerned with the stochastic linear free vibration study of laminated composite plate embedded with piezoelectric layers with random material properties. The system equations are derived using higher order shear deformation theory. The lamina material properties of the laminate are modeled as basic random variables for accurate prediction of the system behavior. A C^0 finite element is used for spatial descretization of the laminate. First order Taylor series based mean centered perturbation technique in conjunction with finite element method is outlined for the problem. The outlined probabilistic approach is used to obtain typical numerical results, i.e., the mean and standard deviation of natural frequency. Different combinations of simply supported, clamped and free boundary conditions are considered. The effect of side to thickness ratio, aspect ratio, lamination scheme on scattering of natural frequency is studied. The results are compared with those available in literature and an independent Monte Carlo simulation.

Keywords: natural frequency; smart composite materials; random variables; finite element method.

1. Introduction

The composite materials are finding extensive applications in primary and secondary structures in aeronautical and space projects, like advance aircraft, helicopters, launch vehicles, satellite, space stations etc. because they have outstanding mechanical properties, such as high strength and stiffness to weight ratio, excellent corrosion resistance and very good fatigue characteristics.

The development of new class of smart materials has improved the performance and reliability of structural systems made of composite materials. Piezoelectric material is one of the smart materials. Such materials combine the superior mechanical properties of composite materials as well as incorporate the additional inherent capability to sense and adapt their static and dynamic responses. The piezoelectric materials have the property of converting mechanical energy in electrical energy and vice versa. They are employed as both sensors and actuators in the development of smart structures by taking advantage of direct and converse piezoelectric effects.

The smart composite structures have more uncertainties and variability in their structural properties compared to conventional isotropic structures as a large number of parameters is associated with their complex manufacturing and fabrication processes. Some variations of structural

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parameters, such as material properties, fiber angles, thickness and curvatures, are inevitable even for the smart composite laminates manufactured carefully in a laboratory. In common deterministic analysis, these structural parameters are taken to be deterministic and the variations in these are ignored, leading to approximation in analysis and design.

Keeping in view the above aspects, the mechanical properties of materials may be modeled as basic random variables (RVs) for accurate prediction of the structural system behavior. In spite of well-developed theory and computational methods for structural responses with deterministic material properties, structural analysis in which the material properties are considered random, is still underdeveloped.

Considerable efforts have been made in the past by researchers and investigators on the predictions of the dynamic and buckling response of smart composite structures considering the material properties as deterministic. Notably among them are due to Tzou and Tseng (1991), Huang and Tseng (1996), Correia Franco *et al*. (1997), Saravanos *et al*. (1997), Correia Franco *et al*. (2000), Zhou and Chattopadhayay (2000), Chen *et al*. (2000), Sadek *et al*. (2003), Simoes Moita *et al*. (2004), and Shu, (2005). Extensive literature is available on the response analysis of the deterministic material properties to random excitations (Nigam and Narayanan, 1994). However, a limited literature is available on the analysis of the structures with random system properties. Much of the published work on random system properties is based on structures made of isotropic materials. A relatively little work is available on the structures made of composites and absolutely no work is reported on structures made of composites with embedded sensors and /or actuators with random system properties to the best of authors' knowledge. A comprehensive summary of the state-of-the art on structural dynamics with parameters uncertainties has been presented by Ibrahim (1987) and Manohar and Ibrahim (1998). Liessa and Martin (1990) analyzed free vibration and buckling of rectangular composite plates and determined the variation in fibre spacing or redistribution of fibre tend to increase fundamental frequency by 21% and the buckling load by 38%. Nakagiri *et al*. (1990) studied simply supported graphite epoxy plates with stochastic finite element method taking fibre orientation, layer thickness, and number of layers as random variables, and found that the overall stiffness of fibre reinforced plastic laminated plates is largely dependent on the fibre orientation. Zang and Chen (1991) presented a method to estimate the standard deviations of eigen-value and eigenvector of random multiple degree of freedom (MDOF) systems. The methods can be applied not only to the MDOF systems with distinct eigenvalues, but also to MDOF systems with uncertainties in mass and stiffness matrix elements. They used both the sensitivity and the perturbation technique to develop the methodology. Salim *et al*. (1993) employed first order perturbation technique (FOPT) with Rayleigh-Ritz technique for analyzing composite plates using classical laminate theory with random material properties. They analyzed specially orthotropic composite laminates with all edges simply supported with deterministic loading to obtain the SD of deflections. Englested and Reddy (1994) studied metal matrix composites based on probabilistic micromechanics nonlinear analysis. They used Monte Carlo simulation (MCS) and different probability distributions to incorporate the uncertainty in basic material properties. Raj *et al*. (1998) obtained the static response of graphite-epoxy composite laminates with randomness in material properties subjected to deterministic loading. The material properties have been modeled as independent random variables. They used a higher order shear deformation theory to model the plate behavior. They used the finite element method (FEM) in conjunction with MCS for obtaining the second order statistics of static response. The sensitivity of variation of the SD of the response towards the uncertainties in material properties has been examined for composite plates with different support conditions. Singh *et al.* (2001) obtained the second order statistics of first five natural frequencies for two stacking sequences of cross ply laminates using Perturbation Technique. They found that the fundamental mean frequency for anti-symmetric cross-ply is always lower than symmetric cross ply. However, the relative value of the other natural frequencies depends on the a/h ratio, mode of vibration and modular ratio. Also they found out that the classical laminate theory over predicts the scatter in the fundamental frequency. Even for the thin plates, the use of classical laminate theory may not be appropriate for study of scatter in the fundamental frequency. Onkar and Yadav (2003) investigated non-linear response statistics of composite laminates using classical approach with random material properties under random loading. Tripathi *et al.* (2007) analyzed the free vibration of laminated composite conical shells with random material properties using stochastic finite element method.

This paper presents probabilistic methodology for application of the HSDT based FEM in conjunction with Taylor series based perturbation technique to random eigen value problem arising from free vibration of the laminated composite plates integrating distributed piezoelectric sensors due to random variation of lamina material properties. The C^0 finite element as proposed by Singh *et al.* (2002) for the random buckling analysis of the laminated composite curved shallow panels is extended to free vibration analysis of composite plates embedded with piezoelectric layers with random material properties. The formulation is in the framework of linear elasticity for small deflection of smart composite laminates. The lamina material properties are modeled as basic RVs, while other system parameters are taken as deterministic. To show the applicability of the proposed outlined probabilistic methodology illustrative numerical examples are presented and discussed.

2. General formulation

Consider a laminated composite plate embedded with piezoelectric layers with perfect bonding between laminae (Fig. 1). Assuming the transverse shear stresses and corresponding transverse shear strains vanish on top and bottom surfaces of the laminate, the displacement field at a point in the laminated composite plate based on HSDT is given as (Reddy 1984)

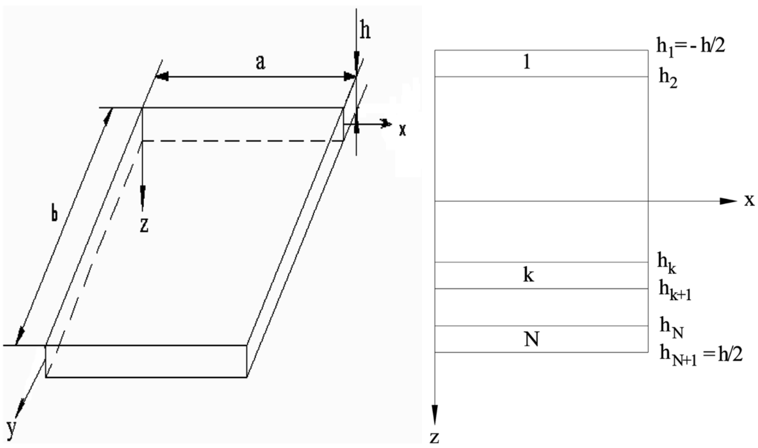

Fig. 1 Geometry of the laminated composite plate with piezoelectric layer

$$\bar{u} = u + z\psi_x - z^3 4/3h^2(\psi_x + \partial w/\partial x) = u + f_1(z)\psi_x + f_2(z)\partial w/\partial x$$
$$\bar{v} = v + z\psi_y - z^3 4/3h^2(\psi_y + \partial w/\partial y) = v + f_1(z)\psi_y + f_2(z)\partial w/\partial y$$
$$\bar{w} = w \tag{1}$$

where u, v and w denote the displacements of a point on the mid surface in x, y and z directions, respectively, and ψ_x and ψ_y are the rotations of plane normal to midsurface about the y and x axes, respectively. The displacement functions are represented as $f_1(z) = C_1 z - C_2 z^3$ an $f_2(z) = -C_4 z^3$, where $C_1 = 1$ and $C2 = C4 = 4/3h^2$.

The displacement field model given in Eq. (1) can be rewritten as

$$\bar{u} = u + f_1(z)\psi_x + f_2(z)\theta_x = u + f_1(z)\psi_1 + f_2(z)\theta_1$$
$$\bar{v} = u + f_1(z)\psi_y + f_2(z)\theta_y = v + f_1(z)\psi_2 + f_2(z)\theta_2$$
$$\bar{w} = w \tag{2}$$

where $\theta_x = \theta_1 = \partial w/\partial x$ and $\theta_y = \theta_2 = \partial w/\partial y$

It can be seen that the degrees of freedom per node, by treating θ_x and θ_y as separate independent DOFs, increase from 5 to 7. However, the strain vector will be having only first order derivative, and hence a C^0 continuous element would be sufficient.

Considering the linear kinematics, the strain-displacement relations can be expressed as

$$\varepsilon_{xx} = \varepsilon_1 = \partial\bar{u}/\partial x = \varepsilon_1^0 + z(k_1^0 + z^2 k_1^2)$$
$$\varepsilon_{yy} = \varepsilon_2 = \partial\bar{v}/\partial y = \varepsilon_1^0 + z(k_1^0 + z^2 k_1^2)$$
$$\gamma_{xy} = \varepsilon_6 = \partial\bar{u}/\partial y + \partial\bar{v}/\partial x = \varepsilon_6^0 + z(k_6^0 + z^2 k_6^2)$$
$$\gamma_{yz} = \varepsilon_4 = \partial\bar{v}/\partial z + \partial\bar{w}/\partial y = \varepsilon_4^0 + z^2 k_4^2$$
$$\gamma_{xz} = \varepsilon_5 = \partial\bar{u}/\partial z + \partial\bar{w}/\partial x = \varepsilon_5^0 + z^2 k_5^2 \tag{3}$$

where

$$\varepsilon_1^0 = \frac{\partial u}{\partial x}; \quad k_1^0 = C_1\frac{\partial \psi_x}{\partial x}; \quad k_1^2 = -C_2\frac{\partial \psi_x}{\partial x} - C_4\frac{\partial \theta_x}{\partial x}$$
$$\varepsilon_2^0 = \frac{\partial v}{\partial y}; \quad k_2^0 = C_1\frac{\partial \psi_y}{\partial y}; \quad k_1^2 = -C_2\frac{\partial \psi_y}{\partial y} - C_4\frac{\partial \theta_y}{\partial y}$$
$$\varepsilon_4^0 = C_1\psi_y + \frac{\partial w}{\partial y}; \quad k_4^0 = -3C_2\psi_y - 3C_4\theta_y$$
$$\varepsilon_5^0 = C_1\psi_x + \frac{\partial w}{\partial x}; \quad k_4^0 = -3C_2\psi_x - 3C_4\theta_x$$
$$\varepsilon_6^0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}; \quad k_6^0 = C_1\left(\frac{\partial \psi_y}{\partial x} + \frac{\partial \psi_x}{\partial y}\right); \quad k_6^2 = -C_2\left(\frac{\partial \psi_y}{\partial x} + \frac{\partial \psi_x}{\partial y}\right) - C_4\left(\frac{\partial \theta_y}{\partial x} + \frac{\partial \theta_x}{\partial y}\right) \tag{4}$$

Assuming that a piezoelectric composite laminate consists of several layers, including the piezoelectric layers, the linear constitutive equation for an orthotropic lamina of the laminate substrate, is

$$\{\sigma\} = [\bar{Q}]\{\varepsilon\} \tag{5}$$

and the constitutive equations of a deformable piezoelectric material, coupling the elastic and the electric fields are given as (Tiersten 1969)

$$\{\sigma\} = [\bar{Q}]\{\varepsilon\} - [e]\{E\} \tag{6}$$

$$\{D\} = [e]^T\{\varepsilon\} + [k]\{E\} \tag{7}$$

with

$$[\bar{Q}] = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & 0 & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}, \quad [e] = \begin{bmatrix} 0 & 0 & 0 & e_{14} & e_{15} \\ 0 & 0 & 0 & e_{24} & e_{25} \\ e_{31} & e_{32} & e_{36} & 0 & 0 \end{bmatrix}^T$$

$$[k] = \begin{bmatrix} k_{11} & k_{12} & 0 \\ k_{12} & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix} \tag{8}$$

where $(\sigma) = [\sigma_x \ \ \sigma_y \ \ \tau_{xy} \ \ \tau_{yz} \ \ \tau_{zx}]^T$ is the elastic stress vector, $(\varepsilon) = [\varepsilon_{xx} \ \ \varepsilon_{yy} \ \ \varepsilon_{xy} \ \ \varepsilon_{yz} \ \ \varepsilon_{zx}]^T$ is the elastic strain vector, $\bar{Q}_{ij}$ is the symmetric reduced elastic constitutive matrix, $[e]$ is the piezoelectric stress coefficient matrix, $\{E\} = [E_x \ \ E_y \ \ E_z]^T$ is the electric field vector, $\{D\} = [D_x \ \ D_y \ \ D_z]^T$ is the electric displacement vector and $[k]$ is the symmetric dielectric matrix, in the element local system (x, y, z) of the laminate.

The electric field vector is the negative gradient of the electric potential ϕ, which is assumed to be applied and varying quadratically in the all three directions

$$\{E\} = \begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial\phi}{\partial x} \\ -\dfrac{\partial\phi}{\partial y} \\ -\dfrac{\partial\phi}{\partial z} \end{Bmatrix}; \quad \phi(x,y,z) = \phi^{(0)}(x,y) + z\phi^{(1)}(x,y) + z^2\phi^{(2)}(x,y) \tag{9}$$

Hence

$$E_x = -\frac{\partial\phi}{\partial x} = -\left(\frac{\partial\phi^{(0)}}{\partial x} + z\frac{\partial\phi^{(1)}}{\partial x} + z^2\frac{\partial\phi^{(2)}}{\partial x}\right) = E_x^{(0)} + zE_x^{(1)} + z^2E_x^{(2)} \tag{10a}$$

$$E_y = -\frac{\partial\phi}{\partial y} = -\left(\frac{\partial\phi^{(0)}}{\partial y} + z\frac{\partial\phi^{(1)}}{\partial y} + z^2\frac{\partial\phi^{(2)}}{\partial y}\right) = E_y^{(0)} + zE_y^{(1)} + z^2E_y^{(2)} \tag{10b}$$

$$E_z = -\frac{\partial\phi}{\partial z} = -(\phi^{(1)} + 2z\phi^{(2)}) = X^{(1)} + zX^{(2)} \tag{10c}$$

Using Eqs. (10a-c), the electric field vector $\{E\}$ can be written as

$$\{E\} = [T_\phi]\{E^{(0)}\} \tag{11}$$

where

$$[T_\phi] = \begin{bmatrix} 1 & 0 & z & 0 & z^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & z & 0 & z^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & z \end{bmatrix} \tag{12}$$

and

$$\{E^0\} = \{E_x^{(0)} \;\; E_y^{(0)} \;\; E_x^{(1)} \;\; E_y^{(1)} \;\; E_x^{(2)} \;\; E_y^{(2)} \;\; X^{(1)} \;\; X^{(2)}\}^T \tag{13}$$

The vector $\{E^0\}$ can also be expressed as

$$\{E^0\} = [L_\phi]\{\phi\} \tag{14}$$

$$[L_\phi] = \begin{bmatrix} -\frac{\partial}{\partial x} & 0 & 0 \\ -\frac{\partial}{\partial y} & 0 & 0 \\ 0 & -\frac{\partial}{\partial x} & 0 \\ 0 & -\frac{\partial}{\partial y} & 0 \\ 0 & 0 & -\frac{\partial}{\partial x} \\ 0 & 0 & -\frac{\partial}{\partial y} \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}; \quad \{\phi\} = \{\phi^{(0)} \;\; \phi^{(1)} \;\; \phi^{(2)}\}^T \tag{15}$$

The potential energy of a piezoelectric laminated composite plate undergoing small deformation is given as

Potential energy U = Strain energy – Electrical energy

$$U = \frac{1}{2}\int_V \{\varepsilon\}^T\{\sigma\}dV - \frac{1}{2}\int_V \{E\}^T\{D\}dV \tag{16}$$

Using constitutive relations as given in Eqs. (6)-(7), the potential energy U can be expressed as

$$U = \frac{1}{2}\int_v (\varepsilon^T[\overline{Q}\varepsilon - eE] - E^T[e^T\varepsilon + kE])dV \tag{17}$$

Using Eq. (3), the strain vector can be expressed as

$$\{\varepsilon\} = [T]\{\overline{\varepsilon}\} \tag{18}$$

where

$$[k] = \begin{bmatrix} 1 & 0 & 0 & z & 0 & 0 & z^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & z & 0 & 0 & z^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & z & 0 & 0 & z^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & z^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & z^2 \end{bmatrix} \tag{19}$$

and

$$\{\bar{\varepsilon}\} = (\varepsilon_1^0 \ \ \varepsilon_2^0 \ \ \varepsilon_6^0 \ \ k_1^0 \ \ k_2^0 \ \ k_6^0 \ \ k_1^2 \ \ k_2^2 \ \ k_6^2 \ \ \varepsilon_4^0 \ \ \varepsilon_5^0 \ \ k_4^2 \ \ k_5^2) \tag{20}$$

Using Eqs. (11) & (18), the Eq. (17) becomes

$$U = \frac{1}{2}\int_V (\bar{\varepsilon}^T T^T \overline{Q} T \bar{\varepsilon} - \bar{\varepsilon}^T T^T e T_\phi E^0 - E^{0T} T_\phi^T e^T T \bar{\varepsilon} - E^{0T} T_\phi^T k T_\phi E^0) dV \tag{21}$$

This can further be rewritten as

$$U = \frac{1}{2}\int_A (\bar{\varepsilon}^T D \bar{\varepsilon} - \bar{\varepsilon}^T D_1 E^0 - (E^0)^T D_1^T \bar{\varepsilon} - (E^0)^T D_2 E^0) dA \tag{22}$$

where

$$D = \sum_{k=1}^{NL} \int_{z_{k-1}}^{z_k} [T]^T [\overline{Q}][T] dz = \begin{bmatrix} [A_1] & [B] & [E] & 0 & 0 \\ [B] & [C_1] & [F_1] & 0 & 0 \\ [E] & [F_1] & [H] & 0 & 0 \\ 0 & 0 & 0 & [A_2] & [C_2] \\ 0 & 0 & 0 & [C_2] & [F_2] \end{bmatrix} \tag{23a}$$

with

$$(A_{1_{ij}}, B_{ij}, C_{1_{ij}}, E_{ij}, F_{1_{ij}}, H_{ij}) = \sum_{k=1}^{NL} \int_{z_{k-1}}^{z_k} \overline{Q}_{ij}^{(k)} (1, z, z^2, z^3, z^4, z^6) dz, \quad i,j = 1, 2, 6$$

$$(A_{2_{ij}}, C_{2_{ij}}, F_{2_{ij}}) = \sum_{k=1}^{NL} \int_{z_{k-1}}^{z_k} \overline{Q}_{ij}^{(k)} (1, z^2, z^4) dz, \quad i,j = 4, 5 \tag{23b}$$

$$[D_1] = \sum_{k=1}^{NL} \int_{z_{k-1}}^{z_k} [T]^T [e][T] dz = \begin{bmatrix} [0] & [0] & [0] & [M_1] & [N_1] \\ [0] & [0] & [0] & [N_1] & [P_1] \\ [0] & [0] & [0] & [Q_1] & [R_1] \\ [M_2] & [N_2] & [P_2] & [0] & [0] \\ [P_2] & [Q_2] & [R_2] & [0] & [0] \end{bmatrix} \tag{24a}$$

with

$$(M_{1_{ij}}, N_{1_{ij}}, P_{1_{ij}}, Q_{1_{ij}}, R_{1_{ij}}) = \sum_{k=1}^{NL} \int_{z_{k-1}}^{z_k} e_{ij}^{(k)}(1, z, z^2, z^3, z^4)dz, \quad i = 3, j = 1, 2, 6$$

$$(M_{2_{ij}}, N_{2_{ij}}, P_{2_{ij}}, Q_{2_{ij}}, R_{2_{ij}}) = \sum_{k=1}^{NL} \int_{z_{k-1}}^{z_k} e_{ij}^{(k)}(1, z, z^2, z^3, z^4)dz \quad i = 1, 2, j = 4, 5 \tag{24b}$$

$$[D_2] = \sum_{k=1}^{NL} \int_{z_{k-1}}^{z_k} [T_\phi]^T [k][T_\phi]dz = \begin{bmatrix} [S_1] & [T_1] & [U_1] & [0] & [0] \\ [T_1] & [U_1] & [V] & [0] & [0] \\ [U_1] & [V] & [W] & [0] & [0] \\ [0] & [0] & [0] & [S_2] & [T_2] \\ [0] & [0] & [0] & [T_2] & [U_2] \end{bmatrix} \tag{25a}$$

with

$$(S_{1_{ij}}, T_{1_{ij}}, U_{1_{ij}}, V_{ij}, W_{ij}) = \sum_{k=1}^{NL} \int_{z_{k-1}}^{z_k} k_{ij}^{(k)}(1, z, z^2, z^3, z^4)dz \quad i, j = 1, 2$$

$$(S_{2_{ij}}, T_{2_{ij}}, U_{2_{ij}}) = \sum_{k=1}^{NL} \int_{z_{k-1}}^{z_k} k_{ij}^{(k)}(1, z, z^2)dz, \quad i, j = 3 \tag{25b}$$

The Eq. (4) can be further written in matrix form as

$$\{\bar{\varepsilon}\} = [L]\{\Lambda\} \tag{26}$$

where

$$\{\Lambda\} = \{u \;\; v \;\; w \;\; \theta_y \;\; \theta_x \;\; \psi_y \;\; \psi_x\}^T, \text{ and}$$

$$[L] = \begin{bmatrix} \partial/\partial x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial/\partial y & 0 & 0 & 0 & 0 & 0 \\ \partial/\partial y & \partial/\partial x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C1\partial/\partial x \\ 0 & 0 & 0 & 0 & 0 & C1\partial/\partial y & 0 \\ 0 & 0 & 0 & 0 & -C4\partial/\partial x & C1\partial/\partial x & C1\partial/\partial y \\ 0 & 0 & 0 & 0 & 0 & 0 & -C2\partial/\partial x \\ 0 & 0 & 0 & -C4\partial/\partial y & -C4\partial/\partial y & -C2\partial/\partial y & 0 \\ 0 & 0 & 0 & -C4\partial/\partial x & 0 & -C2\partial/\partial x & -C2\partial/\partial y \\ 0 & 0 & C1\partial/\partial y & 0 & 0 & C1 & 0 \\ 0 & 0 & C1\partial/\partial x & 0 & 0 & 0 & C_1 \\ 0 & 0 & 0 & -3C4 & 0 & -3C2 & 0 \\ 0 & 0 & 0 & 0 & -3C4 & 0 & -3C2 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ \theta_y \\ \theta_x \\ \psi_y \\ \psi_x \end{Bmatrix} \tag{27}$$

Using Eqs. (14) and (26), the Eq. (22) can be further written as

$$U = \frac{1}{2}\int_A (\Lambda^T L^T D L \Lambda - \Lambda^T L^T D_1 L_\phi \phi - \phi^T L_\phi^T D_1^T L \Lambda - \phi^T L_\phi^T D_2 L_\phi \phi) dA \tag{28}$$

For *NL* number of layers of composite plate embedded with piezoelectric layer, the kinetic energy of a vibrating plate in bending is given as

$$T = \frac{1}{2}\int_A \left(\sum_{k=1}^{NL} \int_{Z_{k-1}}^{Z_k} \rho^{(k)} \{\dot{u}\}^T \{\dot{u}\} dz \right) dA \tag{29}$$

where, $\{u\} = \{u\ v\ w\}^T$ is global displacement vector, and $\rho^{(k)}$ is k^{th} layer density.

The $\{u\}$ vector can be written as

$$\{u\} = [\overline{N}]\{\Lambda\} \tag{30}$$

where

$$[\overline{N}] = \begin{bmatrix} 1 & 0 & 0 & 0 & f_2(z) & 0 & f_1(z) \\ 0 & 1 & 0 & f_2(z) & 0 & f_1(z) & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{31}$$

Hence the global velocity vector reduces to

$$\{\dot{u}\} = [\overline{N}]\{\dot{\Lambda}\} \tag{32}$$

Using the above equations, the Eq. (29) can be written as

$$T = \frac{1}{2}\int_A \{\dot{\Lambda}\}^T [m] \{\dot{\Lambda}\} dA \tag{33}$$

where

$$[m] = \sum_{k=1}^{NL} \int_{z_{k-1}}^{z_k} \rho^{(k)} [\overline{N}]^T [\overline{N}] dz$$

$$= \begin{bmatrix} p & 0 & 0 & 0 & q_2 & 0 & q_1 \\ 0 & p & 0 & q_2 & 0 & q_1 & 0 \\ 0 & 0 & p & 0 & 0 & 0 & 0 \\ 0 & q_2 & 0 & I_2 & 0 & I_3 & 0 \\ q_2 & 0 & 0 & 0 & I_2 & 0 & I_3 \\ 0 & q_1 & 0 & I_3 & 0 & I_1 & 0 \\ q_1 & 0 & 0 & 0 & I_3 & 0 & I_1 \end{bmatrix} \tag{34a}$$

with

$$(p, q_1, q_2, I_1, I_2, I_3) = \sum_{k=1}^{NL} \int_{z_{k-1}}^{z_k} \rho^{(k)} (1, f_1(z), f_2(z), f_1^2(z), f_2^2(z), f_1(z) f_2(z)) dz \tag{34b}$$

3. Probabilistic methodology of solution

3.1 Finite element formulation

In the present a nine noded C^0 isoparametric flat plate element is used to carry out the free vibration analysis of general multi-layered composite plates embedded with piezoelectric layer. The element has nine nodes and seven degrees of freedom per node, $\{\Lambda\} = \{u\ v\ w\ \theta_y\ \theta_x\ \psi_y\ \psi_x\}^T$.

For the finite element analysis, the Eq. (28) can be written as

$$U = \sum_{e=1}^{NE} U^{(e)} \tag{35}$$

$$U = \sum_{e=1}^{NE} \frac{1}{2} \int_{A^{(e)}} (\Lambda^{(e)^T} L^T D L \Lambda^{(e)} - \Lambda^{(e)^T} L^T D_1 L_\phi \phi^{(e)} - \phi^{(e)^T} L_\phi^T D_1^T L \Lambda^{(e)} - \phi^{(e)^T} L_\phi^T D_2 L_\phi \phi^{(e)}) dA \tag{36}$$

where NE is number of elements used for meshing the plate.

The displacement vector Λ and the electric potential ϕ can be written in terms of shape functions N_i and N_ϕ, respectively as

$$\{\Lambda\} = \sum_{i=1}^{NN} [N_i]\{\Lambda_i\}, \{\phi\} = \sum_{i=1}^{NN} [N_{\phi i}]\{\phi_i\} \tag{37}$$

For an element, these can be written in matrix form as

$$\{\Lambda\}^{(e)} = [N]^{(e)}\{q\}^{(e)}, \quad \{\phi\}^{(e)} = [N_\phi]^{(e)}\{q_\phi\}^{(e)} \tag{38}$$

Substituting the above expressions of $\{\Lambda\}^{(e)}$ and $\{\phi\}^{(e)}$ in the Eq. (36), one obtains as

$$U = \sum_{e=1}^{NE} \frac{1}{2} \int_{A^{(e)}} \begin{pmatrix} q^{(e)^T} N^{(e)^T} L^T D L N^{(e)} q^{(e)} - q^{(e)^T} N^{(e)^T} L^T D_1 L_\phi N_\phi^{(e)} q_\phi^{(e)} - \\ q_\phi^{(e)^T} N_\phi^{(e)^T} L_\phi^T D_1^T L N^{(e)} q^{(e)} - q_\phi^{(e)^T} N_\phi^{(e)^T} L_\phi^T D_2 L_\phi N_\phi^{(e)} q_\phi^{(e)} \end{pmatrix} dA \tag{39}$$

Elemental potential energy can be written as

$$U^{(e)} = \frac{1}{2} \int_{A^{(e)}} \begin{pmatrix} q^{(e)^T} B^{(e)^T} D B^{(e)} q^{(e)} - q^{(e)^T} B^{(e)^T} D_1 B_\phi^{(e)} q_\phi^{(e)} - \\ q_\phi^{(e)^T} B_\phi^{(e)^T} D_1^T B^{(e)} q^{(e)} - q_\phi^{(e)^T} B_\phi^{(e)^T} D_2 B_\phi^{(e)} q_\phi^{(e)} \end{pmatrix} dA \tag{40}$$

where

$$[B]^{(e)} = [L][N]^{(e)} \tag{41a}$$

here

$$[B]^{(e)} = [B_1\ B_2\ B_3\ \ldots\ B_{NN}], \quad \text{with} \quad [B_i] = [L][N_i] \quad i = 1, 2, \ldots, NN \tag{41b}$$

Similarly

$$[B_\phi]^{(e)} = [L_\phi][N_\phi]^{(e)} \tag{42}$$

Using Eqs. (41a) and (42), the Eq. (40) can be written as

$$U^{(e)} = \frac{1}{2}q^{(e)T}K^{(e)}q^{(e)} - \frac{1}{2}q^{(e)T}K1^{(e)}q_{\phi}^{(e)} - \frac{1}{2}q_{\phi}^{(e)T}K1^{(e)T}q^{(e)} - \frac{1}{2}q_{\phi}^{(e)T}K2^{(e)}q_{\phi}^{(e)} \tag{43}$$

where element bending stiffness matrix

$$K^{(e)} = \int_{A^{(e)}} B^{(e)T}DB^{(e)}dA$$

coupling matrix

$$K1^{(e)} = \int_{A^{(e)}} B^{(e)T}D_1B_{\phi}^{(e)}dA$$

and dielectric stiffness matrix

$$K2^{(e)} = \int_{A^{(e)}} B_{\phi}^{(e)T}D_2B_{\phi}^{(e)}dA \tag{44a-c}$$

Here $K^{(e)}$, $K_1^{(e)}$ and $K_2^{(e)}$ are computed numerically by transforming existing coordinate system to natural coordinate system ξ and η using Gauss quadrature rule.

Thus potential energy (Eq. (39)) of the laminate becomes,

$$U = \sum_{e=1}^{NE}\frac{1}{2}q^{(e)T}K^{(e)}q^{(e)} - \frac{1}{2}q^{(e)T}K1^{(e)}q_{\phi}^{(e)} - \frac{1}{2}q_{\phi}^{(e)T}K1^{(e)T}q^{(e)} - \frac{1}{2}q_{\phi}^{(e)T}K2^{(e)}q_{\phi}^{(e)} \tag{45}$$

For NL layers, the kinetic energy of the laminate can be written as

$$T = \sum_{e=1}^{NE} T^{(e)} = \frac{1}{2}\sum_{e=1}^{NE}\{\dot{q}^{(e)}\}^T[M^{(e)}]\{\dot{q}^{(e)}\} \tag{46}$$

where

$$[M^{(e)}] = \int_{A^{(e)}} [N^{(e)}]^T[m][N^{(e)}]dA \tag{47}$$

Here the elemental mass matrix can also be obtained by numerical Gauss quadrature rule.

The governing equation for free vibration of piezoelectric laminated composite plate can be derived using Variational principle, which is generalization of the principle of virtual displacement. Lagrange equation for a conservative system can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\Delta}_i}\right) + \frac{\partial U}{\partial \Delta_i} = 0 \quad \text{for} \quad i = 1, 2, \ldots \tag{48}$$

here Δ_i and $\dot{\Delta}_i$ are the generalized coordinates and velocities, respectively.

By substituting Eqs. (45) and (46) into Eq. (48), the following is obtained

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \ddot{q} \\ \ddot{q}_{\phi} \end{Bmatrix} + \begin{bmatrix} K_{qq} & K_{q\phi} \\ K_{\phi q}^T & K_{\phi\phi} \end{bmatrix}\begin{Bmatrix} q \\ q_{\phi} \end{Bmatrix} = 0 \tag{49}$$

where

$$M = \sum_{e=1}^{NE} M^{(e)}; \quad K_{qq} = \sum_{e=1}^{NE} (K^{(e)} + K_1^{(e)} + K_2^{(e)})$$

$$K_{\phi q} = K_{q\phi} = \sum_{e=1}^{NE} (K1^{(e)} + K_4^{(e)}), \quad K_{\phi\phi} = \sum_{e=1}^{NE} K2^{(e)} \tag{50}$$

By eliminating q_ϕ from Eq. (49), one obtains as

$$M\ddot{q} + K^* q = 0 \tag{51}$$

where $K = K_{qq} - K_{q\phi} K_{\phi\phi}^{-1} K_{q\phi}^T$

Assuming the displacements to vary sinusoidal (i.e., principle mode) with respect to time with natural frequency ω, the Eq. (51) can be expressed in the form of generalized eigen-value problem as

$$K\Delta = \lambda M \Delta \tag{52}$$

where $\lambda = \omega^2$.

In deterministic environment, the natural frequency is obtained using Eq. (52). However, in random environment, the elements of the stiffness matrix K, being dependent on the system material properties, are random in nature, and it is not possible to directly obtain the statistics of natural frequency by using Eq. (52). To achieve this, further analysis is required and an probabilistic approach with perturbation technique and Taylor series is outlined in next sub-section.

3.2 Taylor series based perturbation technique

It can be seen that the eigenvalues and eigen vectors (Eq. (52)) are random. For sensitive application, it is appropriate to assume that strict quality control would be exercised and the dispersion of material properties around their mean is small. Consequently, the dispersion in derived quantities like K, ω, etc. can also be assumed small as compared to their mean values.

Without any loss of generality, the derived random variables like K, ω, etc. may be split up as the sum of its mean and zero mean random part as (Singh 2001, Nigam and Narayan 1994)

$$\begin{aligned} K^R &= \overline{K} + K^r, \ \lambda_i^R = \overline{\lambda}_i + \lambda_i^r \\ \omega_i^R &= \overline{\omega}_i + \omega_i^r, \ \Delta_i^R = \overline{\Delta}_i + \Delta_i^r \end{aligned} \tag{53a-d}$$

where overbar denotes the mean value and superscript 'r' denotes the zero mean random part.

By substituting these in the governing random Eq. (52) and collecting same order terms up to first order, assuming also small random variations in derived random variables, one obtains as

Zeroth order: $$K\overline{\Delta}_i = \overline{\lambda}_i M \overline{\Delta}_i \tag{54}$$

First order: $$(\overline{K} - \overline{\lambda} I_i)\Delta_i^r = -(K^r - \lambda_i^r M)\overline{\Delta}_i \tag{55}$$

Eq. (54) is a deterministic equation relating the mean quantities. The mean eigenvalues and corresponding mean eigenvectors can be determined by conventional eigen solution procedures. However, Eq. (55) cannot be solved by the conventional eigen solution procedure. For obtaining the solution of the random Eq. (55), it is assumed that the eigenvalues are all distinct. The normalized eigenvectors meet the orthogonality conditions (Singh 2001). Any vector in the space can be expressed as a linear combination of eigenvectors. Hence, the i^{th} random part of the eigenvectors can be written as

$$\Delta_i^r = \sum_{j=1}^{p} C_{ij}^r \bar{\Delta}_i, \quad i \neq j, \quad C_{ii}^r = 0, \quad \text{with} \quad i = 1, 2, 3 \ldots p \tag{56}$$

where C_{ij}^r are small random coefficients to be determined.

Substituting Eq. (56) into the Eq. (55), pre multiplying the first by $\bar{\Delta}_i^T$ and the second by $\bar{\Delta}_j^T$ $(j \neq i)$, respectively and applying orthogonality conditions (Singh 2001), one obtains as

$$\lambda_i^r = \bar{\Delta}_i^T K^r \bar{\Delta}_i \tag{57}$$

$$C_{ij}^r = \frac{\bar{\Delta}_j^T K^r \bar{\Delta}_i}{(\bar{\lambda}_i - \bar{\lambda}_j)}, \quad i \neq j \tag{58}$$

By substituting the Eq. (58) into the Eq. (56), one obtains as

$$\Delta_i^r = \sum_{j=1}^{p} \frac{\bar{\Delta}_j^T K^r \bar{\Delta}_i}{(\bar{\lambda}_i - \bar{\lambda}_j)} \bar{\Delta}_i, \quad i \neq j \tag{59}$$

Let $d_1^R, d_2^R, d_3^R, \ldots, d_q^R$ denote the random lamina material properties. The d_j^R can also be expressed as

$$d_j^R = \bar{d}_j + d_j^r \tag{60}$$

By using first order Taylor's series expansion centered at the mean value of d_j (Kleiber and Hein 1992, Liu *et al.* 1986), the derived random variables can be expressed as

$$\lambda_i^r = \sum_{j=1}^{p} \bar{\lambda}_{i,j} d_j^r, \quad \Delta_i^r = \sum_{j=1}^{p} \bar{\Delta}_{i,j} d_j^r$$
$$K^r = \sum_{j=1}^{p} \bar{K}_{,j} d_j^r \tag{61a-c}$$

where, j denotes the partial differentiation with respect to d_j^R

By substituting Eqs. (61a) and (61c) into Eq. (57), one obtains as

$$\bar{\lambda}_{i,j} = \bar{\Delta}_i^T \bar{K}_{,j} \bar{\Delta}_i \tag{62}$$

On taking the expected value of Eq. (53b), an expression for the mean value of eigenvalues is expressed as

$$E[\lambda_i] = E[\bar{\lambda}_i] + E[\lambda_i^r] = \bar{\lambda}_i \tag{63}$$

On squaring Eq. (53b) and taking the expectation, an expression for the mean square value of the eigenvalues is expressed as

$$E[\lambda_i^2] = E[\bar{\lambda}_i^2] + E[\lambda_i^{r2}] \tag{64}$$

Using Eq. (63), the variance of λ_i is obtained as (Nigam and Narayan 1994)

$$\mathrm{var}(\lambda_i) = E[\lambda_i^2] - (E[\lambda_i])^2 = E[\lambda_i^{r2}] \tag{65}$$

Using the basic definition of variance, the $\mathrm{var}(\lambda_i)$ is written as

$$\mathrm{var}(\lambda_i) = \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \lambda_i^{r2} p(d_j, d_k) D_{d_j} D_{d_k} \tag{66}$$

By substituting Eq. (61a) in Eq. (66), one can write as

$$\mathrm{var}(\lambda_i) = \sum_{j=1}^{m}\sum_{k=1}^{m} \lambda_{i,j}\lambda_{i,k}\mathrm{cov}(d_j d_k) \tag{67}$$

where $\mathrm{cov}(d_j d_k)$ is the covariance between d_j and d_k and defined as

$$\mathrm{cov}(d_j d_k) = \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} d_{rj} d_{rk} p(d_j, d_k) D_{d_j} D_{d_k} = \rho_{jk}\sigma_{bj}\sigma_{bk} \tag{68}$$

where $p(d_j, d_k)$ is the joint probability density function for d_j and d_k, and ρ_{jk} is the correlation coefficient for d_j and d_k, which ranges from −1 to 1, and σ_{bj} is the standard deviation for d_j, D_{bj} is the differential variable of d_j.

4. Numerical examples and discussion

The results for some examples of free vibration of laminated composite plate embedded with piezoelectric layers with random material properties have been presented to illustrate the technique. The approach has been validated by comparison with the results available in the literature and an independent MCS results. A comparative study for the mean and standard deviation of natural frequency of composite plate and smart composite plate is also presented. A nine noded simple C^0 Lagrangian isoparametric finite element, having seven degrees of freedom (DOFs) per node with 63 DOFs per element for the HSDT model has been used for spatial descretization of the laminate. These elements are found to be quite stable. Based on convergence study, a (5 × 5) mesh has been adopted throughout the study. The results have been computed by employing full (3 × 3) integration rule. The nondimensional natural frequency is taken as

$$\varpi = (\omega a^2 \sqrt{\rho/Ett})/h$$

Various combinations of plate edge support conditions of clamped (C), free (F) and simple (S) have been used for the analysis. The notation CSCS, for examples, implies clamped edges at $x = 0$, a, and simple support edges at $y = 0$, b. The boundary conditions considered for the plate are

Clamped edges: $u = v = w = \theta_1 = \psi_x = \theta_2 = \psi_y = 0$ at $y = 0$, b and $x = 0$, a
Simply supported edges: $v = w = \theta_2 = \psi_y = 0$, at $x = 0$, a and $u = w = \theta_1 = \psi_x = 0$, at $y = 0$, b
Free edges: $u \neq v \neq w \neq \theta_1 \neq \psi_x \neq \theta_2 \neq \psi_y \neq 0$ at $y = 0$, b and $x = 0$, a

The ratio of the standard deviation (SD) and mean of material properties considered in this study is assumed to vary from 0% to 20% (Liu *et al.* 1986). The lamina material properties modeled as RVs are longitudinal and transverse modulii E_{11} and E_{22}, in plane shear modulus G_{12}, out-of-plane shear modulii G_{13} and G_{23}, and Poisson ratio ν_{12}. The materials used for present investigations are Graphite/Epoxy composite material and Lead Zirconate Titanate (PZT-4) piezoelectric material. The material properties for these materials are shown below. These properties are in the direction of fiber orientation (Saravanos *et al.* 1997, Singh *et al.* 2001):

Material-1: $E_{11} = 25\ E_{22}$, $G_{12} = G_{13} = 0.5\ E_{22}$, $G_{23} = 0.2\ E_{22}$, $E_{22} = 10.3$ GPa, $\nu_{12} = 0.25$.
Material-2: $E_{11} = 40\ E_{22}$, $G_{12} = G_{13} = 0.6\ E_{22}$, $G_{23} = 0.5\ E_{22}$, $E_{22} = 6.92$ GPa, $\nu_{12} = 0.25$.
PZT-4: Elastic properties: $E_{p11} = 81.3$ GPa, $G_{p12} = 30.6$ GPa, $G_{p13} = 25.6$ GPa, $G_{p23} = 25.6$ GPa, $E_{p22} = 81.3$ GPa, $\nu_{p12} = 0.33$.
Piezoelectric coefficients (10^{-12} m/V): $e_{31} = e_{32} = -122.0$, $e_{33} = -285.0$, $e_{24} = 0$
Electric permittivity ($\varepsilon_0 = 8.85 \times 10^{-12}$ Farad/m): $k_{11}/\varepsilon_0 = 1475$, $k_{22}/\varepsilon = 1475$, $k_{33}/\varepsilon_0 = 1300$

4.1 Convergence study

In this section, the convergence study of the laminated composite plate embedded with piezoelectric layers is discussed.

Convergence of the normalized fundamental frequency with various mesh refinement is as shown in Table 1. Here Material-2, PZT-4, aspect ratio $(a/b) = 1$, $a/h = 10$ and 100 and SSSS boundary conditions. From the Table, it can be concluded that a 5×5 mesh gives sufficiently converged values of the fundamental frequency for moderately thick and thin plates.

4.2 Numerical validation

The validation study of the C^0 finite element based probabilistic plate model is accomplished by

Table 1 Convergence of normalized fundamental frequency $\varpi = (\omega a^2\sqrt{\rho/E_{22}})/h$ for a square [0/P/90] plate (Material-2 and SSSS) with mesh

Mesh	$a/h = 10$	$a/h = 100$
1 × 1	10.2090	11. 07098
2 × 2	9.962654	11. 02690
4 × 4	9.568518	10. 19674
5 × 5	9.547060	10. 10596
6 × 6	9.539077	10. 10243

comparing the results of the following set of problems with the results published in the literature and an independent MCS. Here for all cases Material-1 & 2, aspect ratio (a/b) = 1, a/h = 10 and 100, boundary condition SSSS and laminate lay-ups [0/90], [0/90/90/0], [0/P/90] and [0/90/P/0/90]. Based on convergence, 5000 samples have been taken for the MCS approach. Each layer is having equal thickness and density, $\rho = 1$ kg/m^3.

The comparative study of the mean results from the present investigation with the results available in the published literature is presented in Tables 2 and 3 without and with piezoelectric layer, respectively.

From the Table 2, it is seen that the difference in the results obtained from the present study and those of Ref. (Singh *et al*., 2001) is less than 2.5% in all the cases. Hence one can conclude that the present formulation gives reasonably good results.

From the Table 3, it is observed that the difference in the results obtained from the present study and those of Ref. (Umrao *et al*. 2008) is less than 1% in all the cases. Hence one can conclude that the present formulation for smart laminates gives quite good results.

Fig. 2 shows a comparison of the normalized standard deviation (SD) results of the fundamental frequency obtained by the present investigation and an independent MCS approaches for a [0/P/90] square laminate having SSSS boundary condition with all lamina material properties changing simultaneously, keeping same value at a time. The influence of scattering in the lamina material properties on fundamental frequency has been obtained by allowing the ratio of SD (σ) to mean (μ) to vary from 0 to 20%.

From the Fig. 2, it can be seen that the difference between two results are very small. Hence it can be concluded that the present outlined probabilistic approach for smart laminate gives reasonably accurate results as the MCS is considered to be exact approach for random analysis.

Table 2 Validation of normalized fundamental frequency, $\varpi = (\omega a^2 \sqrt{\rho/E_{22}})/h$ for a square laminate with SSSS

Stacking Sequence	Material-1		Material-2	
	a/h = 10	a/h = 100	a/h = 10	a/h = 100
0/90	9.172616*	10.25518*	10.8111*	11.74685*
	8.98083^	10.47657^	10.56565^	11.90491^
0/90/90/0	11.94181*	15.73490*	15.39339*	19.56791*
	11.77252^	15.17055^	15.10799^	19.13029^

*Present Study; ^ Singh *et al*. (2001).

Table 3 Validation of normalized fundamental frequency, $\varpi = (\omega a^2 \sqrt{\rho/E_{22}})/h$ for a square (0/90/P/0/90) laminate with SSSS

	Material-1		Material-2	
	a/h = 10	a/h = 100	a/h = 10	a/h = 100
Umrao *et al*. (2008)	13.7854	15.2992	16.9900	18.9341
Present	13.8560	15.3987	17.1208	18.9954

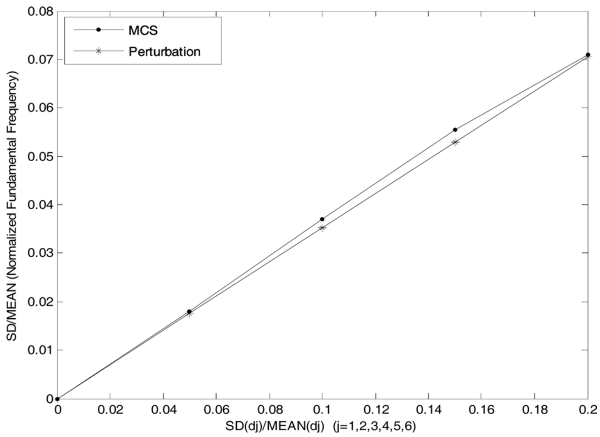

Fig. 2 Validation of results for variance of normalized fundamental frequency of square (0/P/90) SSSS laminate (Material-1 and a/h = 10) with all basic material properties varying simultaneously

4.3 Parametric study of the smart composite laminates

4.3.1 Mean natural frequency

Tables 4-7 show the first five normalized mean natural frequencies of a square (0/P/90) plate, Material-1 and 2 and a/h = 10 and 100, for SSSS, CCSC, CCCC, CFCF, CSCS boundary conditions, respectively. The Tables show that the natural frequencies increase as the thickness ratio increase. The increment is more for higher modes as compared with the increment in the fundamental frequency. A look at the frequencies from orthotropicity point of view clearly states that with increase in modular ratio the natural frequencies of free vibration increases. The natural frequencies are clearly affected by the end conditions. The frequencies of the plate with CCCC boundary conditions are the highest for the two thickness ratios considered when compared with other end conditions. The Plates subject to different end conditions in ascending order of natural frequencies are CFCF, CSCS, CCSC, and CCCC.

Fig. 3 shows the variation of the first five normalized frequencies with side to thickness (a/h) ratio for a square (0/P/90) plate with CCCS for Material-1 and 2. It is seen that a/h ratio has a strong effect on frequency of a smart composite plate. Normalized frequency increases as a/h increases. This can be attributed to the difference in the elastic properties between fiber filament and

Table 4 Normalized natural frequencies, $\varpi = (\omega a^2 \sqrt{\rho/E_{22}})/h$ for a square (0/P/90) laminate with SSSS

Mode	Material-1		Material-2	
	a/h = 10	a/h = 100	a/h = 10	a/h = 100
1	9.547060	10.19674	11.07451	11.78057
2	25.87808	31.68508	30.53770	37.24854
3	25.88645	31.69032	30.54754	37.25517
4	32.13499	43.86338	37.98730	50.87662
5	3214505	77.75460	37.99890	92.11673

Table 5 Normalized natural frequencies, $\varpi = (\omega a^2\sqrt{\rho/E_{22}})/h$ for a square (0/P/90) laminate with CCSC

Mode	Material-1		Material-2	
	a/h = 10	a/h = 100	a/h = 10	a/h = 100
1	16.98614	19.96802	19.85710	23.48224
2	31.77430	40.95112	36.91307	48.20580
3	34.44293	48.74458	40.20100	57.55658
4	43.50641	60.45696	50.40527	73.1122
5	54.33292	87.41765	67.1829	103.5228

Table 6 Normalized natural frequencies, $\varpi = (\omega a^2\sqrt{\rho/E_{22}})/h$ for a square (0/P/90) laminate with CCCC

Mode	Material-1		Material-2	
	a/h=10	a/h=100	a/h=10	a/h=100
1	19.39275	23.40323	22.67836	27.59243
2	35.55004	50.34827	41.29375	59.43067
3	35.73644	50.36979	41.74078	59.48750
4	46.33182	67.16863	53.72116	78.60659
5	58.23301	100.3574	67.57400	118.8477

Table 7 Normalized natural frequencies, $\varpi = (\omega a^2\sqrt{\rho/E_{22}})/h$ for a square (0/P/90) laminate with CFCF

Mode	Material-1		Material-2	
	a/h = 10	a/h = 100	a/h = 10	a/h = 100
1	13.53673	16.30042	15.84367	19.34540
2	14.05095	16.98313	16.36652	20.01812
3	21.11531	24.69709	24.61743	28.87544
4	30.79387	46.99903	36.44279	55.77712
5	32.55048	47.56282	37.87311	56.18690

matrix materials which leads to a high ratio of in-plane Young's modulus to transverse shear modulus for the composite plates. It is also seen that this increase is more in case of higher modes of vibration as compared with fundamental mode. In general, frequency having higher orthotropicity ratio is more as compared to lower ratio.

Fig. 4 shows the effect of the modulii ratio on the normalized fundamental frequency of a square (0/P/90) plate, a/h = 10, for CCCS boundary condition. It is observed that the frequency increases as the modulus ratio increases. This follows from the fact that the higher the modulus ratio the stiffer the plate will be.

Free vibration analysis of the composite plates with five different boundary conditions was carried out for a square (0/P/90) plate (Material-1) with a/h. The results for fundamental frequency are presented in Fig. 5. From the figure, it is seen that the smart plate is most affected by the CCCC and least affected by the SSSS boundary conditions.

Effect of stacking sequences on free vibration response was also studied. The variation of

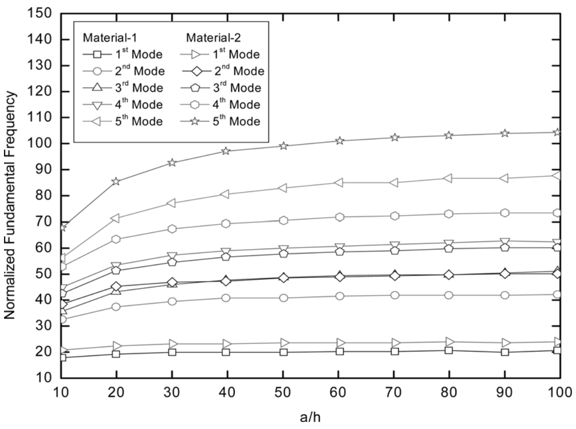

Fig. 3 Variation of the first five normalized natural frequency of square (0/P/90) laminate (CCCS and Material-1 and 2) with side to thickness ratio (a/h)

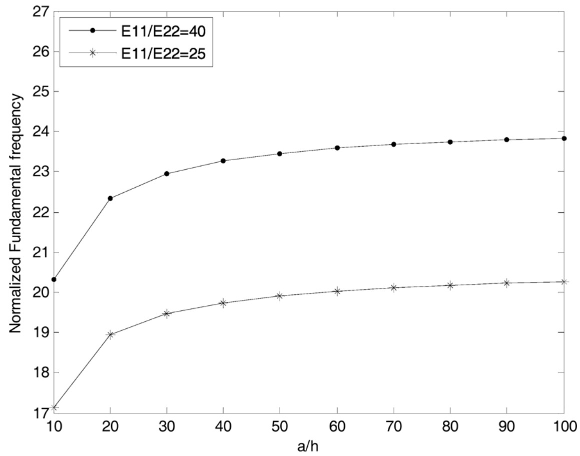

Fig. 4 Variation of normalized fundamental frequency of square (0/P/90) laminate (CCCS) with side to thickness ratio (a/h)

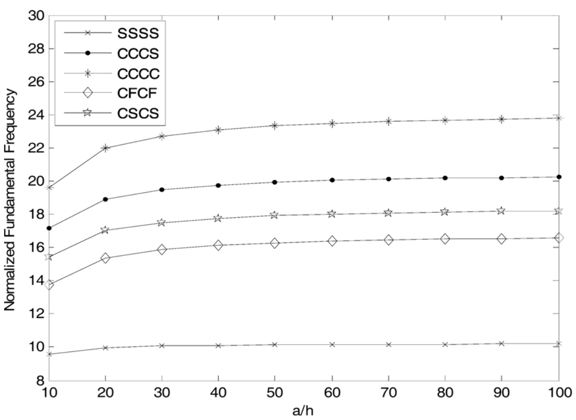

Fig. 5 Variation of normalized fundamental frequency of square (0/P/90) laminate (Material-1) with side to thickness ratio (a/h) for different support conditions

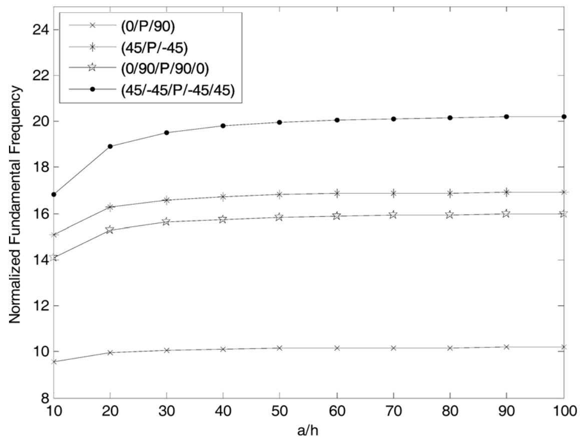

Fig. 6 Variation of normalized fundamental frequency of square (0/P/90) laminate (SSSS and Material-1) with side to thickness ratio (a/h) for different stacking sequences

fundamental frequency of the plate (Material-1) with a/h having four stacking sequences for SSSS boundary condition is presented in Fig. 6. The results show similar trend for cross ply and angle ply laminates. It is seen that symmetrical angle and cross ply laminates have higher frequency when compared with the other two respective sequences. The (45/-45/P/-45/45) plate gives the highest frequency, while the (0/P/90) plate gives the lowest frequency.

Fig. 7 shows the variation of normalized fundamental frequency with fiber orientation of a square ($\theta/P/\theta$) laminate, a/h = 10 and 100 and Material-1 and Material-2 for SSSS boundary conditions. It is seen that variations in lamination angle may result in large changes of frequency. The maximum value of frequency occurs at angle 45 degrees.

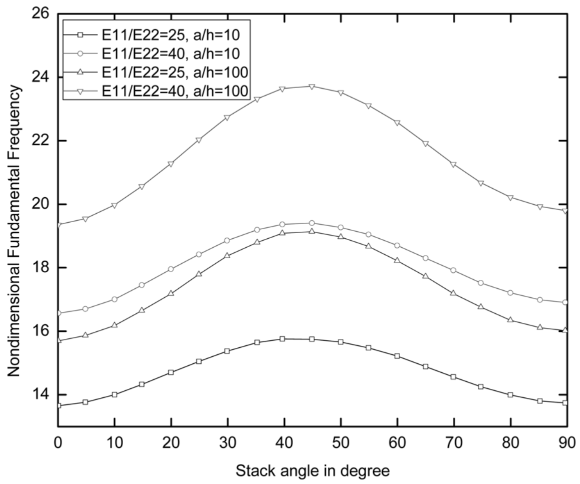

Fig. 7 Variation of normalized fundamental frequency of square (θ/P/θ) laminate (SSSS and a/h = 10 and 100) with stack angle in degree

4.3.2 Standard deviation of fundamental frequency

The influence of scattering in the material properties on fundamental frequency has been obtained by allowing the ratio Standard deviation (SD or σ) to mean (μ) to vary from 0 to 20% for piezoelectric laminated composite plates. Plots of $\sigma_{\omega^2}/\mu_{\omega^2}$ vs σ (material property)/μ (material property) have been obtained considering the lamina material properties E_{11}, E_{22}, G_{12}, G_{13}, G_{23} and ν_{12} as basic random variables (RVs). These random variables are sequenced as: $d_1 = E_{11}$, $d_2 = E_{22}$, $d_3 = G_{12}$, $d_4 = G_{13}$, $d_5 = G_{23}$ and $d_6 = \nu_{12}$.

The scattering in the fundamental frequency of a piezocomposite (0/P/90) plate, Material-1 and a/h = 10 for different boundary conditions with all the material properties varying simultaneously is shown in Fig. 8. It is revealed that the frequency of the plate with SSSS boundary conditions has the highest scatter and that with CCCC boundary conditions has lowest scatter. It is also revealed that the plate with CCCS, CCCC and CSCS shows almost same scattering up to 12 percent

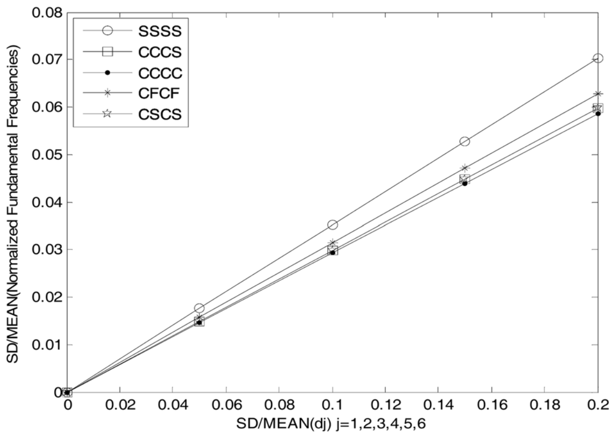

Fig. 8 Variation of standard deviation (SD)/mean of normalized fundamental frequency of square (0/P/90) laminate (Material 1 and a/h = 10) with all basic material properties changing simultaneously

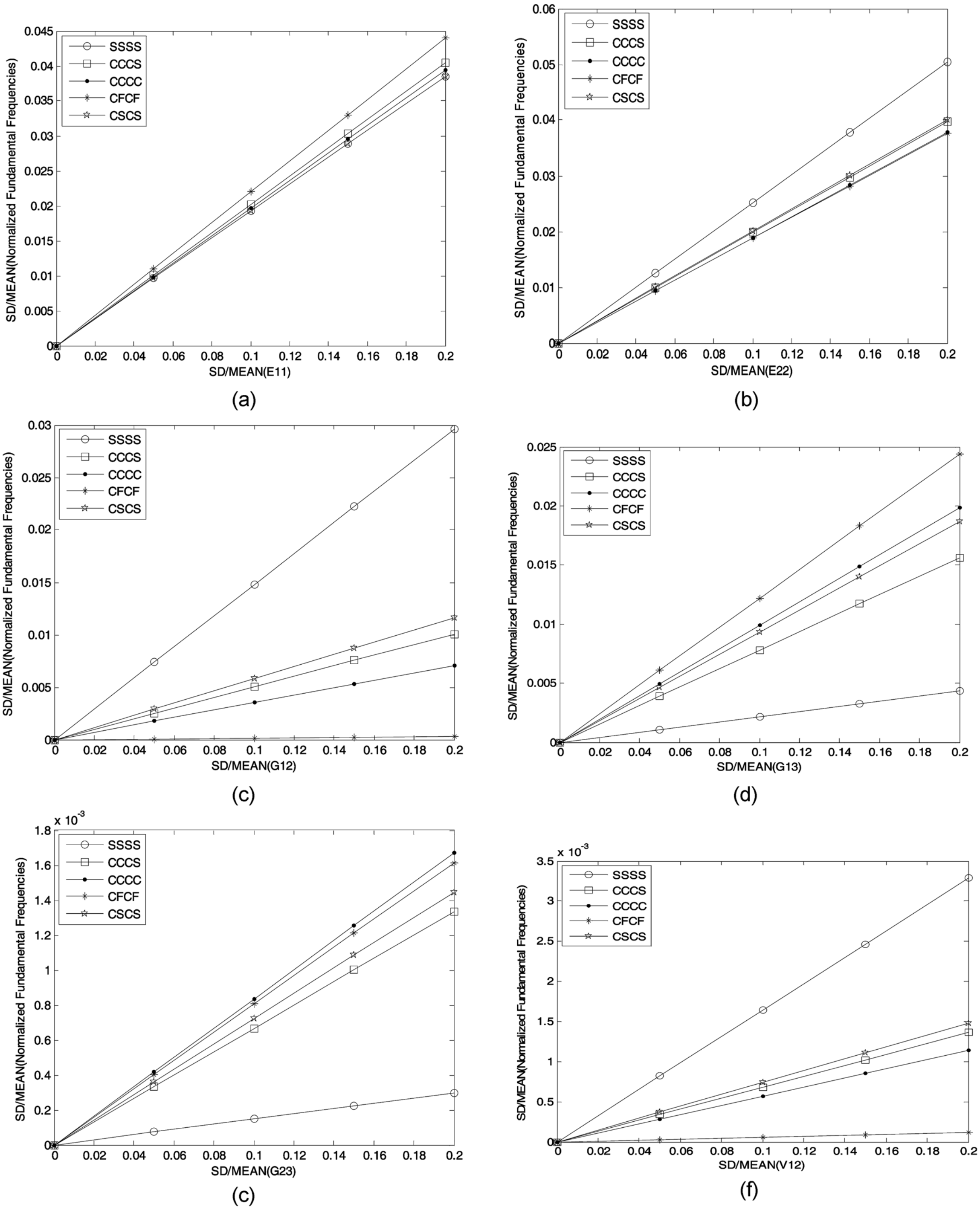

Fig. 9 Variation of standard deviation (SD)/mean of normalized fundamental frequency of square (0/P/90) laminate (Material 1, a/h = 10 and SSSS) with SD of material property, (a) E_{11}, (b) E_{22}, (c) G_{12}, (d) G_{13}, (e) G_{23}, and (f) v_{12}

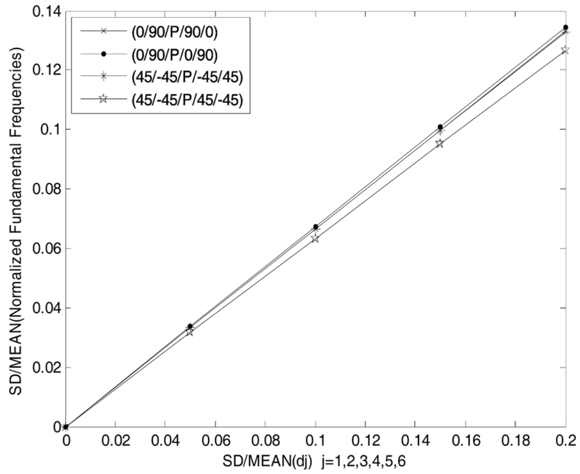

Fig. 10 Variation of standard deviation (SD) of fundamental frequency of square smart laminate (SSSS, Material-1, and a/h = 10) with all basic material properties changing simultaneously

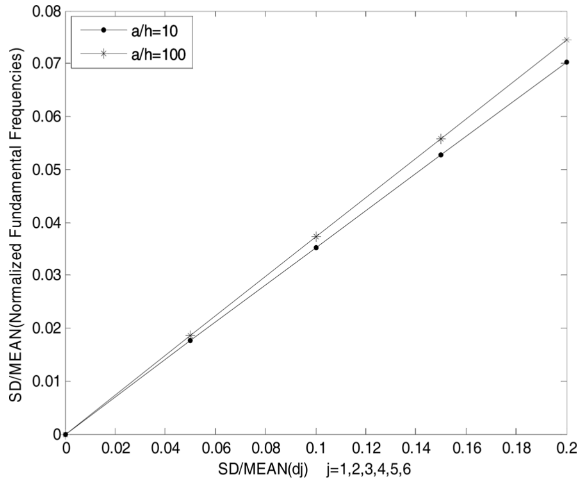

Fig. 11 Variation of standard deviation (SD) of normalized fundamental frequency of square (0/P/90) laminate (SSSS and Material-1) with all basic material properties changing simultaneously

scattering in all material properties.

Figs. 9(a)-(f) plot normalized SD of fundamental frequency with individual random change in material property, keeping the other deterministic, Material-1 and a/h = 10 of the (0/P/90) plate with different boundary conditions. It is observed that the frequency is most affected by randomness in E_{11} and E_{22}, while it is least affected by changes in G_{23} and ν_{12}.

Scattering in fundamental for different stacking sequence of a plate with SSSS, Material-1 and $a/h = 10$ with material properties varying simultaneously is shown in Fig. 10. The stacking sequences considered are (0/90/P/90/0), (0/90/P/0/90), (45/-45/P/-45/45) and (45/-45/P/45/-45). It is seen that for all material properties varying scatter in fundamental frequency is more or less similar for all the four sequences. Anti- symmetric and symmetric cross ply show higher scattering of frequency for all material properties other than G_{23} and G_{13} for which angle ply plates show higher scatter.

The scattering of fundamental frequency for a variation in material properties for a (0/P/90) plate with SSSS and Material-1 for a/h ratios 10 and 100 was studied and the results are shown in Figs. 11 and 12. The thin plate has higher scatter in its frequency for a variation in all material properties. This also follows for all properties when considered alone other than for G_{23} and G_{13} for which thicker plates have higher scatter in their frequency.

4.4 Comparison with composite laminate

Fig. 13 shows a comparison of the scattering in normalized fundamental frequency with normalized SD of material properties varying simultaneously of a conventional composite (0/90) plate and a piezoelectric composite (0/P/90) plate, with a/h = 10 and Material-1 having SSSS boundary conditions. It is seen that the piezoelectric composite plate shows less scattering as compared with a conventional composite plate.

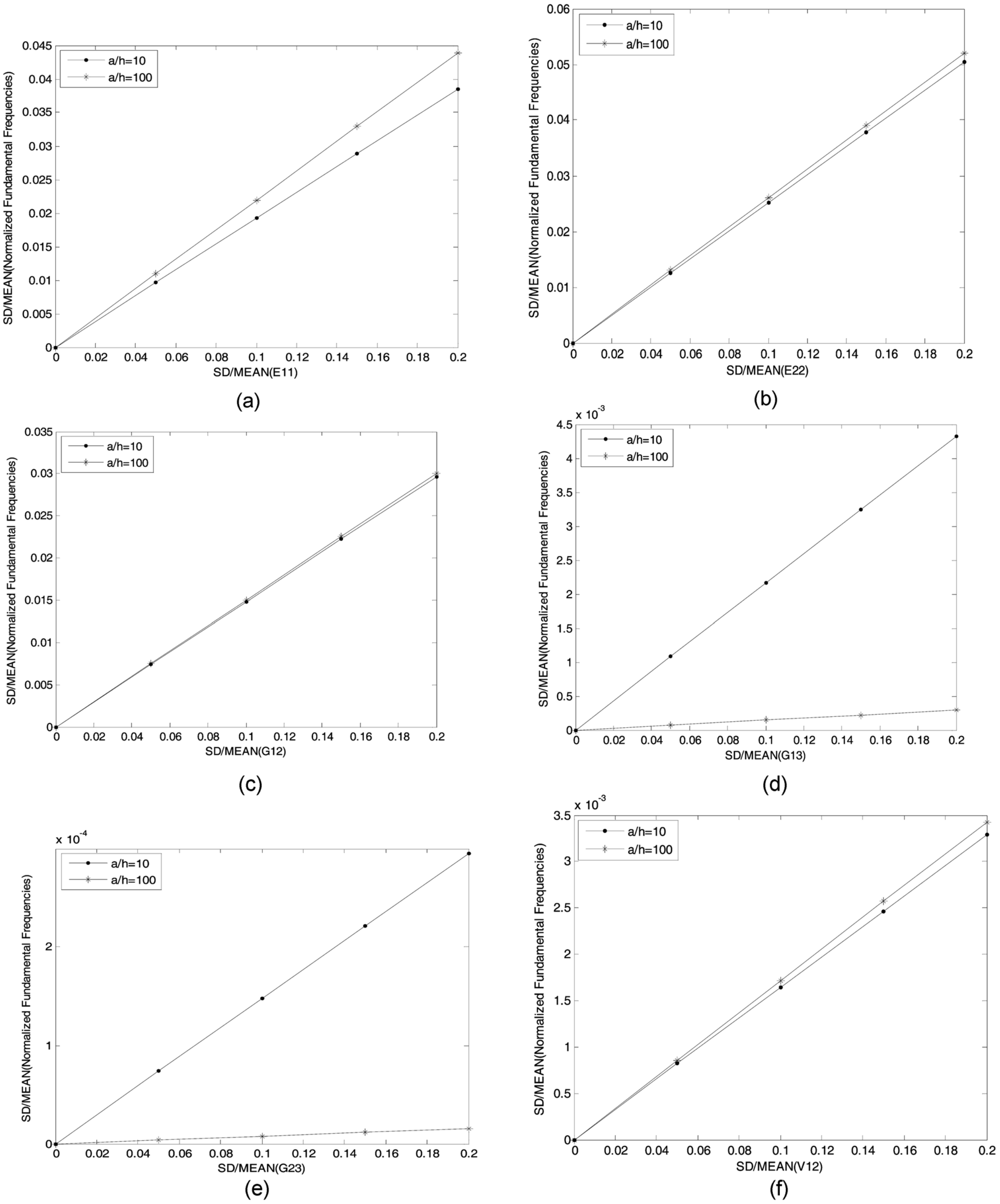

Fig. 12 Variation of standard deviation (SD)/mean of normalized fundamental frequency of square (0/P/90) laminate (Material 1 and SSSS) with SD of material property (a) E_{11}, (b) E_{22}, (c) G_{12}, (d) G_{13}, (e) G_{23}, and (f) v_{12}

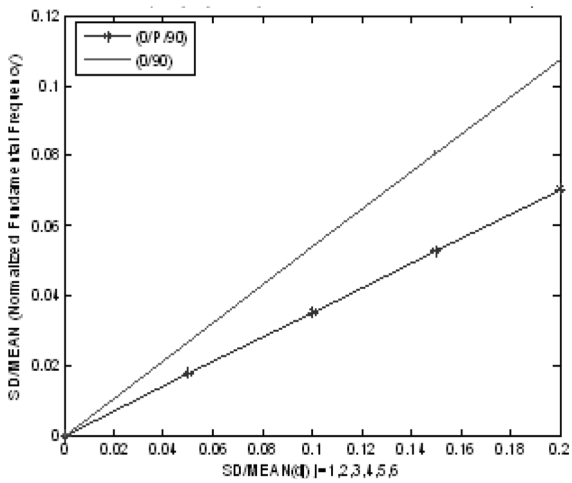

Fig. 13 Comparison of composite plate fundamental frequency scattering with smart composite laminate with all basic material properties changing simultaneously (Material-1, $a/h = 10$, SSSS and $a/b = 1$)

Table 8 Normalized natural frequencies, $\varpi = (\omega a^2\sqrt{\rho/E_{22}})/h$ for a square (0/P/90) laminate with CSCS

Mode	Material-1		Material-2	
	$a/h = 10$	$a/h = 100$	$a/h = 10$	$a/h = 100$
1	15.24570	18.05818	17.74388	21.09884
2	28.05016	33.35139	32.71793	39.12659
3	32.21801	47.87848	38.08360	56.54519
4	33.41458	55.54581	38.73973	64.78235
5	40.70200	69.27845	47.01246	81.81233

5. Conclusions

The linear free vibration response of laminated composite plates embedded with piezoelectric sensors with random material properties has been investigated for different thickness ratios (a/h), material properties stacking sequences and boundary conditions. The mean for natural frequencies and standard deviation for fundamental frequency of free vibration of piezoelectric laminated composite plate has been obtained using an outlined probabilistic approach. The following conclusions are noted from this limited study.

- The natural frequency of free vibration of piezoelectric laminated composite plate increases as a/h ratio increases. The frequency also increases as the modulus ratio increases.
- A smart plate with all edges clamped has the highest frequency and the one with all edges simply supported has the least when mode wise comparison is made.
- The laminated composite plates with piezoelectric layers shows less scattering in fundamental frequency with simultaneous variation in material properties as compared to plates without piezoelectric layers.
- The SD of fundamental frequency shows different sensitivity to different material properties. The sensitivity also changes with the boundary condition, stacking sequence, and a/h ratio.

- The dispersion in fundamental frequency is least affected with scatter in Poisson's ratio ν_{12} and most affected with scatter in E_{11}.

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