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Dynamic behaviour of multi-stiffened plates

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Abstract. The paper investigates the dynamic behaviour of stiffened panels. The coupled differential equations for eccentric stiffening configuration are first derived. Then a semi-analytical procedure for dynamic analysis of stiffened panels is presented. Unlike finite element or finite strip methods, where the plate is discretized into a set of elements or strips, the plate in this procedure is treated as a single element. The potential energy of the structure is first expressed in terms generalized functions that describe the longitudinal and transverse displacement profiles. The resulting non-linear strain energy functions are then transformed into unconstrained optimization problem in which mathematical programming techniques are employed to determine the magnitude of the lowest natural frequency and the associated mode shape for pre-selected plate/stiffener geometric parameters. The described procedure is verified with other numerical methods for several stiffened panels. Results are then presented showing the variation of the natural frequency with plate/stiffener geometric parameters for various stiffening configurations.

Keywords: dynamic analysis; plates; free vibration.

1. Introduction

Dynamic analysis of stiffened plates has been a subject of interest for many years. Several analytical and numerical methods were developed in the past by several researches. The philosophy of each method depends upon the structural idealization of the plate and stiffener elements. Wah (1964) presented a semi-analytical procedure for the analysis of equally spaced, concentric stiffeners with identical cross sectional properties. Asku and Ali (1976, 1982) presented alternative procedure for analysis of equally spaced stiffeners. The method is based on the variational principles in conjunction with finite difference techniques. They illustrated the method for the analysis of plate with one longitudinal stiffener and a plate with one longitudinal and one transverse stiffener. Mukhopadhyay (1989) presented a finite difference procedure for dynamic analysis of stiffened plates. The governing differential equations of the structure are derived by assuming the stiffeners are symmetric about the mid-plane of the plate and ignoring the torsional stiffness of the stiffeners. A displacement function satisfying the boundary condition is then substituted into the governing differential equations and the resulting equations are transformed into ordinary differential equations with constant coefficients that were solved by a finite

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difference scheme. Park and Cho (2006) presented approximate formulations to predict the structural damage stiffened plates under explosion loads. Qing and Qiu (2006) presented numerical formulations for free vibration analysis of stiffened plates. Displacement and stress compatibility boundary conditions were used at the plate and the stiffeners interface. They presented schemes to economize the computer time during the solution process. Peng and Kitipornchai (2006) used the Galerkin method to analyze stiffened panels. Other approximate methods were also used by other researchers (e.g., Kirk (1970), Long (1971), Mead *et al.* (1988), Mukherjee and Chattopadhyay (1994), Fletcher (1959), Wu and Cheung (1974).

Finite element formulations were also presented by Mukherjee and Chattopadhyay (1994) and Barrette *et al*. (2000) for free vibration analysis of stiffened plates. They suggested various meshing strategies to economize the computer time. Ghosh and Biswal (1996) used standard four-noded rectangular element with seven degrees of freedom to model the plate. Magnified stiffness values were used at the plate/stiffener points of contacts. Koko (1990), Koko and Olson (1991) used a "super" plate element to represent each panel bay, and a single beam element to represent each of the stiffeners. Each plate element has nine displacement nodes with a total of 59 degrees of freedom, while the beam element has three displacement nodes with 19 degrees of freedom. Rikards *et al*. (2001), developed alternative Finite Element Formulations for buckling and vibration of stiffened plates. Kumar, and Mukhopadhyay (2002) presented a numerical model for linear transient response analysis of laminated stiffened plates with arbitrarily oriented stiffeners. Zhang *et al.* (2005) used the method for dynamic analysis of stiffened plates under heavy fluid loading. They incorporated in their formulations the radiation damping effect due to the fluid loading. Patel *et al*. (2006) used Finite Element formulations for static and dynamic analysis of stiffened shells subjected to uniform in-plane harmonic edge loading. Wittrick (1968) used the Finite Strip Method for stability and vibration analysis of stiffened plates. A sinusoidal distribution of forces and moments is assumed in the longitudinal direction for each plate. By solving the differential equation of each plate element, a sinusoidal stiffness matrix, of undetermined coefficient for each plate, is generated. By equating the edge displacements of each panel, the problem reduces to finding the solution of the determinant, which provides the natural frequency, or buckling load of the structure. Other Finite Strip formulations for analysis of stiffened plates were also presented by Harik and Salamoun (1988), Peng-Cheng *et al.* (1987) and Chan *et al.* (1991).

While these investigations focused on the analysis of stiffened plates, little attention was given to the behaviour of the structure. Objectives of this paper are to study the dynamic behaviour of stiffened plates. In other words, how changing plate/stiffener geometric proportions affect the dynamic response of the structure. This part of the investigation will provide valuable information from the design point of view.

2. Analysis

Consider the plate shown in Fig. 1, of length a and width b, stiffened orthogonally by eccentric stiffeners of number NS^x and NS^y, where NS denotes the number of stiffeners and the superscripts x and y denote the directions along which the stiffeners span. The origin of the global axis of the plate is chosen at the lower left hand corner denoted by O. The distance from the global co-ordinate system to the centroid of stiffeners along the x-axis is denoted by η_i and for stiffeners along the y-axis is denoted by ξ_i as shown in the figure. The spacing of the stiffeners along the x direction is

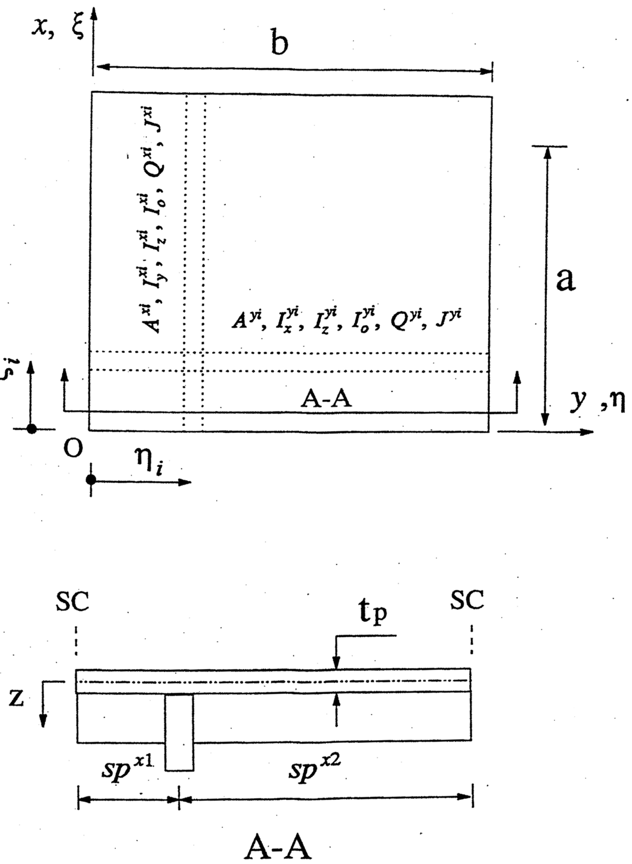

Fig. 1 Orthogonally stiffened plate

also denoted by sp^{xi}, $i = 1, 2, 3,... NS^x$, and for stiffeners along the y axis, is denoted by sp^{yi}, $i = 1, 2, 3,... NS^y$. Each stiffener is described by a set of geometric properties, cross-sectional area A, first moment of inertia Q, second moments of inertia about the major and minor axes i.e., I_y, I_z for stiffeners along the x-axis and I_x, I_z for stiffeners along the y axis, polar moment of inertia I_o and torsional rigidity, J. The subsequent sections describe two approaches for dynamic analysis of eccentrically stiffened plates. The plate and stiffeners for both cases are treated as rigidly connected at their junctions. In the first approach, the governing differential equations are derived and, in the second, an energy formulation, which will be used in subsequent sections as a method of analysis, will be presented.

2.1 Governing differential equations

By taking the mid-plane of the top plate as the axis of reference, the resultant forces and moments for the interval $-t_p/2 < z < t_p/2$ is given by

$$\begin{Bmatrix} N_{xx}^{p} \\ N_{yy}^{p} \\ N_{xy}^{p} \end{Bmatrix}_{-t_p/2<z<t_p/2} = E_p A_p \begin{Bmatrix} \frac{\partial U}{\partial x} + \nu \frac{\partial V}{\partial y} \\ \frac{\partial V}{\partial y} + \nu \frac{\partial U}{\partial x} \\ \frac{1}{2}(1-\nu)\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) \end{Bmatrix}, \begin{Bmatrix} M_{xx}^{p} \\ M_{yy}^{p} \\ M_{xy}^{p} \end{Bmatrix}_{-t_p/2<z<t_p/2} = E_p I_p \begin{Bmatrix} \frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2} \\ \frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2} \\ (1-\nu)\left(\frac{\partial^2 W}{\partial x \partial y}\right) \end{Bmatrix} \tag{1}$$

where I_p is the plate flexural rigidity per unit width = $t_p^3/12(1-\nu^2)$, A_p is the area per unit width of the section = $t_p/(1-\nu^2)$. The product E_pI_p in the conventional plate rigidity D, is so designated since the plate and stiffeners might have different modulus of elasticity. By Assuming a compatible strain distribution at this line of junction, the resultant forces and moments within this interval can be written as

$$\begin{Bmatrix} N_{xx}^{xi} \\ N_{yy}^{yi} \end{Bmatrix} = E_{st} \begin{bmatrix} A^{xi} & -Q^{xi} \\ A^{yi} & -Q^{yi} \end{bmatrix} \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{\partial V}{\partial y} \\ \frac{\partial^2 W}{\partial x^2} & \frac{\partial^2 W}{\partial y^2} \end{bmatrix}, \begin{Bmatrix} M_{xx}^{xi} \\ M_{yy}^{yi} \end{Bmatrix} = E_{st} \begin{bmatrix} Q^{xi} & -I_y^{xi} \\ Q^{yi} & -I_x^{yi} \end{bmatrix} \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{\partial V}{\partial y} \\ \frac{\partial^2 W}{\partial x^2} & \frac{\partial^2 W}{\partial y^2} \end{bmatrix} \tag{2}$$

where Q^{xi} and Q^{yi} denote the first moment of area of the stiffeners per unit width about the mid-plane of the plate; their magnitudes, therefore, depend upon the stiffener profile, i.e., I-section, rectangular, ...etc. As an example, for a typical i^{th} rectangular stiffener of height, h, and distance from the mid-plane of the plate to the centroid of the stiffener, e_x, these quantities become

$$A^{xi} = \int_{t/2}^{t/2+h} dz = h, \quad Q^{xi} = \int_{t/2}^{t/2+h} z dz = \frac{h}{2}[t_p + h] = A^{xi} e_x \tag{3}$$

$$I_y^{xi} = \int_{t/2}^{t/2+h} z^2 dz = \frac{h^3}{3} + \frac{h}{2}(t^2 + th) = \frac{h^3}{12} + \frac{h}{4}(t^2 + 2ht + h^2) = \frac{h^3}{12} + he_x^2 \tag{4}$$

The resultant forces and moments for the assembled structure are, therefore,

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix}_{total} = \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix}_{-t_p/2<z<t_p/2} + \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix}_{t_p/2<z<h}, \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}_{total} = \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}_{-t_p/2<z<t_p/2} + \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}_{t_p/2<z<h} \tag{5}$$

When substituting Eqs. (1), (2) into Eq. (5), the total system of forces for the assembled structure can be written in the following matrix form as

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = E_p \begin{bmatrix} \left(A_p + \frac{E_{st}}{E_p}A^{xi}\right) & \nu A_p & -\frac{E_{st}}{E_p}Q^{xi} & 0 & 0 \\ \nu A_p & \left(A_p + \frac{E_{st}}{E_p}A^{yi}\right) & 0 & -\frac{E_{st}}{E_p}Q^{yi} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-\nu)A_p \end{bmatrix} \begin{Bmatrix} \frac{\partial U}{\partial x} \\ \frac{\partial V}{\partial y} \\ \frac{\partial^2 W}{\partial x^2} \\ \frac{\partial^2 W}{\partial y^2} \\ \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \end{Bmatrix} \tag{6}$$

and the moment resultants are given by

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = E_p \begin{bmatrix} -\frac{E_{st}}{E_p}Q^{xi} & 0 & \left(I_p + \frac{E_{st}}{E_p}I_y^{xi}\right) & \nu I_p & 0 \\ 0 & -\frac{E_{st}}{E_p}Q^{yi} & \nu I_p & \left(I_p + \frac{E_{st}}{E_p}I_x^{yi}\right) & 0 \\ 0 & 0 & 0 & 0 & 2(1-\nu)I_p + \frac{G_{st}}{2E_p}(J^{xi} + J^{yi}) \end{bmatrix} \begin{Bmatrix} \frac{\partial U}{\partial x} \\ \frac{\partial V}{\partial y} \\ \frac{\partial^2 W}{\partial x^2} \\ \frac{\partial^2 W}{\partial y^2} \\ \frac{\partial^2 W}{\partial x \partial y} \end{Bmatrix} \tag{7}$$

The force and moment equilibrium equations are given by

$$\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} = [m_p + m_{st}^{xi} + m_{st}^{yi}] \begin{bmatrix} \frac{\partial^2 U}{\partial t^2} \\ \frac{\partial^2 V}{\partial t^2} \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} \end{bmatrix} \begin{bmatrix} M_{xx} \\ M_{xy} \\ M_{yy} \end{bmatrix} = -[m_p + m_{st}^{xi} + m_{st}^{yi}, m_{st}^{xi} I_o^{xi}, m_{st}^{yi} I_o^{yi}] \begin{bmatrix} \frac{\partial^2 W}{\partial t^2} \\ \frac{\partial^4 W}{\partial y^2 \partial t^2} \\ \frac{\partial^4 W}{\partial x^2 t^2} \end{bmatrix} \tag{9}$$

where m^p and m_{st}^{xi} and m_{st}^{yi} are the mass density of the plate and stiffeners along the x and y axes, respectively, and I_o^{xi} and I_o^{yi} are the polar moment of inertia for typical x and y stiffeners, respectively.

Substituting Eq. (6) into the force equilibrium Eq. (8), results in the following pair of differential equations

$$\beta^{-1}\left[A_p+\frac{E_{st}}{E_p}A^{xi}\right]\frac{\partial^2 U}{\partial\xi^2}+\frac{1}{2}(1+\nu)A_p\frac{\partial^2 V}{\partial\xi\partial\eta}+\frac{\beta}{2}(1-\nu)A_p\frac{\partial^2 U}{\partial\eta^2}-\frac{\beta^{-1}}{a}\frac{E_{st}}{E_p}Q^{xi}\frac{\partial^3 W}{\partial\xi^3}=[m_p+m_{st}^{xi}+m_{st}^{yi}]\frac{\partial^2 U}{\partial t^2} \tag{10}$$

$$\beta\left[A_p+\frac{E_{st}}{E_p}A^{yi}\right]\frac{\partial^2 V}{\partial\eta^2}+\frac{1}{2}(1+\nu)A_p\frac{\partial^2 U}{\partial\xi\partial\eta}+\frac{\beta^{-1}}{2}(1-\nu)A_p\frac{\partial^2 V}{\partial\xi^2}-\frac{\beta E_{st}}{bE_p}Q^{yi}\frac{\partial^3 W}{\partial\eta^3}=[m_p+m_{st}^{xi}+m_{st}^{yi}]\frac{\partial^2 V}{\partial t^2} \tag{11}$$

Similarly, by substituting Eq. (7) into Eq. (9), the moment equilibrium results in an additional differential equation given by

$$\left[I_p+\frac{E_{st}}{E_p}I_y^{xi}\right]\frac{\partial^4 W}{\partial\xi^4}+2\beta^2\left[I_p+\frac{G_{st}}{2E_p}(J^{xi}+J^{yi})\right]\frac{\partial^4 W}{\partial\xi^2\partial\eta^2}+\beta^4\left[I_p+\frac{E_{st}}{E_p}I_x^{yi}\right]\frac{\partial^4 W}{\partial\eta^4}-a\frac{E_{st}}{E_p}\left[Q^{xi}\frac{\partial^3 U}{\partial\xi^3}+\beta^3 Q^{yi}\frac{\partial^3 V}{\partial\eta^3}\right]$$

$$=-a^2\frac{m_p}{E_p}\left[a^2\frac{\partial^2 W}{\partial t^2}+\frac{m_{st}^{xi}}{m_p}\left(a^2\frac{\partial^2 W}{\partial t^2}+I_o^{xi}\beta^2\frac{\partial^4 W}{\partial\eta^2 t^2}\right)+\frac{m_{st}^{yi}}{m_p}\left(a^2\frac{\partial^2 W}{\partial t^2}+I_o^{yi}\frac{\partial^4 W}{\partial\xi^2 t^2}\right)\right] \tag{12}$$

Therefore, for given plate and stiffener properties, the solution of the three coupled differential Eqs. (10)-(12) gives the natural frequency for the assembly

2.2 Semi-analytical formulation

The energy method affords an alternative means of analyzing stiffened plates. In this section a non-linear energy based approach is presented for the analysis of stiffened plates. In dealing with the structure as assembled plate and beam (or stiffener) elements, the membrane strain energy, in the interval $-t_p/2< z < t_p/2$, is given by

$$U_m=\frac{Et}{2(1-\nu^2)}\int_0^1\!\!\int_0^1\beta^{-1}\left(\frac{\partial U}{\partial\xi}\right)^2+\beta\left(\frac{\partial V}{\partial\eta}\right)^2+\frac{\beta^{-1}}{a}\left(\frac{\partial U}{\partial\xi}\right)\left(\frac{\partial W}{\partial\xi}\right)^2+\beta\left(\frac{\partial V}{\partial\eta}\right)\left(\frac{\partial W}{\partial\eta}\right)^2$$

$$\frac{\nu}{a}\left[2a\left(\frac{\partial U}{\partial\xi}\right)\left(\frac{\partial V}{\partial\eta}\right)+\beta\left(\frac{\partial U}{\partial\xi}\right)\left(\frac{\partial W}{\partial\eta}\right)^2+\left(\frac{\partial V}{\partial\eta}\right)\left(\frac{\partial W}{\partial\xi}\right)^2\right]+\frac{1}{4ab}\left[\left(\beta^{-1}\frac{\partial W}{\partial\xi}\right)^2+\beta\left(\frac{\partial W}{\partial\eta}\right)^2\right]^2$$

$$+\frac{1}{2}(1-\nu)\left[\beta\left(\frac{\partial U}{\partial\eta}\right)^2+\beta^{-1}\left(\frac{\partial V}{\partial\xi}\right)^2+2\beta\left(\frac{\partial U}{\partial\eta}\right)\left(\frac{\partial W}{\partial\xi}\right)+\frac{2}{b}\frac{\partial U}{\partial\eta}\frac{\partial W}{\partial\xi}\frac{\partial W}{\partial\eta}+\frac{2}{a}\frac{\partial V}{\partial\xi}\frac{\partial W}{\partial\xi}\frac{\partial W}{\partial\eta}\right]d\xi d\eta \tag{13}$$

and the bending strain energy is

$$U_b=\frac{D}{2ab}\int_0^1\!\!\int_0^1\beta^{-2}\left[\left(\frac{\partial^2 W}{\partial\xi^2}\right)+\left(\frac{\partial^2 W}{\partial\eta^2}\right)\right]^2-2(1-\nu)\left[\frac{\partial^2 W}{\partial\xi^2}\frac{\partial^2 W}{\partial\eta^2}-\left(\frac{\partial^2 W}{\partial\xi\partial\eta}\right)^2\right]d\xi d\eta \tag{14}$$

The strain energy of the longitudinal and transverse stiffeners is composed of two parts, the axial and shear strain components, i.e.

$$U_{st}^{xi} = \frac{E_{st}}{2}\iiint_V (\varepsilon_{xx}^{xi})^2 dV + \frac{(GJ^{xi})}{2a}\int_0^1 \left(\frac{\partial\theta}{\partial\xi}\right)^2 d\xi \tag{15}$$

$$U_{st}^{yi} = \frac{E_{st}}{2}\iiint_V (\varepsilon_{yy}^{yi})^2 dV + \frac{(GJ^{yi})}{2b}\int_0^1 \left(\frac{\partial\theta}{\partial\eta}\right)^2 d\eta \tag{16}$$

where U_{st}^{xi} and U_{st}^{yi} are the strain energies of stiffeners spanning along the x and y axes, respectively. Consider a typical i^{th} stiffener spanning in the x direction, the axial strain is given by

$$\varepsilon_{xx}^{xi} = (\varepsilon_{xx}^{p})_{z=t_p/2} - (z - t_p/2)\left(\frac{d^2W}{dx^2}\right) - y_i\left(\frac{d^2V}{dx^2}\right) + c_w\left(\frac{d^2\theta}{dx^2}\right) \tag{17}$$

where c_w is the warping rigidity of the stiffener and θ is the angle of rotation. The first component is the plate strain evaluated at $z = t_p/2$, the second and third components are the axial strains due to bending about the major and minor axes of their local co-ordinates, and the last component is the axial strain due to warping. The plate kinetic energy is given by

$$T_p = \frac{m_p\omega^2}{2}ab\int_0^1\int_0^1 \left(\frac{\partial W}{\partial t}\right) d\xi d\eta \tag{18}$$

where m_p is the mass density of the plate and ω is the natural frequency. The kinetic energy of a typical stiffener along the x-axis is given by

$$T_{st}^{xi} = \frac{m_{st}^{xi}\omega^2}{2}A^{xi}a\int_0^1 \left(\frac{\partial W}{\partial t}\right)^2 d\xi + \frac{m_{st}^{xi}\omega^2}{2b}I_o^{xi}\beta\int_0^1 \left(\frac{\partial^2 W}{\partial t\partial\eta}\right)^2 d\xi \tag{19}$$

where I_o^{xi} is the polar moment of inertia of the stiffener. Similarly, the kinetic energy of a typical stiffener along the y-axis is given by

$$T_{st}^{yi} = \frac{m_{st}^{yi}\omega^2}{2}A^{yi}b\int_0^1 \left(\frac{\partial W}{\partial t}\right)^2 d\eta + \frac{m_{st}^{yi}\omega^2}{2a}I_o^{yi}\beta^{-1}\int_0^1 \left(\frac{\partial^2 W}{\partial t\partial\xi}\right)^2 d\eta \tag{20}$$

The total potential of the assembled structure is composed of the strain energy of plate bending, U_b, the membrane strain energy, U_m, the strain energies of the stiffeners in the x and y directions, U_{st}^{xi}, U_{st}^{yi}, respectively, and the kinetic energy of the plate and stiffeners. The out-of and in-plane displacement or shape functions, $W(\xi, \eta)$, $U(\xi, \eta)$ and $V(\xi, \eta)$, can be expressed in the following forms

$$W(\xi, \eta) = \sum_{i=1}^{M_1}\sum_{j=1}^{N_1} w_{ij}F_i(\xi)G_j(\eta) \tag{21}$$

$$U(\xi, \eta) = \sum_{m=1}^{M_2}\sum_{n=1}^{N_2} u_{mn}B_m(\xi)D_n(\eta) \tag{22}$$

$$V(\xi, \eta) = \sum_{r=1}^{M_3}\sum_{s=1}^{N_3} v_{rs}E_r(\xi)H_s(\eta) \tag{23}$$

where $F_i(\xi)$ and $G_j(\eta)$ are generalized functions that may be polynomials, harmonics...etc., satisfying out-of-plane boundary conditions at $\xi = 0,1$ and $\eta = 0$, 1, respectively, and w_{ij} are associated coefficients for the $F_i(\xi)$ and $G_j(\eta)$ functions. Similarly $B_m(\xi)$, $D_n(\eta)$, $E_r(\xi)$ and $H_s(\eta)$ are generalized functions that satisfy the in-plane boundary conditions and $\{u_{mn}, v_{rs}\}$ are their corresponding amplitudes. The integers N_1, M_1, N_2, M_2, N_3, M_3 denote the number of generalized functions used to define the displacement functions, $W(\xi,\eta)$, $U(\xi,\eta)$ and $V(\xi,\eta)$, respectively. Substituting these displacement functions into Eq. (14), the plate bending strain energy can be written as

$$U_b = \frac{E_p t_p^3}{24(1-\nu^2)} \sum_{i=1}^{M_1}\sum_{j=1}^{N_1}\sum_{k=1}^{M_1}\sum_{l=1}^{N_1} [\{I_{ijkl}^{(1)}\} \quad \{I_{ijkl}^{(2)}\} \quad \{I_{ijkl}^{(3)}\} \quad \{I_{ijkl}^{(4)}\}] \begin{Bmatrix} w_{ij}w_{kl} \\ w_{ij}w_{kl} \\ w_{ij}w_{kl} \\ w_{ij}w_{kl} \end{Bmatrix} \tag{24}$$

where the integrals $I^{(T)}$, $T = 1...4$, are given by

$$\begin{Bmatrix} \{I_{ijkl}^{(1)}\} \\ \{I_{ijkl}^{(2)}\} \\ \{I_{ijkl}^{(3)}\} \\ \{I_{ijkl}^{(4)}\} \end{Bmatrix} = \begin{Bmatrix} \beta^{-2} \\ \beta^2 \\ 2\nu \\ 2(1-\nu) \end{Bmatrix} \int_0^1\int_0^1 \begin{Bmatrix} \dfrac{\partial^2 F_i(\xi)}{\partial\xi^2}\dfrac{\partial^2 F_k(\xi)}{\partial\xi^2} G_j(\eta) G_l(\eta) \\ F_i(\xi)F_k(\xi)\dfrac{\partial^2 G_j(\eta)}{\partial\eta^2}\dfrac{\partial^2 G_l(\eta)}{\partial\eta^2} \\ \dfrac{\partial^2 F_i(\xi)}{\partial^2\xi}\dfrac{\partial^2 F_k(\xi)}{\partial\xi^2}\dfrac{\partial^2 G_j(\eta)}{\partial\eta^2}\dfrac{\partial^2 G_l(\eta)}{\partial\eta^2} \\ \dfrac{\partial F_i(\xi)}{\partial\xi}\dfrac{\partial F_k(\xi)}{\partial\xi}\dfrac{\partial G_j(\eta)}{\partial\eta}\dfrac{\partial G_l(\eta)}{\partial\eta} \end{Bmatrix} d\xi d\eta \tag{25}$$

The superscripts denote the integral number and the subscripts the label of the associated displacement function. Eq. (24) can be written in compact form as

$$U_b = \frac{D}{2}[\{I_{ijkl}^{(T)}\}]\{[w_{ij}w_{kl}]^T\} \quad (T = 1, \ldots, 4) \tag{26}$$

By substituting Eqs. (21)-(23) into Eq. (13), the membrane strain energy, U_m, can be written as

$$U_m = \frac{E_p t_p}{2(1-\nu^2)}\left\{\sum_{m=1}^{M_2}\sum_{n=1}^{N_2}\left(\sum_{p=1}^{M_2}\sum_{q=1}^{N_2}[\{I_{mnpq}^{(1)}\} \quad \{I_{mnpq}^{(2)}\}]\{u_{mn}u_{pq}\} + \sum_{r=1}^{M_3}\sum_{s=1}^{N_3}[\{I_{mnrs}^{(1)}\} \quad \{I_{mnrs}^{(2)}\}]\{u_{mn}v_{rs}\} + \right.\right.$$
$$\left.\left(\sum_{i=1}^{M_1}\sum_{j=1}^{N_1}\sum_{k=1}^{M_1}\sum_{l=1}^{N_1}[\{I_{mnijkl}^{(1)}\} \quad \{I_{mnijkl}^{(2)}\} \quad \{I_{mnijkl}^{(2)}\}]\{u_{mn}w_{ij}w_{kl}\}\right)\right)$$
$$+\sum_{i=1}^{M_1}\sum_{j=1}^{N_1}\sum_{k=1}^{M_1}\sum_{l=1}^{N_1}\sum_{t=1}^{M_1}\sum_{f=1}^{N_1}\sum_{z=1}^{M_1}\sum_{e=1}^{N_1}[\{I_{ijkltfze}^{(1)}\} \quad \{I_{ijkltfze}^{(2)}\} \quad \{I_{ijkltfze}^{(2)}\}]\{w_{ij}w_{kl}w_{tf}w_{ze}\}$$
$$\left.+\sum_{r=1}^{M_3}\sum_{s=1}^{N_3}\left(\sum_{g=1}^{M_3}\sum_{h=1}^{N_3}[\{I_{rsgh}^{(1)}\} \quad \{I_{rsgh}^{(2)}\}]\{v_{rs}v_{gh}\} + \sum_{i=1}^{M_1}\sum_{j=1}^{N_1}\sum_{k=1}^{M_1}\sum_{l=1}^{N_1}[\{I_{rsijkl}^{(1)}\} \quad \{I_{rsijkl}^{(2)}\} \quad \{I_{rsijkl}^{(2)}\}]\{v_{rs}w_{ij}w_{kl}\}\right)\right\} \tag{27}$$

where the integrals I are given by

$$\left\{\begin{matrix} I^{(1)}_{ijkltfze} \\ I^{(2)}_{ijkltfze} \\ I^{(3)}_{ijkltfze} \\ I^{(1)}_{rsgh} \\ I^{(2)}_{rsgh} \\ I^{(1)}_{rsijkl} \\ I^{(2)}_{rsijkl} \\ I^{(3)}_{rsijkl} \end{matrix}\right\} = \left\{\begin{matrix} \dfrac{\beta^{-2}}{4ab} \\ \dfrac{\beta^{2}}{4ab} \\ \dfrac{1}{2ab} \\ \beta \\ \dfrac{\beta^{-2}}{2}(1-\nu) \\ \beta \\ \dfrac{\nu}{a} \\ \dfrac{1}{a}(1-\nu) \end{matrix}\right\} \int_0^1\int_0^1 \left[\begin{matrix} \dfrac{\partial F_i(\xi)}{\partial \xi}\dfrac{\partial F_k(\xi)}{\partial \xi}\dfrac{\partial F_t(\xi)}{\partial \xi}\dfrac{\partial F_z(\xi)}{\partial \xi} G_j(\eta)G_l(\eta)G_f(\eta)G_e(\eta) \\ F_i(\xi)F_k(\xi)F_t(\xi)F_z(\xi)\dfrac{\partial G_j(\eta)}{\partial \eta}\dfrac{\partial G_l(\eta)}{\partial \eta}\dfrac{\partial G_f(\eta)}{\partial \eta}\dfrac{\partial G_e(\eta)}{\partial \eta} \\ \dfrac{\partial F_i(\xi)}{\partial \xi}\dfrac{\partial F_k(\xi)}{\partial \xi}\dfrac{\partial F_t(\xi)}{\partial \xi}\dfrac{\partial F_z(\xi)}{\partial \xi}\dfrac{\partial G_j(\eta)}{\partial \eta}\dfrac{\partial G_l(\eta)}{\partial \eta}\dfrac{\partial G_f(\eta)}{\partial \eta}\dfrac{\partial G_e(\eta)}{\partial \eta} \\ E_r(x)E_g(\xi)\dfrac{\partial H_s(\eta)}{\partial \eta}\dfrac{\partial H_h(\eta)}{\partial \eta} \\ \dfrac{E_r(\xi)}{\partial \xi}\dfrac{E_g(\xi)}{\partial \xi}H_s(\eta)H_h(\eta) \\ E_r(\xi)F_i(\xi)F_k(\xi)\dfrac{\partial H_s(\eta)}{\partial \eta}\dfrac{\partial G_j(\eta)}{\partial \eta}\dfrac{\partial G_l(\eta)}{\partial \eta} \\ E_r(\xi)\dfrac{\partial F_i(\xi)}{\partial \xi}\dfrac{\partial F_k(\xi)}{\partial \xi}\dfrac{\partial H_s(\eta)}{\partial \eta}G_j(\eta)G_l(\eta) \\ \dfrac{\partial E_r(\xi)}{\partial \xi}\dfrac{\partial F_i(\xi)}{\partial \xi}F_k(\xi)H_s(\eta)G_j(\eta)\dfrac{\partial G_l(\eta)}{\partial \eta} \end{matrix}\right] d\xi d\eta \qquad (28)$$

$$\left\{\begin{matrix} I^{(1)}_{mnpq} \\ I^{(2)}_{mnpq} \\ I^{(1)}_{mnrs} \\ I^{(2)}_{mnrs} \\ I^{(1)}_{mnijkl} \\ I^{(2)}_{mnijkl} \\ I^{(3)}_{mnijkl} \end{matrix}\right\} = \left\{\begin{matrix} \beta^{-1} \\ \dfrac{\beta}{2}(1-\nu) \\ 2\nu \\ (1-\nu) \\ \dfrac{\beta^{-1}}{a} \\ \nu\dfrac{\beta}{a} \\ \dfrac{1-\nu}{b} \end{matrix}\right\} \int_0^1\int_0^1 \left[\begin{matrix} \dfrac{\partial B_m(\xi)}{\partial \xi}\dfrac{\partial B_p(\xi)}{\partial \xi}D_n(\eta)D_q(\eta) \\ B_m(\xi)B_p(\xi)\dfrac{\partial D_n(\eta)}{\partial \eta}\dfrac{\partial D_q(\eta)}{\partial \eta} \\ \dfrac{\partial B_m(\xi)}{\partial \xi}E_r(\xi)D_n(\eta)\dfrac{\partial H_s(\eta)}{\partial \eta} \\ B_m(\xi)\dfrac{\partial E_r(\xi)}{\partial \xi}\dfrac{\partial D_n(\eta)}{\partial \eta}H_s(\eta) \\ \dfrac{\partial B_m(\xi)}{\partial \xi}\dfrac{\partial F_i(\xi)}{\partial \xi}\dfrac{\partial F_k(\xi)}{\partial \xi}D_n(\eta)G_j(\eta)G_l(\eta) \\ \dfrac{\partial B_m(\xi)}{\partial \xi}F_i(\xi)F_k(\xi)D_n(\eta)\dfrac{\partial G_j(\eta)}{\partial \eta}\dfrac{\partial G_l(y)}{\partial \eta} \\ B_m(\xi)\dfrac{\partial F_i(\xi)}{\partial \xi}F_k(\xi)\dfrac{\partial D_n(\eta)}{\partial \eta}G_j(\eta)\dfrac{\partial G_l(\eta)}{\partial \eta} \end{matrix}\right] d\xi d\eta \qquad (29)$$

Also, by substituting the expressions for U, V and W into Eq. (15) the strain energy of a typical stiffener spanning in the x-direction reads

$$U^{xi}_{st} = \frac{E_{st}}{2}\left\{\sum_{i=1}^{M_1}\sum_{j=1}^{N_1}\left(\sum_{k=1}^{M_1}\sum_{l=1}^{N_1}[\{I^{(5)}_{ijkl}\}\ \{I^{(6)}_{ijkl}\}]\{w_{ij}w_{kl}\} + \sum_{m=1}^{M_2}\sum_{n=1}^{N_2}[\{I^{(1)}_{mnij}\}]\{u_{mn}w_{ij}\} + \sum_{m=1}^{M_2}\sum_{n=1}^{N_2}\sum_{k=1}^{M_1}\sum_{l=1}^{N_1}[\{I^{(4)}_{mnijkl}\}]\{u_{mn}w_{ij}w_{kl}\}\right)\right\}$$

$$+\sum_{k=1}^{M_1}\sum_{l=1}^{N_1}\sum_{t=1}^{M_1}\sum_{f=1}^{N_1}\sum_{z=1}^{M_1}\sum_{e=1}^{N_1}[\{I^{(4)}_{ijkltfze}\}]\{w_{ij}w_{kl}w_{tf}w_{ze}\}+\sum_{k=1}^{M_1}\sum_{l=1}^{N_1}\sum_{t=1}^{M_1}\sum_{f=1}^{N_1}[\{I^{(1)}_{ijketf}\}]\{w_{ij}w_{kl}w_{tf}\}\Big)$$

$$+\sum_{m=1}^{M_2}\sum_{n=1}^{N_2}\sum_{p=1}^{M_2}\sum_{q=1}^{N_2}[\{I^{(3)}_{mnpq}\}]\{u_{mn}u_{pq}\}+\sum_{r=1}^{M_3}\sum_{s=1}^{N_3}\sum_{g=1}^{M_3}\sum_{h=1}^{N_3}[\{I^{(3)}_{rsgh}\}]\{v_{rs}v_{gh}\}\Bigg\} \tag{30}$$

where

$$\begin{Bmatrix} I^{(3)}_{ijkl} \\ I^{(3)}_{rsgh} \\ I^{(3)}_{mnpq} \\ I^{(1)}_{mnij} \\ I^{(6)}_{ijkl} \\ I^{(4)}_{mnijkl} \\ I^{(1)}_{ijkltf} \\ I^{(4)}_{ijkltfze} \end{Bmatrix} = \frac{1}{a^3}\begin{Bmatrix} I^{xi}_y \\ I^{xi}_z \\ a^2A^{xi} \\ -2Q^{xi} \\ \dfrac{J^{xi}b^2\beta^3}{2(1+\nu)} \\ aA^{xi} \\ -Q^{xi} \\ \dfrac{A^{xi}}{4} \end{Bmatrix}\begin{Bmatrix} \dfrac{\partial^2 F_i(\xi)}{\partial\xi^2}\dfrac{\partial^2 F_k(\xi)}{\partial\xi^2}G_j(\eta)G_l(\eta) \\ \dfrac{\partial^2 E_r(\xi)}{\partial\xi^2}\dfrac{\partial^2 E_g(\xi)}{\partial\xi_2}H_s(\eta)H_h(\eta) \\ \dfrac{\partial B_m(\xi)}{\partial\xi}\dfrac{\partial B_p(\xi)}{\partial\xi}D_n(\eta)D_q(\eta) \\ \dfrac{\partial B_m(\xi)}{\partial\xi}\dfrac{\partial^2 F_i(\xi)}{\partial\xi^2}D_n(\eta)G_j(\eta) \\ \dfrac{\partial F_i(\xi)}{\partial\xi}\dfrac{\partial F_k(\xi)}{\partial\xi}\dfrac{\partial G_j(\eta)}{\partial\eta}\dfrac{\partial G_l(\eta)}{\partial\eta} \\ \dfrac{\partial B_m(\xi)}{\partial\xi}\dfrac{\partial F_i(\xi)}{\partial\xi}\dfrac{\partial F_k(\xi)}{\partial\xi}D_n(\eta)G_j(\eta)G_l(\eta) \\ \dfrac{\partial^2 F_i(\xi)}{\partial\xi^2}\dfrac{\partial F_k(\xi)}{\partial\xi}\dfrac{\partial F_t(\xi)}{\partial\xi}G_j(\eta)G_l(\eta)G_f(\eta) \\ \dfrac{\partial F_i(\xi)}{\partial\xi}\dfrac{\partial F_k(\xi)}{\partial\xi}\dfrac{\partial F_t(\xi)}{\partial\xi}\dfrac{\partial F_z(\xi)}{\partial\xi}G_j(\eta)G_l(\eta)G_f(\eta)G_e(\eta) \end{Bmatrix}\delta(\eta-\eta_i)d\xi \tag{31}$$

The displacement functions of Eqs. (21)-(23) are two dimensional, i.e., compatible with the two dimensional integrals for the plate strain energies, while the strain energy integrals for the stiffeners are one dimensional. Since these functions are also used for the stiffener strain energies, the transformation delta function, $\delta(\eta-\eta_i)$, has been introduced into Eq. (31) that evaluates the displacement function at the location of the stiffener. For example, when this transformation function is used with the out of plane displacement function, then

$$\int_0^1 W(\xi,\eta)\delta(\eta-\eta_i)d\xi = \int_0^1 W(\xi,\eta)d\xi \tag{32}$$

where η_i is the location of the stiffener spanning along the x or ξ-axis. By similar analogy, the strain energy of the stiffeners spanning in the y direction becomes

$$U_{st}^{y}=\frac{E_p^{y}}{2}\Bigg\{\sum_{i=1}^{M_1}\sum_{j=1}^{N_1}\Bigg(\sum_{k=1}^{M_1}\sum_{l=1}^{N_1}[\{I_{ijkl}^{(7)}\}\ \{I_{ijkl}^{(8)}\}]\{w_{ij}w_{kl}\}+\sum_{r=1}^{M_3}\sum_{s=1}^{N_3}[\{I_{rsij}^{(1)}\}]\{v_{rs}w_{ij}\}+\sum_{r=1}^{M_3}\sum_{s=1}^{N_3}\sum_{k=1}^{M_1}\sum_{l=1}^{N_1}[\{I_{rsijkl}^{(4)}\}]\{v_{rs}w_{ij}w_{kl}\}\Bigg)$$
$$+\sum_{k=1}^{M_1}\sum_{l=1}^{N_1}\sum_{t=1}^{M_1}\sum_{f=1}^{N_1}\sum_{z=1}^{M_1}\sum_{e=1}^{N_1}[\{I_{ijkltfze}^{(5)}\}]\{w_{ij}w_{kl}w_{tf}w_{ze}\}+\sum_{k=1}^{M_1}\sum_{l=1}^{N_1}\sum_{t=1}^{M_1}\sum_{f=1}^{N_1}[\{I_{ijkltf}^{(2)}\}]\{w_{ij}w_{kl}w_{tf}\}\Bigg)$$
$$+\sum_{m=1}^{M_2}\sum_{n=1}^{N_2}\sum_{p=1}^{M_2}\sum_{q=1}^{N_2}[\{I_{mnpq}^{(4)}\}]\{u_{mn}u_{pq}\}+\sum_{r=1}^{M_3}\sum_{s=1}^{N_3}\sum_{g=1}^{M_3}\sum_{h=1}^{N_3}[\{I_{rsgh}^{(4)}\}]\{v_{rs}v_{gh}\}\Bigg\} \tag{33}$$

where the values of the I integrals are given by

$$\begin{Bmatrix} I_{ijkl}^{(7)} \\ I_{mnpq}^{(4)} \\ I_{rsgh}^{(4)} \\ I_{rsij}^{(1)} \\ I_{ijkl}^{(8)} \\ I_{rsijkl}^{(4)} \\ I_{ijkltf}^{(2)} \\ I_{ijkltfze}^{(5)} \end{Bmatrix}=\frac{1}{b^3}\begin{Bmatrix} I_x^{yi} \\ I_z^{yi} \\ b^2A^{yi} \\ -2bQ^{yi} \\ \dfrac{J^{yi}a^2}{2(1+\nu)\beta^3} \\ bA^{yi} \\ -Q^{yi} \\ \dfrac{1}{4}A^{yi} \end{Bmatrix}\int_0^1\begin{Bmatrix} F_i(\xi)F_k(\xi)\dfrac{\partial^2 G_j(\eta)}{\partial\eta^2}\dfrac{\partial^2 F_l(\eta)}{\partial\eta^2} \\ B_m(\xi)B_p(\xi)\dfrac{\partial^2 D_n(\eta)}{\partial\eta^2}\dfrac{\partial^2 D_q(\eta)}{\partial\eta_2} \\ E_r(\xi)E_g(\xi)\dfrac{\partial H_s(\eta)}{\partial\eta}\dfrac{\partial H_s(\eta)}{\partial\eta} \\ E_r(\xi)F_i(\xi)\dfrac{\partial H_s(\eta)}{\partial\eta}\dfrac{\partial^2 G_i(\eta)}{\partial\eta^2} \\ \dfrac{\partial F_i(\xi)}{\partial\xi}\dfrac{\partial F_k(\xi)}{\partial\xi}\dfrac{\partial G_j(\eta)}{\partial\eta}\dfrac{\partial G_l(\eta)}{\partial\eta} \\ E_r(\xi)F_i(\xi)F_k(\xi)\dfrac{\partial H_s(\eta)}{\partial\eta}\dfrac{\partial G_j(\eta)}{\partial\eta}\dfrac{\partial G_l(\eta)}{\partial\eta} \\ F_i(\xi)F_k(\xi)F_t(\xi)\dfrac{\partial^2 G_j(\eta)}{\partial\eta^2}\dfrac{\partial G_l(\eta)}{\partial\eta}\dfrac{\partial G_f(\eta)}{\partial\eta} \\ F_i(\xi)F_k(\xi)F_t(\xi)F_z(\xi)\dfrac{\partial G_j(\eta)}{\partial\eta}\dfrac{\partial F_l(\eta)}{\partial\eta}\dfrac{\partial G_f(\eta)}{\partial\eta}\dfrac{\partial G_e(\eta)}{\partial\eta} \end{Bmatrix}\delta(\xi-\xi_i)d\eta \tag{34}$$

The plate and stiffeners kinetic energies can also be written in terms of the displacement functions as

$$T=\frac{\omega^2 m_p}{2t_p}\sum_{i=1}^{M_1}\sum_{j=1}^{N_1}\sum_{k=1}^{M_1}\sum_{l=1}^{N_1}\left[\{I_{ijkl}^{(9)}\},\sum_{i=1}^{NS^{xi}}\{I_{ijkl}^{(10)}+I_{ijkl}^{(11)}\},\sum_{i=1}^{NS^{y}}\{I_{ijkl}^{(12)}+I_{ijkl}^{(13)}\}\right]\begin{Bmatrix} w_{ij}w_{kl} \\ w_{ij}w_{kl} \\ w_{ij}w_{kl} \\ w_{ij}w_{kl} \\ w_{ij}w_{kl} \end{Bmatrix} \tag{35}$$

where

$$\begin{Bmatrix} I_{ijkl}^{(9)} \\ I_{ijkl}^{(10)} \\ I_{ijkl}^{(11)} \\ I_{ijkl}^{(12)} \\ I_{ijkl}^{(13)} \end{Bmatrix} = \begin{Bmatrix} ab \\ \dfrac{m_{st}A^{xi}a}{m_p t_p} \\ \dfrac{m_{st}I_o^{xi}\beta}{m_p t_p b} \\ \dfrac{m_{st}^{yi}A^{yi}b}{m_p t_p} \\ \dfrac{m_{st}^{yi}\beta^{-1}}{m_p a} \end{Bmatrix} \begin{Bmatrix} \int_0^1\int_0^1 F_i(\xi)F_k(\xi)G_j(\eta)G_l(\eta)d\xi d\eta \\ \int_0^1\int_0^1 F_i(\xi)F_k(\xi)G_j(\eta)G_l(\eta)\delta(\eta-\eta_i)d\xi \\ \int_0^1 F_i(\xi)F_k(\xi)\dfrac{\partial G_j(\eta)}{\partial\eta}\dfrac{\partial G_l(\eta)}{\partial\eta}\delta(\eta-\eta_i)d\xi \\ \int_0^1 F_i(\xi)F_k(\xi)G_j(\eta)G_l(\eta)\delta(\xi-\xi_i)d\eta \\ \int_0^1 F_i(\xi)F_k(\xi)\dfrac{\partial G_j(\eta)}{\partial\eta}\dfrac{\partial G_l(\eta)}{\partial\eta}\delta(\xi-\xi_i)d\eta \end{Bmatrix} \tag{36}$$

Equating the maximum kinetic and strain energies given by Eqs. (24), (27), (30), (33), the natural frequency of the assembled structure, ω, can be expressed in the following format

$$\omega = \frac{1}{a^2}\sqrt{\frac{D}{m_p t_p}} \bullet \Omega \tag{37}$$

where Ω is a natural frequency parameter that is a function of the plate/stiffener geometric properties, the shape functions $\{F_i(\xi), G_j(\eta), B_m(\xi), D_n(\eta), E_r(\xi), H_s(\eta)\}$ that satisfy the boundary conditions along the four edges and their associated coefficients, $\{w_{ij}, u_{mn}, v_{rs}\}$. The objective now is to find, for prescribed $\{F_i(\xi), G_j(\eta), B_m(\xi), D_n(\eta), E_r(\xi)$ and $H_s(\eta)\}$, the coefficients $\{w_{ij}, u_{mn}, v_{rs}\}$ that minimize the parameter Ω. Since this parameter is a non linear function of these coefficients, analytical treatment of the problem becomes difficult especially as the number of coefficients increases. In this investigation Mathematical Programming Techniques was used as the optimization algorithm for the vibration analysis of stiffened plates. The mathematical statement of the problem is stated as

$$\mathit{Minimize}\ \Omega\ \{w_{ij}, u_{mn}, v_{rs}\} \tag{38}$$

$$\mathit{Subject\ to} \begin{bmatrix} \{w_{ij}\} \\ \{u_{mn}\} \\ \{v_{rs}\} \end{bmatrix}^L \le \begin{bmatrix} \{w_{ij}\} \\ \{u_{mn}\} \\ \{v_{rs}\} \end{bmatrix} \le \begin{bmatrix} \{w_{ij}\} \\ \{u_{mn}\} \\ \{v_{rs}\} \end{bmatrix}^U \tag{39}$$

where the superscript U and L denote the upper and lower bounds on these variables. The optimization strategy for the non-linear function is performed iteratively by generating and solving a sequence of Quadratic sub-problems. The optimization strategy is described by

$$\{x_i^{k+1}\} = \{x_i^k\} + \{\alpha^k\} + \{P^k\} \tag{40}$$

where the superscript denotes the iteration number and the subscript i denotes the design variable, x

is the vector containing the displacement coefficients, α is the step size and P is the search direction. For each iteration, the search direction and step size is computed to produce a sufficient decrease in the objective function. At a typical k^{th} iteration, the search direction is computed from the solution of the quadratic, Taylor expansion of the frequency parameter Ω

$$Minimize\ \Omega\ \{w_{ij}, u_{mn}, v_{rs}\}\ \bullet P + \frac{1}{2}P^T \bullet \Delta^2\Omega(w_{ij}, u_{mn}, v_{rs}) \bullet P \tag{41}$$

where $\nabla\Omega$ is the gradient of the natural frequency parameter at the k^{th} iteration and $\nabla^2\Omega$ is the Hessian matrix. After obtaining the search direction from this Quadratic approximation of the function, each iteration proceeds by determining a step length, α, that produces a decrease in the objective function. The process continues until there is no further decrease in Π, or the decrease is of negligible order

3. Result

The first part of this section presents verifications to the described procedure for several stiffened and unstiffened panels that have been analyzed by other authors, using alternative numerical methods, such as Finite Element, Finite Difference and Finite Strip. In the second part, the dynamic behaviour of the structure is investigated for several panels.

3.1 Verification examples

Table 1 shows a numerical comparison of the values obtained using the present formulations and the analytical values of Harris and Crede (1961), for four aspect ratios, b/a = 1, 1.5, 2, 2.5. The natural frequencies are presented in terms of the non-dimensional parameter Ω given by

$$\Omega = \omega a^2\sqrt{\frac{m_p t_p}{D}} \tag{42}$$

where $D = E_p t_p^3/12(1 - v^2)$ is the plate flexural rigidity, t_p is the plate thickness, m_p is the mass density, ω the natural frequency in rad/sec. and a is the length of plate. As can be seen that both values are in good agreement.

The second verification example is a simply supported plate with one eccentric stiffener spanning along the centerline shown in Fig. 2. The Modulus of Elasticity of the plate and the stiffener are $E_p = E_{st} = 30 \times 10^6$ psi (2.07×10^5 N/mm^2), and the mass densities $m_p = m_{si}^{xi} = m_{si}^{yi} = 0.28$ Ib/in^3 (7.83×10^{-6} kg/mm^3). The dimensions of the plate are a = 16 in. (410 mm), b = 24 in. (600 mm)

Table 1 Comparison of Ω obtained using present formulations with Harris and Crede (1961)

b/a	Ω (rad/s)	
	Harris and Crede (1961)	Present
1	19.74	19.73
1.5	14.26	14.17
2	12.34	12.33
2.5	11.45	11.43

and the thickness t_p = 0.25 in. (6.33 mm). The stiffener depth h^{x1} = 0.875 in. (22.22 mm) and the thickness t_s^{xi} = 0.5 in. (12.7 mm). This panel was analyzed by Mukherjee and Mukhopadhy (1989) using the Finite Element method, by Asku (1982) using Finite Difference formulation and by Harik and Guo (1993) using Finite Element method. The lowest natural frequency for the structure is shown in Table 2 and as can be seen they are in reasonable agreement.

The next verification example is a 2 × 2 Bay continuous plate shown in Fig. 3. The line support are placed at the $sp^{x1} = sp^{y1} = 0.5$. This plate was analyzed by using the Finite Strip Method by Wu and Cheung (1974) and by and by Koko (1990) using a refined Finite Element formulation. The lowest natural frequencies Ω are compared in Table 3 with these references. It can be seen that the agreement is reasonable.

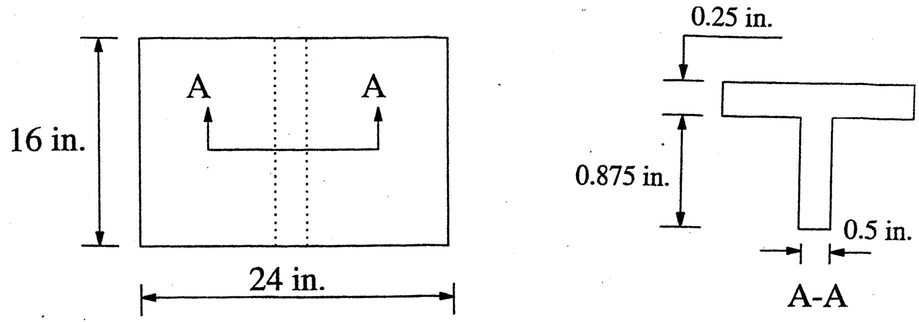

Fig. 2 Geometric details of plate with one longitudinal stiffener

Table 2 Comparison of Ω obtained using present formulations with other references

Reference	ω (Hz)
Asku (1982)	254.94
Harik and Guo (1993)	253.6
Mukherjee and Mukhopadhyay (1989)	257.05
Present	256.2

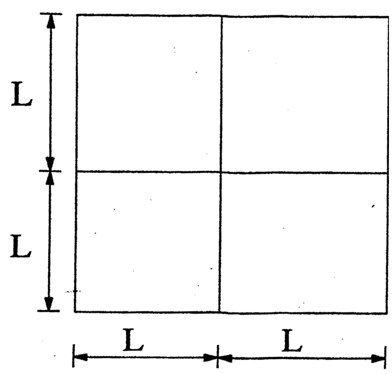

Fig. 3 Geometric details of 2 × 2 continuous plate

Table 3 Comparison of Ω obtained using present formulations with other references

Reference	Ω (rad/s)
Koko (1990)	20.05
Wu and Cheung (1974)	19.74
Present	19.6

3.2 Dynamic Behaviour of Stiffened panels

The interest in this section is to show the influence of the geometric parameters on the natural frequency of the structure. Several stiffening configurations are used in order to illustrate the structural efficiency of the system. Fig. 4(a) shows the natural frequency parameter, Ω vs. h^{x1}/t_p ratio for a centrally stiffened panel. The solid curve represents the eccentric while the dashed curve represents the concentric configuration. The solid circles are the numerical values obtained from the analysis using SQP. Starting with an unstiffened plate with unit aspect ratio, the natural frequency parameter Ω equals 19.73, and the mode shape is half-sine wave. Increasing h^{x1}/t_p, while setting the contribution of the torsional strain energy GJ^{x1}/Db to zero, the natural frequency parameter, Ω, increases until it attains a constant value of 49.3 at $h^{x1}/t_p = 6.1$ for the eccentric and 13.7 for the concentric configuration. At this stage, the stiffener subdivides the plate into two sub-panels and freely rotates, since $GJ^{x1}/Db = 0$. The EI^{x1}/Db ratio for this h^{x1}/t_p value is about 17. The h^{x1}/t_p for the concentric configuration is larger since the axis of bending of the plate and the stiffener in this case coincides while, for the eccentric configuration, the value of the inertia of the stiffener is larger. The natural frequency parameter Ω at this stage equals 49.3 which can also be obtained by replacing the paramter β by $(\beta)_{sub}$ in Eq. (37), i.e.

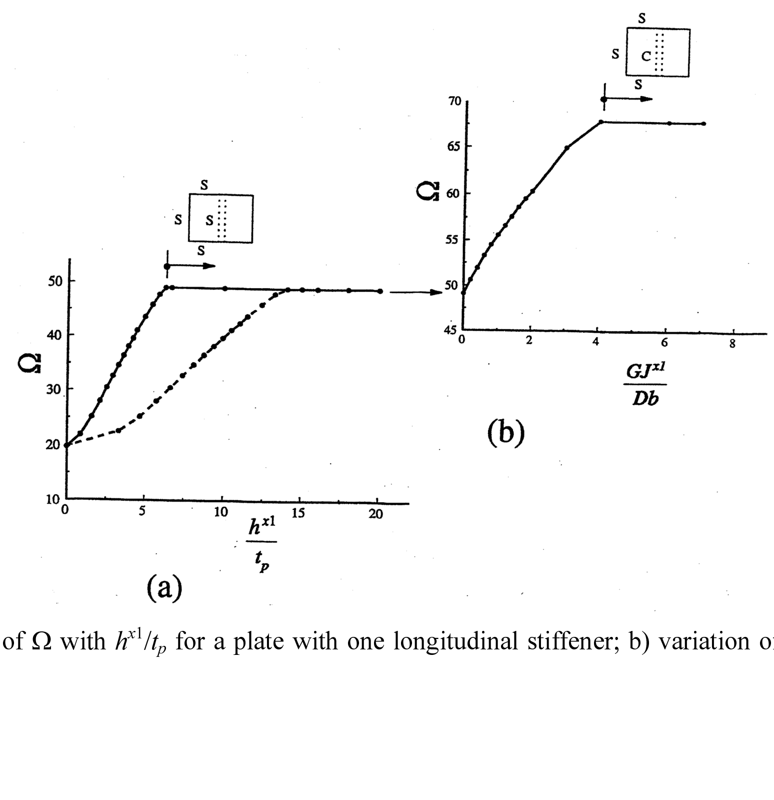

Fig. 4 Variation of Ω with h^{x1}/t_p for a plate with one longitudinal stiffener; b) variation of Ω with GJ^{x1}/Db

$$(\omega)_{sub} = \pi^2(i^2 + j^2(\beta)^2_{sub})\frac{1}{a^2}\sqrt{\frac{D}{m_p t_p}} = \pi^2(1 + 1*4)\frac{1}{a^2}\sqrt{\frac{D}{m_p t_p}} = 49.3\frac{1}{a^2}\sqrt{\frac{D}{m_p t_p}} = (\omega)_{PL} \quad (43)$$

The first quantity of the above equation is the natural frequency parameter of the sub-panel i.e., a simply supported plate along the four edges of width sp^{x1} and $(\beta)_{sub}$ is the aspect ratio of the sub-panel = (a/sp^{x1}). Therefore, these values of h^{x1}/t_p are the points where the natural frequencies of the plate and the sub-panels coincide and hence represent the optimum values.

Now, by fixing the EI^{x1}/Db or h^{x1}/t_p ratio at any value along the constant $\Omega = 49.3$ line and increasing the torsional stiffness parameter GJ^{x1}/Db of the stiffener, a further increase in the natural frequency parameter Ω can be obtained as shown Fig. 4(b). Note that, by fixing h^{x1}/t_p, t_s^{x1}/t_p needs to be increased to increase the J^{x1} value of the stiffener. The stiffener, at this stage, partially restrains the plate against rotation along $b/2$ until it clamps the plate along this side and the Ω value becomes constant at 68 for $GJ^{x1}/Db \approx 3.8$. This corresponds to the natural frequency of a plate with three simply supported edges and clamped along the fourth longitudinal edge. Any further increase in t_s^{x1}/t_p produces no further increase in the natural frequency of the plate.

To give numerical insight to the advantage of stiffened plates, if we assume that the Modulus of Elasticity of the plate and the stiffener are $E_p = E_{st} = 30 \times 10^6$ psi (2.07×10^5 N/mm^2), the mass densities $m_p = m_{st}^{xi} = m_{st}^{yi} = 0.28$ Ib/in^3 (7.83×10^{-6} kg/mm^3) and the dimensions of the plate are $a = 30$ in. (762 mm), $b = 30$ in. (762 mm). If we further assume that the natural frequency to be attained is $\omega = 266$ Hz, therefore for unstiffened plate the required plate thickness to achieve this natural frequency is $t_p = 0.5$ in. (12.7 mm) and thus the total volume of material required is 450 in^3 (7.4×10^6 mm^3). By adding a stiffener along the centerline of the plate, this natural frequency can be attained at $h^{x1} = 1.22$ in. (30.5 mm), $t_s^{x1} = 0.5$ in. (12.7 mm) and a reduced plate thickness t_p of 0.2 in. (5.1 mm). Thus the total volume of the stiffened plate is 198 in^3 (3.24×10^6 mm^3). Therefore for the same natural frequency, the stiffened plate requires less than one half the material the unstiffened plate requires. This shows the great efficiency of stiffened plates in material savings.

Fig. 5 illustrates the variation of the stiffener spacing with the natural frequency parameter Ω for

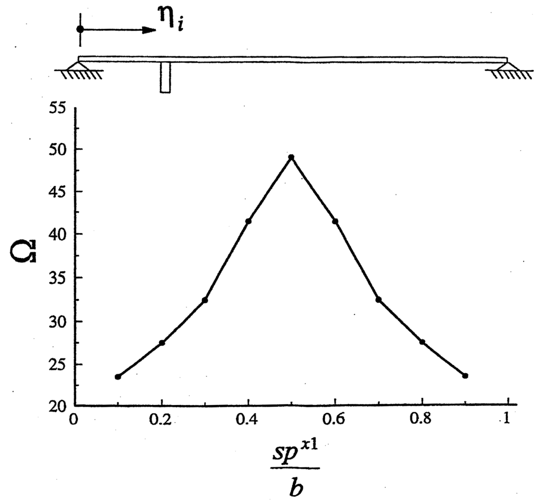

Fig. 5 Variation of Ω with the stiffener spacing, sp^{x1}/b

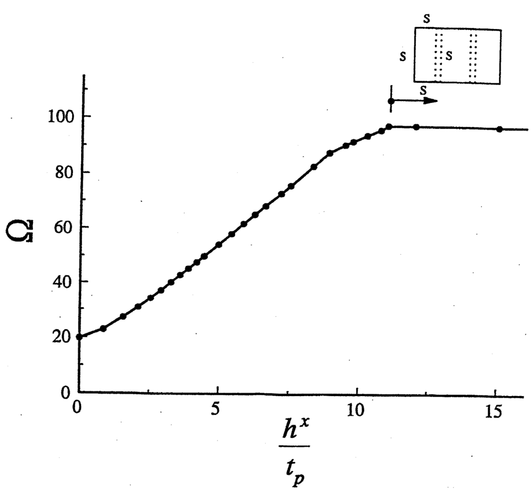

Fig. 6 Variation of Ω with h^x/t_p for a plate with two longitudinal stiffeners

$0 < (sp^{x1}/b) < 1$ for eccentric configuration for $EI^{x1}/Db = 20$, i.e., $h^{x1}/t_p = 6.65$. Note that this value of h^{x1}/t_p is along the constant Ω line. The incremental value of sp^{x1} was $b/10$. The value of GJ^{x1}/Db is taken to be zero for this case. Note that the curve is symmetric about the centerline, $sp^{x1} = b/2$. As can be seen, the best location for the stiffener, which produces the highest Ω value, is at the centerline of the plate. A possible explanation for this is that when the stiffener is off centre, the plate is divided into two sub-panels with different widths, sp^{x1} and sp^{x2}, as shown in Fig. 1. The critical one will correspond to the larger sp^{x1} since it will produce the smallest Ω value. If, for example, the stiffener is at $sp^{x1} = 0.4b$, the panel with $sp^{x2} = 0.6b$ will have a lower natural frequency since it has a lower aspect ratio. Therefore, the largest Ω can be obtained when the natural frequency of both sub-panels coincides, i.e., $sp^{x1} = sp^{x2} = 0.5$.

When considering two equally spaced longitudinal stiffeners, with $h^{x1}/t_p = h^{x2}/t_p$ and eccentric configuration the optimum stiffener slenderness is increased to 10.8 as shown in Fig. 6. Note that since the parameters h^{x1}/t_p and h^{x2}/t_p are increased equally, they are denoted by h^x/t_p in the graph. At this value of h^x/t_p, the stiffeners sub-divide the plate into three sub-panels each of length a and width $b/3$. The maximum Ω value at this stage, ignoring the contribution of the torsional strain energy, is 98.6. This value can also be obtained by replacing the plate parameters, β and a by $(\beta)_{\text{sub}}$ and $(a)_{\text{sub}}$ in Eq. (37), i.e.

$$(\omega)_{sub} = \pi^2(i^2 + j^2(\beta)^2_{sub})\frac{1}{(a)^2_{sub}}\sqrt{\frac{D}{m_p t_p}} = \pi^2\left(1 + 1*\left[\frac{1}{3}\right]^2\right)\frac{1}{a^2}\sqrt{\frac{D}{m_p t_p}} = 98.6\frac{1}{a^2}\sqrt{\frac{D}{m_p t_p}} = (\omega)_{PL} \quad (44)$$

Noting that in this case $(\beta)_{\text{sub}} = 3\beta$. It can be seen that the value of the maximum natural frequency parameter is almost doubled by adding additional transverse stiffener.

By adding a transverse stiffener along the centerline of the plate, i.e., $sp^{y1} = a/2$, and increasing h^x/t_p and h^y/t_p equally, the optimum h/t_p for the longitudinal and transverse stiffeners is increased to 10.5 for the eccentric and to 21 for the concentric configuration as shown in Fig. 7. The natural frequency parameter of the structure, in this case, increases to 128.3. At this h^x/t_p value, the stiffeners subdivide the plate into six sub-panels each with $(\beta)_{\text{sub}} = 1.5$. This value of Ω can be obtained by replacing β and a in Eq. (37) by $(\beta)_{\text{sub}}$ and $(a)_{\text{sub}}$ to obtain

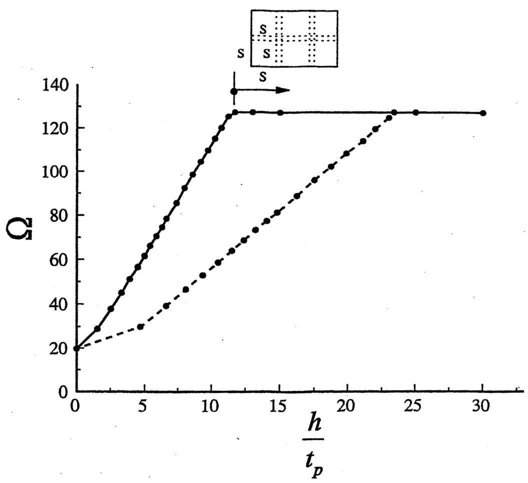

Fig. 7 Variation of Ω with h/t_p for a plate with two longitudinal and one transverse stiffeners

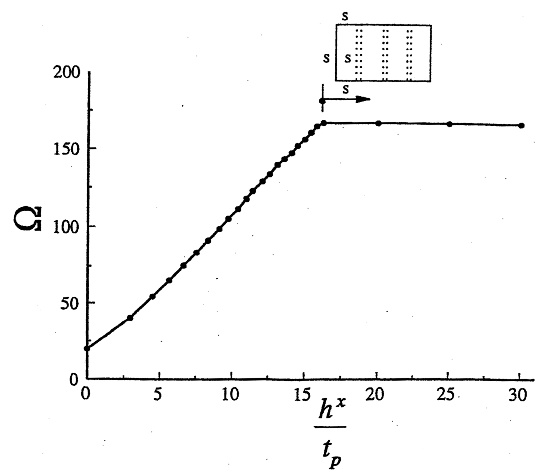

Fig. 8 Variation of Ω with h^x/t_p for a plate with three longitudinal stiffeners

$$(\omega)_{sub} = \pi^2(i^2 + j^2(\beta)_{sub}^2)\frac{1}{(a)_{sub}^2}\sqrt{\frac{D}{m_p t_p}} = \pi^2\left(1 + 1*\left[\frac{2}{3}\right]^2\right)\frac{1}{\left(\frac{a}{2}\right)^2}\sqrt{\frac{D}{m_p t_p}} = 128.3\sqrt{\frac{D}{m_p t_p}} = (\omega)_{PL} \quad (45)$$

If the transverse stiffener is added to in the longitudinal direction instead of the transverse direction, the maximum value of natural frequency parameter Ω of the structure is 167.7, i.e., higher by about 30% from the previous configuration, as shown in Fig. 8. The amount of h^x/t_p to subdivide the plate into four equally sub-panels on the other hand is 16.1 for the eccentric configuration.

4. Conclusions

The paper investigated the dynamic behaviour of multi-stiffened plates. As a first stage, the coupled differential equations for eccentric stiffening were derived and expressed in terms of the plate/stiffeners geometric properties. Generalized energy formulations were then derived for the computation of the natural frequency of the assembled plate and stiffener elements. The resulting non-linear strain energy functions are then transferred into unconstrained optimization problem and Mathematical Programming is used to determine the magnitude of the lowest natural frequency and the associated mode shape. Verification examples are then presented for several panels. The dynamic behaviour was then detailed for several stiffening configurations. The variation of the natural frequency was shown with the plate stiffener geometric proportions. From these graphs it is now possible to determine the finite values of h/t_p which maximize the natural frequency of the structure.

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Nataion

A_p	: area of the plate;
A^{xi}, A^{yi}	: areas of i^{th} stiffener along the x and y axes, respectively;
a, b	: length and width of the plate;
E_p, E_{st}	: Young's modulus for the plate and the stiffeners, respectively;
e_x	: eccentricity of the stiffeners;
$F_i(\xi)$, $G_j(\eta)$	: out of plane displacement functions;
$B_m(\xi)$, $D_n(\eta)$, $E_r(\xi)$, $H_s(\eta)$	: in-plane displacement functions;
G	: shear modulus;
h^{xi}, h^{yi}	: depth of i^{th} stiffener along the x and y axes, respectively;
I^p	: moment of inertia of the plate;

I_y^{xi}, I_x^{yi}	: second moment of inertia about the major axis of i^{th} stiffener along the x and y axes, respectively;
I_z^{xi}, I_z^{yi}	: second moment of inertia, about the minor axis, of i^{th} stiffener along the x and y axes, respectively;
I_o^{xi}, I_o^{yi}	: polar moment of inertia of a typical i^{th} stiffener along the x and y axes, respectively;
J^{xi}, J^{yi}	: torsional rigidity of a typical i^{th} stiffener along the x and y axes, respectively;
M_{xx}^p, M_{yy}^p, M_{xy}^p	: Moment components of the plate;
M_{xx}^{xi}, M_{yy}^{yi}	: Moment components of the stiffeners;
m_p	: mass density of the plate;
m_{st}^{xi}, m_{st}^{yi}	: mass densities of stiffeners along the x and y axes, respectively;
$M_{xx}^p, M_{yy}^p, M_{xy}^p$	: force components of the plate;
N_{xx}^{xi}, N_{yy}^{yi}	: force components of the plate;
Q^{xi}, Q^{yi}	: first moment of inertia of a typical i^{th} stiffener along the x and y-axes, respectively;
W, U, V	: out of plane and in-plane displacements;
SC	: support condition;
sp^{xi}, sp^{yi}	: stiffener's spacing in the x and y-direction, respectively;
t_p	: thickness of the plate;
t_s^{xi}, t_s^{yi}	: thickness of i^{th} stiffener along the x and y axes, respectively;
$U_{st}^{xi}, tv_{st}^{yi}$	: total strain energies of stiffeners along the x and y axes, respectively;
Ω	: non-dimensional natural frequency parameter;
ω	: natural frequency of the structure.