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Technical Note

Free vibrations of delaminated beams in prebuckled states: Lower and upper bounds

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1. Introduction

Wang *et al.* (1982) developed a 'free mode' model for the free vibrations of delaminated beams. The model assumed that the delaminated layers vibrate 'freely' and will have different transverse deformations. Mujumdar and Suryanarayan (1988) developed a 'constrained mode' model, where the delaminated layers are assumed to be 'constrained' to have identical transverse deformations but are free to slide over each other in the axial direction except at their ends. Chen *et al.* (1995) used the 'constrained mode' for prebuckled composite beams with a single delamination. Lee *et al.* (2003) used the 'free mode' to study multiple-delamination problem, and their results showed a linear relation between the square of the frequency of a clamped-clamped beam and the axial compressive load. Bokaian (1988) showed that the linear relation between the natural frequency of a simply supported perfect beam and the axial compressive load can be expressed as $(\omega/\omega_0)^2 = 1 - P_1^0/P_{cr}$, where ω and ω_0 are the natural frequency of the loaded and the unloaded beam, respectively, P_1^0 is the axial compressive load and P_{cr} is the buckling load of the beam.

In this technical note, we present analytical solutions for the lower bound and the upper bound of the free vibrations of axially compressed beams with two delaminations. The lower bound uses the 'free mode', whereas the upper bound uses the 'constrained mode'. In addition, we study the linear relation between the square of the natural frequency of a simply supported delaminated beam and the axial compressive load.

2. Formulation

Fig. 1 shows a homogeneous and isotropic beam with length L and thickness H_1 with two delaminations with length a and located at a distance d from the center of the beam. The delaminated beam is analyzed as five interconnected Euler-Bernoulli beams.

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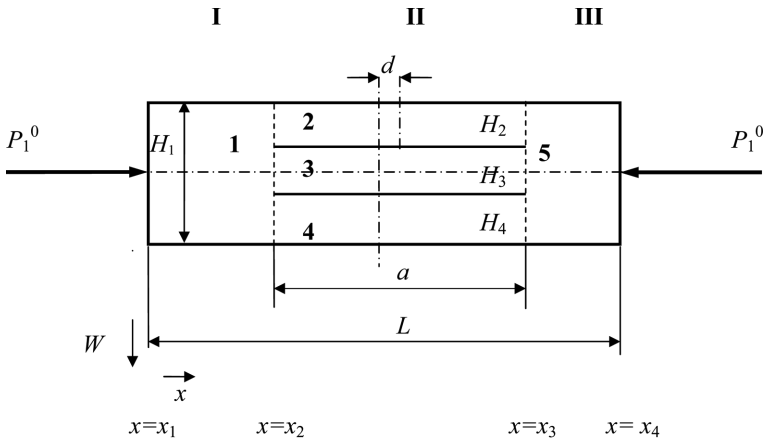

Fig. 1 A homogeneous and isotropic beam with two delaminations under axial compressive loading

For the 'free mode', the governing equations for the free vibrations of a delaminated beam under axial compressive loading are

$$EI_i\frac{\partial^4 w_i}{\partial x^4} + \rho_i A_i\frac{\partial^2 w_i}{\partial t^2} + P_i^0\frac{\partial^2 w_i}{\partial x^2} = 0 \quad (i = 1 - 5) \tag{1}$$

where EI_i (i = 1 − 5) is the bending stiffness of beam i, $w_i(x, t)$ is the midplane deflection, ρ_i is the mass density, A_i is the cross-sectional area, and P_i^0 is the axial load. Substituting $w_i(x, t) = W_i(x) \sin(\omega t)$ in Eq. (1), we have

$$EI_iW_i'''' + P_i^0W_i'' - \rho_iA_i\omega^2W_i = 0 \quad (i = 1 - 5) \tag{2}$$

The generalized solutions for the differential equation in Eq. (2) are

$$W_i(x) = C_i\cos(\alpha_i x/L) + S_i\sin(\alpha_i x/L) + CH_i\cosh(\beta_i x/L) + SH_i\sinh(\beta_i x/L) \tag{3}$$

where

$$\alpha_i^2 = \frac{P_i^0 + \sqrt{(P_i^0)^2 + 4\rho_iA_iEI_i\omega^2}}{2EI_i}L^2 \text{ and } \beta_i^2 = \frac{-P_i^0 + \sqrt{(P_i^0)^2 + 4\rho_iA_iEI_i\omega^2}}{2EI_i}L^2 \tag{4}$$

and where C_i, S_i, CH_i and SH_i (i = 1 − 5) are the 20 unknown coefficients, which can be determined from the 4 boundary conditions and 16 continuity conditions.

The boundary conditions for a simply supported beam are $W_i = 0$ and $W_i'' = 0$ (i = 1, 5). The continuity conditions for deflection, slope, shear and bending moments at delamination junction $x = x_2$ are

$$W_1 = W_2 = W_3 = W_4 \tag{5}$$

$$W_1' = W_2' = W_3' = W_4' \tag{6}$$

$$EI_1W_1''' = EI_2W_2''' + EI_3W_3''' + EI_4W_4''' \tag{7}$$

$$EI_1W_1'' = EI_2W_2'' + EI_3W_3'' + EI_4W_4'' - P_2\left(\frac{H_2}{2}\right) - P_3\left(H_2 + \frac{H_3}{2}\right) - P_4\left(H_2 + H_3 + \frac{H_4}{2}\right) \quad (8)$$

The perturbed axial forces P_i (i = 2 − 4) in Eq. (8) can be solved from the compatibility between the stretching/shortening of the delaminated layers and axial equilibrium (Mujumdar and Suryanarayan 1988, Della and Shu 2005). The boundary conditions and the continuity conditions provide 20 homogeneous equations. A non-trivial solution for the coefficients exists only when the determinant of the coefficient matrix vanishes.

The 'constrained mode' model assumes that the delaminated layers (beams 2, 3 and 4) are 'constrained' to have the same transverse deformations. The delaminated beam is then analyzed as three beam segments I – III (Fig. 1). The governing equations are

$$EI_i\frac{\partial^4 w_i}{\partial x^4} + \rho A_i\frac{\partial^2 w_i}{\partial t^2} + P_1^0\frac{\partial^2 w_i}{\partial x^2} = 0 \quad (i = \text{I} - \text{III}) \quad (9)$$

where

$$EI_I = EI_{III} = EI_I,\ EI_{II} = EI_2 + EI_3 + EI_4 \quad (10)$$

$$\rho A_I = \rho A_{III} = \rho_1 A_1,\ \ \rho A_{II} = \rho_2 A_2 + \rho_3 A_3 + \rho_4 A_4 \quad (11)$$

The generalized solutions for the 'constrained mode' model are identical in form to that of the 'free mode' model. However, the unknown coefficients are reduced to 12 coefficients, which can be determined from the 4 boundary conditions and 8 continuity conditions.

3. Results and discussions

Fig. 2 shows the variation of the square of the fundamental frequency $(\omega/\omega_d)^2$ of the delaminated beam with the normalized buckling load (P_1^0/P_d). The square of the fundamental frequency ω^2 is normalized with respect to square of the frequency of a delaminated beam ω_d^2, while the

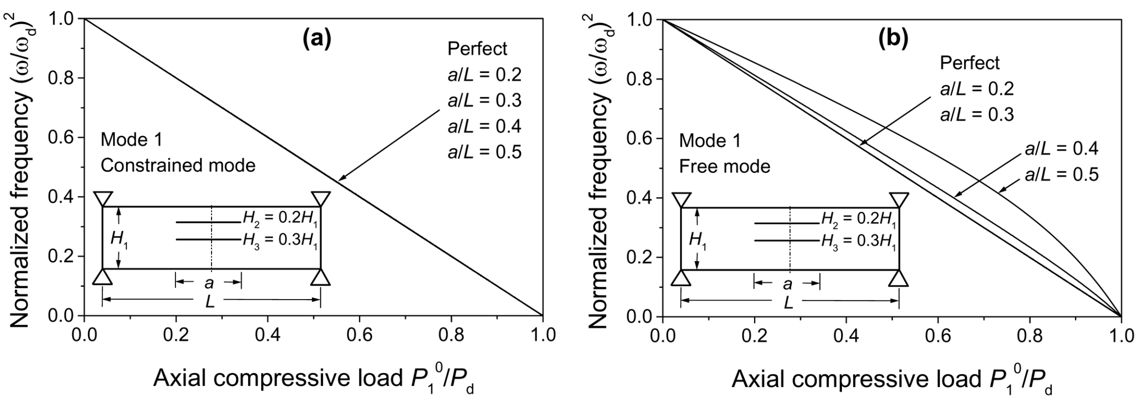

Fig. 2 Variation of the square of the normalized fundamental frequency $(\omega/\omega_d)^2$ of a simply supported beam with the normalized compressive load P_1^0/P_d for various delamination length a/L: (a) 'Constrained mode'; (b) 'Free mode'

normalized compressive load P_1^0 is normalized with respect to the buckling load of the delaminated beam P_d. Fig. 2(a) shows that the 'constrained mode' $(\omega/\omega_d)^2$ varies linearly with (P_1^0/P_d) and the linear variation can be expressed as $(\omega/\omega_d)^2 = 1 - P_1^0/P_d$, which is identical in form to that of an undelaminated beam (Bokaian 1988). This is because the delaminated portion of the beam is modeled as an undelaminated beam with a reduced bending stiffness. Therefore, a delaminated beam of 'constrained mode' is an undelaminated beam with a reduced stiffness at the location of the delamination. In the 'free mode' model, the relation is not valid when one of the delaminated layers is long and thin, as shown in Fig. 2(b); the variation between the 'free mode' $(\omega/\omega_d)^2$ and (P_1^0/P_d) tends to lose its linearity as the delamination length increases.

In this study, we have shown that for simply supported beams, the square of the 'constrained mode' fundamental frequency varies linearly with the axial compressive load. Thus, once the natural frequency and the buckling load of the delaminated beam are identified, an estimate of the fundamental frequency of the delaminated beam under axial compressive loading can be obtained.

References

Bokaian, A. (1988), "Natural frequencies of beams under compressive axial loads", *J. Sound Vib.*, **126**(1), 49-65.

Chen, H.P., Tracy, J.J. and Nonato, R. (1995), "Vibration analysis of delaminated composite laminates in prebuckled states based on a new constrained model", *J. Compos. Mater.*, **29**(2), 229-256.

Della, C.N. and Shu, D. (2005), "Vibration of beams with double delaminations", *J. Sound Vib.,* **282**(3-5), 919-935.

Lee, S., Park, T. and Voyiadjis, G.Z. (2002), "Free vibration analysis of axially compressed laminated composite beam-columns with multiple delaminations", *Compos. Part B*, **33**(8), 605-617.

Mujumdar, P.M. and Suryanarayan, S. (1988), "Flexural vibrations of beams with delaminations", *J. Sound Vib.*, **125**(3), 441-461.

Wang, J.T.S., Liu, Y.Y. and Gibby, J.A (1982), "Vibration of split beams", *J. Sound Vib.*, **84**(4), 491-502.