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Response of a finite beam on a tensionless Pasternak foundation under symmetric and asymmetric loading

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Abstract. The static response of a finite beam resting on a tensionless Pasternak foundation and subjected to a concentrated vertical load is assessed in this study. The concentrated vertical load may be applied at the center of the beam, or it may be offset from the center. The tensionless character of the foundation results in the creation of lift-off regions between the beam and the foundation. An analytical/ numerical solution is obtained from the governing equations of the contact and lift-off regions to determine the extent of the contact region. Although there is no nonlinear term in the equations, the problem shows a nonlinear character since the contact region is not known in advance. Due to that nonlinearity, the essentials of the problem (the coordinates of the lift-off points) are calculated numerically using the Newton-Raphson technique. The numerical results are presented in figures to illustrate the behaviours of the free-free and pinned-pinned beams under symmetric or asymmetric loading. The figures illustrate the effects of the shear foundation parameter and the symmetric and asymmetric loading options on the variation of the contact lengths and the displacement of the beam.

Keywords: Pasternak foundation; finite beam; lift-off.

1. Introduction

The analysis of beams in elastic foundations is very common in engineering. As the mechanical response of the foundation is governed by many factors and cannot be directly calculated, it is necessary to idealize the behaviour of the foundation. The Winkler elastic foundation model, which consists of an infinite number of closed-spaced linear springs, is a one-parameter model used

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extensively in practice. The well known text by Hetényi (1946) provides a thorough treatment of the Winkler model for elastic foundations. However, interactions between springs are not considered in this model, so it does not accurately represent the characteristics of many practical foundations. To overcome this problem, several two-parameter models have been suggested (Filonenko-Borodich 1940, Pasternak 1954, Kerr 1964, Vlasov and Leontev 1966). A comprehensive review of these models has been presented by Dutta and Roy (2002). In the Pasternak foundation model, which will be used in this study, the existence of a shear interaction among the spring elements is assumed. This is accomplished by connecting the ends of the springs to a beam or plate that only undergoes a transverse shear deformation. There are numerous studies dealing with the response of beams on Pasternak foundations in literature. Some recent studies in this area include those of De Rosa and Maurizi (1998), Horibe and Asano (2001), Filipich and Rosales (2002), Rao (2003), Chen *et al.* (2004) and Kargarnovin and Younesian (2004). In most of the studies on the static/dynamic behaviours of beams on an elastic foundation, it has been assumed that the foundation (regardless of whether the model is of the Winkler or two-parameter variety) reacts in tension as well as in compression. That is, if a downward lateral load is applied to a beam resting on a foundation, the beam will be compressed into the foundation. If the direction of the load is reversed, the beam and the foundation are pulled up, creating tension in the foundation. However, this assumption does not hold for many practical problems: i.e., while compressive stresses can be transmitted easily, it is difficult to transmit tensile stresses across the boundary between the beam and the foundation except when the adhesion between the beam and foundation is assured, and thus, no separation is permitted between them. Instead, a model in which the foundation reacts only by compression (one-way or tensionless foundation) would be more realistic. In the case of an absence of tensile forces across the interface between the beam and the foundation, lift-off regions can develop in the system. Therefore, the solution to this type of problem is complicated by the need to determine the contact region, which, in turn, depends on the parameters of the system.

The static/dynamic behaviour of infinite beams resting on a tensionless foundation has been studied by Tsai and Westmann (1967), Weitsman (1970, 1971, 1972), Rao (1974), Choros and Adams (1979), Lin and Adams (1987), Ioakimidis (1996) and Maheshwari *et al.* (2004). In these studies, due to the infinite beam assumption, the system is symmetric and the applied concentrated load must be centered on the beam. Studies involving behaviour of finite beams on tensionless foundations do appear in the literature. Celep *et al.* (1989) studied the dynamic response of a finite beam on a tensionless Winkler foundation by considering eccentric loading. Kerr and Coffin (1991) studied the static behaviour of a finite beam resting on a tensionless Pasternak foundation subjected to a vertical concentrated load. Coşkun and Engin (1999) and Coşkun (2000) studied the nonlinear vibrations of a finite beam resting on a nonlinear tensionless Winkler foundation subjected to a vertical concentrated load. Coşkun (2003) studied the response of a finite beam on a tensionless Pasternak foundation subjected to a vertical dynamic load. Zhang and Murphy (2004) studied the static response of a finite beam resting on a tensionless Winkler foundation subjected to a vertical concentrated load which could be applied symmetrically or asymmetrically. Celep and Demir (2005) studied the static response of a circular rigid beam on a tensionless two-parameter foundation subjected to a vertical load and a moment. The same authors also studied the static response of an elastic beam on such a foundation by considering a uniformly distributed load and concentrated edge loads (Celep and Demir 2007). Finally, Lancioni and Lenci (2007) studied the nonlinear vibrations of a semi-infinite beam on a tensionless Winkler foundation subjected to a uniformly distributed load. The results of all the studies given above mostly include the determination of the coordinates of the lift-off points, i.e., the contact lengths of the beam. On the other hand, there are only a limited number of studies in the literature that consider the lift-off problem of plates on tensionless Pasternak foundations. Recent studies in this field include those of Shen and Yu (2004), Güler (2004), Yu *et al.* (2007) and Celep and Güler (2007).

In this study, the static response of a finite beam on a tensionless Pasternak foundation subjected to a vertical concentrated load is investigated. The load may either be applied at the center of the beam (symmetric loading) or may be offset (asymmetric loading). However, this assumption does not hold for an infinite beam because the load is symmetric by definition and thus, asymmetric loads cannot exist. The finite beam assumption permits the off-center loading, which breaks the symmetry of the system and requires the specification of some appropriate boundary conditions, which influence the results. For the sake of brevity, this study only investigates the responses of free-free and pinned-pinned beams.

2. Formulation of the problem

2.1 Definition of the system and governing equations

Consider a finite beam of length L resting on a tensionless Pasternak foundation and subjected to a vertical load P such that lift-off of the beam is possible. This situation is shown in Fig. 1. The distance to the left (right) end of the beam is $L_1(L_2)$ and is measured from the origin of the coordinate system centered at the load. The vertical deflection is given by $W(x)$. The contact region is defined as $-X_1 < x < X_2$, where X_1 and X_2 represent the lift-off points on the left and right sides of the beam, respectively. In order to investigate the behaviour in both the contact and noncontact regions, the vertical deflection $W(x)$ is broken into the following five distinct regions:

(i) in the noncontact region (for foundation surface): $W(x) = W_1(x)$, $-\infty < x < -X_1$
(ii) in the noncontact region (for beam): $W(x) = W_2(x)$, $-L_1 < x < -X_1$
(iii) in the contact region: $W(x) = W_3(x)$, $-X_1 < x < X_2$
(iv) in the noncontact region (for beam): $W(x) = W_4(x)$, $X_2 < x < L_2$
(v) in the noncontact region (for foundation surface): $W(x) = W_5(x)$, $X_2 < x < \infty$

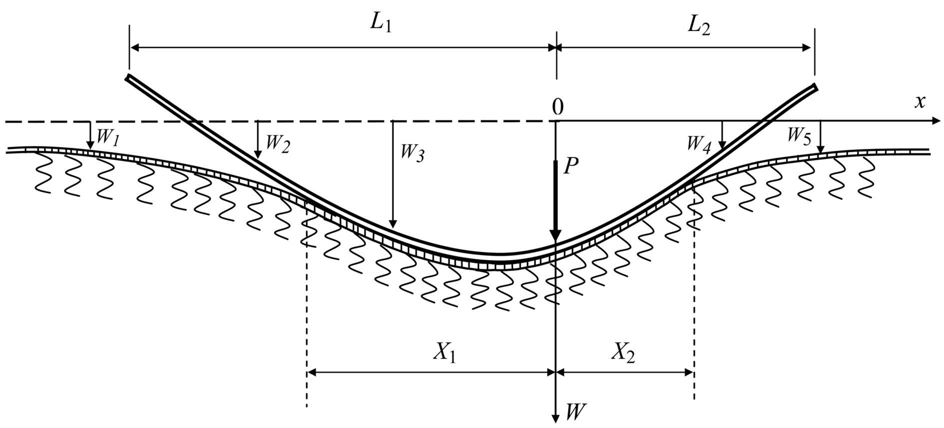

Fig. 1 Finite beam on a tensionless Pasternak foundation subjected to an eccentric load

The differential equations governing the vertical responses in these regions are

$$G\frac{d^2W_1}{dx^2} - kW_1 = 0, \qquad -\infty < x < -X_1 \tag{1}$$

$$EI\frac{d^4W_2}{dx^4} = 0, \qquad -L_1 < x < -X_1 \tag{2}$$

$$EI\frac{d^4W_3}{dx^4} - G\frac{d^2W_3}{dx^2} + kW_3 = P\delta(x), \quad -X_1 < x < X_2 \tag{3}$$

$$EI\frac{d^4W_4}{dx^4} = 0, \quad X_2 < x < L_2 \tag{4}$$

$$G\frac{d^2W_5}{dx^2} - kW_5 = 0, \quad X_2 < x < \infty \tag{5}$$

where EI is the beam flexural rigidity, $\delta(x)$ is the Dirac delta function, k is the Winkler foundation modulus and G is the shear modulus of the shear layer. It may be noted here that the contact pressure $q(x)$ is of the form $q = kW - G(d^2W/dx^2)$ in the Pasternak foundation model. For convenience, the nondimensionalized variable ξ, deflection $w(\xi)$, Winkler foundation constant λ, shear foundation coefficient λ_G, lift-off points ξ_1 and ξ_2, beam length l, and left (right) side beam length $l_1(l_2)$, are introduced as follows

$$\lambda^4 = \frac{k}{4EI}, \quad \xi = \lambda x, \quad \xi_1 = \lambda X_1, \quad \xi_2 = \lambda X_2, \quad l = \lambda L, \quad l_1 = \lambda L_1, \quad l_2 = \lambda L_2$$

$$w = \lambda W, \quad \lambda_G = \frac{k}{G\lambda^2}, \quad F = \frac{P}{\lambda^2 4EI} \tag{6}$$

Introducing these quantities into the governing Eqs. (1)-(5) produces the following non-dimensional equations

$$\frac{d^2w_1}{d\xi^2} - \lambda_G w_1 = 0, \quad -\infty < \xi < -\xi_1 \tag{7}$$

$$\frac{d^4w_2}{d\xi^4} = 0, \quad -l_1 < \xi < -\xi_1 \tag{8}$$

$$\frac{1}{4}\frac{d^4w_3}{d\xi^4} - \frac{1}{\lambda_G}\frac{d^2w_3}{d\xi^2} + w_3 = F\delta(\xi), \quad -\xi_1 < \xi < \xi_2 \tag{9}$$

$$\frac{d^4w_4}{d\xi^4} = 0, \quad \xi_2 < \xi < l_2 \tag{10}$$

$$\frac{d^2w_5}{d\xi^2} - \lambda_G w_5 = 0, \quad \xi_2 < \xi < \infty \tag{11}$$

2.2 Boundary conditions

Eqs. (7) and (11) are second order differential equations and Eqs. (8), (9) and (10) are fourth order differential equations. Therefore, 16 unknown integration constants will appear in the solution of these equations. Because the lift-off points (ξ_1 and ξ_2) are also unknown, there are a total of 18 unknowns to be determined. To obtain these unknowns, there must be an equal number of boundary/matching conditions. At $\xi = -\xi_1$ and $\xi = \xi_2$, the geometric boundary conditions require continuity of the displacement and slope. These are expressed as

$$w_1(-\xi_1) = w_3(-\xi_1), \quad w_1'(-\xi_1) = w_3'(-\xi_1), \quad w_2(-\xi_1) = w_3(-\xi_1), \quad w_2'(-\xi_1) = w_3'(-\xi_1) \tag{12}$$

$$w_3(\xi_2) = w_5(\xi_2), \quad w_3'(\xi_2) = w_5'(\xi_2), \quad w_3(\xi_2) = w_4(\xi_2), \quad w_3'(\xi_2) = w_4'(\xi_2) \tag{13}$$

There are also four natural boundary conditions at $\xi = -\xi_1$, ξ_2. These require continuity of the bending moment and shear force. These are

$$w_2''(-\xi_1) = w_3''(-\xi_1), \quad w_2'''(-\xi_1) = w_3'''(-\xi_1), \quad w_4''(\xi_2) = w_3''(\xi_2), \quad w_4'''(\xi_2) = w_3'''(\xi_2) \tag{14}$$

The displacement $w(\xi)$ at the foundation surface should be finite as $\xi \to \pm\infty$. This gives us two conditions

$$\lim_{\xi \to \infty} \{w_1\} \to \mathit{finite}, \quad \lim_{\xi \to \infty} \{w_5\} \to \mathit{finite} \tag{15}$$

Four additional conditions arise from the boundaries at $\xi = -l_1$, l_2. However, these will differ depending on the types of the boundaries used. In this study, only two cases (free-free and pinned-pinned) are considered. These are

$$w_2''(-l_1) = 0, \quad w_2'''(-l_1) = 0, \quad w_4''(l_2) = 0, \quad w_4'''(l_2) = 0 \tag{16}$$

for the free-free beam and

$$w_2(-l_1) = 0, \quad w_2''(-l_1) = 0, \quad w_4(l_2) = 0, \quad w_4''(l_2) = 0 \tag{17}$$

for the pinned-pinned beam. These are the 18 boundary/matching conditions necessary to determine the 18 unknown constants. However, in the formulation given above, it is assumed that the load may either be applied at the center of the beam or may be offset. If the load is applied at the center of the beam, the number of the boundary/matching conditions reduces to 11 with the use of the symmetry in the system. If the $\xi \geq 0$ region is considered, for instance, the boundary and matching conditions given above for the right side of the system are still valid. But in addition to these conditions, one must use the continuity of the slope of the elastic curve and the symmetry in the system: i.e., in dimensional terms, the slope is zero $(W_3'(0) = 0)$ and the shear force is $W_3'''(0) = P/(2EI)$ at the center of the beam. Apart from this, in some cases the beam may compresses into the foundation completely (no separation develops), or one-sided contact may occur between the beam and the foundation. In the full contact case, the slopes of the free part of the foundation and the foundation beneath the beam are not equal at the free ends of the beam. Thus, the boundary conditions that will be satisfied at $\xi = l_1$ and $\xi = l_2$ are

$$-w_3'''(-l_1) + (4/\lambda_G)w_3'(-l_1) = (4/\lambda_G)w_1'(-l_1)\,, \quad -w_3'''(l_2) + (4/\lambda_G)w_3'(l_2) = (4/\lambda_G)w_5'(l_2) \quad (18)$$

Here, the terms on the left and right sides, respectively, show the generalized shearing force in the foundation beneath the beam and the generalized shearing force in the free part of the foundation. If one-sided contact occurs in the system, the first or the second sections of Eq. (18) can be used, depending on the region where separation fails to develop.

3. Solution

The differential Eqs. (7)-(11) contain only constant coefficients. Therefore, the general solutions are of the form $w = Ae^{s\xi}$. Substituting this into Eq. (9) for a homogeneous solution gives

$$s^4 - \frac{4}{\lambda_G}s^2 + 4 = 0 \quad (19)$$

The roots of this equation are

$$s_{1,2,3,4} = \pm\sqrt{\frac{2}{\lambda_G} \pm \sqrt{\left(\frac{2}{\lambda_G}\right)^2 - 4}} \quad (20)$$

Since the parameters EI, k and G are rigidity parameters of the beam and foundation, they are all non-negative and thus the parameter λ_G is always positive. For this reason, there are only three possible combinations of λ_G that need to be considered, i.e., λ_G larger than, equal to, or smaller than 1. For $\lambda_G > 1$, the general solution of Eq. (9) can be written in the form

$$\begin{aligned} w_3(\xi) = {} & [A_7\cos(n\xi) + A_8\sin(n\xi)]\cosh(t\xi) + [A_9\cos(n\xi) + A_{10}\sin(n\xi)]\sinh(t\xi) \\ & + E_1\sinh|t\xi|\cos(n\xi) + E_2\cosh(t\xi)\sin|n\xi| \end{aligned} \quad (21)$$

where $n = \sqrt{1 - 1/\lambda_G}$, $t = \sqrt{1 + 1/\lambda_G}$; and A_7, A_8, A_9, A_{10}, E_1, E_2 are unknown constants. Now, consider the functions $f_1 = \sinh|t\xi|\cos(n\xi)$ and $f_2 = \cosh(t\xi)\sin|n\xi|$ for the evaluation of the constants E_1 and E_2. The second and fourth derivatives of these functions with $d|\xi|/d\xi = \mathrm{sgn}(\xi)$ and $d\mathrm{sgn}(\xi)/d\xi = 2\delta(\xi)$ are

$$f_1'' = 2t\delta(\xi) + [-2tn\cosh(t\xi)\sin(n\xi) + (t^2 - n^2)\sinh(t\xi)\cos(n\xi)]\mathrm{sgn}(\xi)$$

$$\begin{aligned} f_1^{iv} = {} & 2t\delta''(\xi) + 2t(t^2 - 3n^2)\delta(\xi) + [4tn(n^2 - t^2)\cosh(t\xi)\sin(n\xi) \\ & + (t^4 + n^4 - 6t^2n^2)\sinh(t\xi)\cos(n\xi)]\mathrm{sgn}(\xi) \end{aligned}$$

$$f_2'' = 2n\delta(\xi) + [2tn\sinh(t\xi)\cos(n\xi) + (t^2 - n^2)\cosh(t\xi)\sin(n\xi)]\mathrm{sgn}(\xi)$$

$$\begin{aligned} f_2^{iv} = {} & 2n\delta''(\xi) + 2n(3t^2 - n^2)\delta(\xi) + [4tn(t^2 - n^2)\sinh(t\xi)\cos(n\xi) \\ & + (t^4 + n^4 - 6t^2n^2)\cosh(t\xi)\sin(n\xi)]\mathrm{sgn}(\xi) \end{aligned}$$

By differentiating Eq. (21) with respect to ξ, substituting into Eq. (9) and equating coefficients of $\delta(\xi)$ and $\delta''(\xi)$ we obtain

$$E_1 = -\frac{F}{2t}, \quad E_2 = \frac{F}{2n} \tag{22}$$

The solutions of Eqs. (7)-(11) for $\lambda_G > 1$ $(G < 2\sqrt{kEI})$ are now given as follows

$$w_1(\xi) = A_1 e^{-\mu\xi} + A_2 e^{\mu\xi} \tag{23}$$

$$w_2(\xi) = A_3\xi^3 + A_4\xi^2 + A_5\xi + A_6 \tag{24}$$

$$\begin{aligned} w_3(\xi) = {} & [A_7\cos(n\xi) + A_8\sin(n\xi)]\cosh(t\xi) + [A_9\cos(n\xi) + A_{10}\sin(n\xi)]\sinh(t\xi) \\ & -\frac{F}{2t}\sinh|t\xi|\cos(n\xi) + \frac{F}{2n}\cosh(t\xi)\sin|n\xi| \end{aligned} \tag{25}$$

$$w_4(\xi) = A_{11}\xi^3 + A_{12}\xi^2 + A_{13}\xi + A_{14} \tag{26}$$

$$w_5(\xi) = A_{15} e^{-\mu\xi} + A_{16} e^{\mu\xi} \tag{27}$$

where $\mu = \sqrt{\lambda_G}$ and $A_i (i = 1-16)$ are the unknown constants. For $\lambda_G < 1$ $(G > 2\sqrt{kEI})$, solutions are similar to those given above except for $w_3(\xi)$. So, the general solutions of Eqs. (7)-(11) can be written as

$$w_1(\xi) = B_1 e^{-\mu\xi} + B_2 e^{\mu\xi} \tag{28}$$

$$w_2(\xi) = B_3\xi^3 + B_4\xi^2 + B_5\xi + B_6 \tag{29}$$

$$\begin{aligned} w_3(\xi) = {} & B_7\sinh(m_1\xi) + B_8\cosh(m_1\xi) + B_9\sinh(m_2\xi) + B_{10}\cosh(m_2\xi) \\ & + H_1\sinh|m_1\xi| + H_2\sinh|m_2\xi| \end{aligned} \tag{30}$$

$$w_4(\xi) = B_{11}\xi^3 + B_{12}\xi^2 + B_{13}\xi + B_{14} \tag{31}$$

$$w_5(\xi) = B_{15} e^{-\mu\xi} + B_{16} e^{\mu\xi} \tag{32}$$

where $\mu = \sqrt{\lambda_G}$, $m_1 = \sqrt{2/\lambda_G + \sqrt{(2/\lambda_G)^2 - 4}}$, $m_2 = \sqrt{2/\lambda_G - \sqrt{(2/\lambda_G)^2 - 4}}$, $H_1 = 2F/m_1(m_1^2 - m_2^2)$, $H_2 = -2F/m_2(m_1^2 - m_2^2)$; and $B_i (i = 1-16)$ are unknown constants. It is possible to get the exact solution for $\lambda_G = 1$ $(G = 2\sqrt{kEI})$, but in practice accurate results can be obtained by simply increasing λ_G by a very small amount and then using the solution for $\lambda_G > 1$. For this reason, the exact solution for $\lambda_G = 1$ $(G = 2\sqrt{kEI})$ is not presented here.

As mentioned in section two, there are a total of 18 unknowns to be determined (A_i or $B_i (i = 1-16)$ and ξ_1, ξ_2). Since the case of $\lambda_G > 1$ $(G < 2\sqrt{kEI})$ is satisfied by most physical problems (Scott 1981, Zhaohua and Cook 1983), the solution procedure will be discussed only for this case. Omitting the pinned-pinned beam case for brevity, the constants A_i and the lift-off points ξ_1, ξ_2 for the free-free beam are determined as follows. The boundary conditions in Eqs. (15) and

(16) give $A_1 = A_{16} = 0$ and $A_3 = A_4 = A_{11} = A_{12} = 0$, respectively. Thus, the number of the unknown constants reduces to 10. By using the boundary conditions (12), (13) and (14), 12 nonhomogeneous algebraic equations are obtained. In this case, the coefficients of the constants and the terms that appear on the right hand side of the equations are related to ξ_1 and ξ_2. Since ξ_1 and ξ_2 are not known in advance, the solution is sought by an iterative scheme. First, by choosing ξ_1 and ξ_2, 10 constants are obtained numerically from the solution of 10 equations at each step and substituted into the remaining 2 equations. Then, the lift-off points are determined as the roots of these transcendental equations by using the Newton-Raphson technique. During the solution procedure, the global equilibrium of the beam is checked by considering the vertical equilibrium of the forces as

$$F = \int_{-\xi_1}^{\xi_2} \left(w - \frac{1}{\lambda_G} w'' \right) d\xi \tag{33}$$

4. Numerical results and discussion

4.1 The free-free beam

Fig. 2 shows the variation of the total contact length ($\xi_1 + \xi_2$) with respect to the beam length l for symmetric ($l_1 = 0.5l$) and asymmetric ($l_1 = 0.7l$) cases. First, the symmetric case is considered. For short beam lengths (regardless of whether the foundation is of the Winkler or Pasternak type), the contact length scales linearly with the beam length with a slope of one. This simply means that the entire beam is in contact with the foundation. When the beam length is increased, the contact length quickly levels off depending on the values of the parameter λ_G, and separation develops

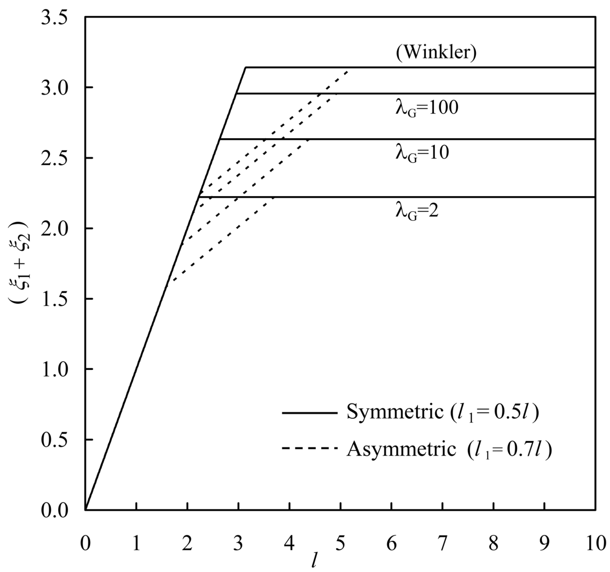

Fig. 2 Variation of total contact length ($\xi_1 + \xi_2$) with beam length l for $F = 0.1$ and some values of λ_G, for the symmetric and asymmetric loading in the free-free beam case

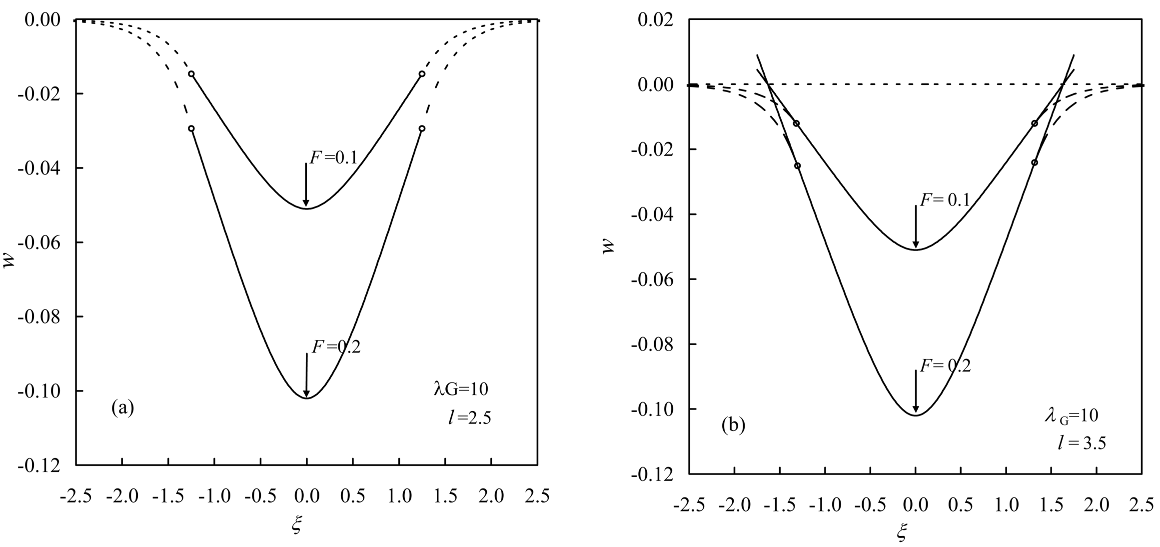

Fig. 3 Deflection curves for the free-free beam under two different symmetric loads with λ_G = 10. (a) l = 2.5, (b) l = 3.5

between the beam and the foundation. In this case, the contact lengths ($2 \times \xi_1$) are: 2.221 for $\lambda_G = 2$, 2.633 for λ_G = 10 and 2.956 for λ_G = 100. With further increases in the beam length, the contact lengths remain constant at these critical values. As seen in the figure, the contact length increases as λ_G increases. Also, if the second parameter G in the two-parameter Pasternak foundation model approaches zero ($\lambda_G \to \infty$), this model reduces to the Winkler model. For this case, separation develops after the beam length approaches π (the critical contact length for a Winkler foundation), and the contact length remains constant at this value as the beam length increases further. This conclusion is consistent with the infinite and finite beam cases presented by Weitsman (1970) and Zhang and Murphy (2004), respectively. The deflection curves of the beam and the shear layer are given in Figs. 3(a) and 3(b) for λ_G = 10. In Fig. 3(a), the entire beam is in contact with the foundation since the beam length is less than the contact length ($l = 2.5 < 2 \times \xi_1 = 2.633$). The beam is compressed into the foundation further when it is subjected to a larger load. The constant contact length behaviour is shown in Fig. 3(b) for a beam length of $l = 3.5$ ($3.5 > 2 \times \xi_1 = 2.633$) with loads F = 0.1 and F = 0.2. From Fig. 3(b), it is evident that the contact length is independent of load. In these figures, the dashed line represents the surface of the shear layer.

Fig. 2 illustrates the asymmetric case ($l_1 = 0.7l$). As in the symmetric case for short beams, the contact length increases linearly (with a slope of one) for all values of λ_G; i.e., the entire beam is in contact with the foundation. With the increase of the beam length, the contact length (shown by dashed lines in the figure), continues to increase. However, this increase occurs more slowly than in the symmetric case. For long beams, the contact lengths level off to the values given for the symmetric case. To explain the more gradual increase in the contact length, consider Figs. 4(a) and 4(b) which show, respectively, the variation of the left and right contact lengths in relation to the beam length for λ_G = 10 and F = 0.1. For small beam lengths, the left side contact length increases linearly at a rate of $\xi_1 = 0.7l$, which is a faster rate of increase than the symmetric case ($\xi_1 = 0.5l$). Thus, ξ_1 in the asymmetric case will reach the asymptotic value of 1.3165 more quickly than its symmetric counterpart: i.e., the left side lift-off occurs sooner than in the symmetric

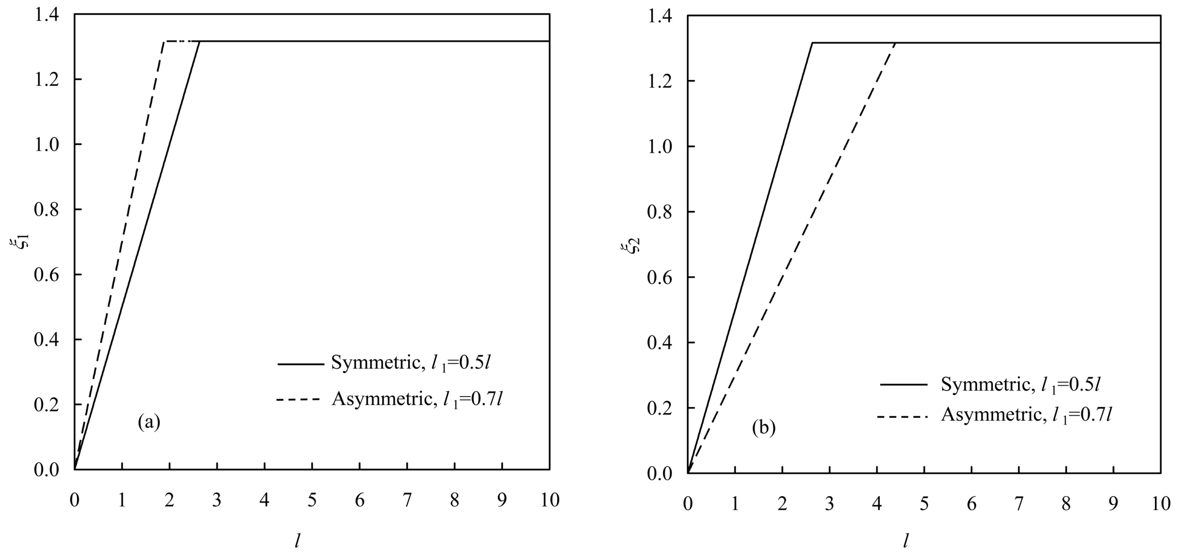

Fig. 4 Contact lengths versus beam length for F = 0.1 and λ_G = 10, for the symmetric and asymmetric case in the free-free beam. (a) Left side (ξ_1); (b) Right side (ξ_2)

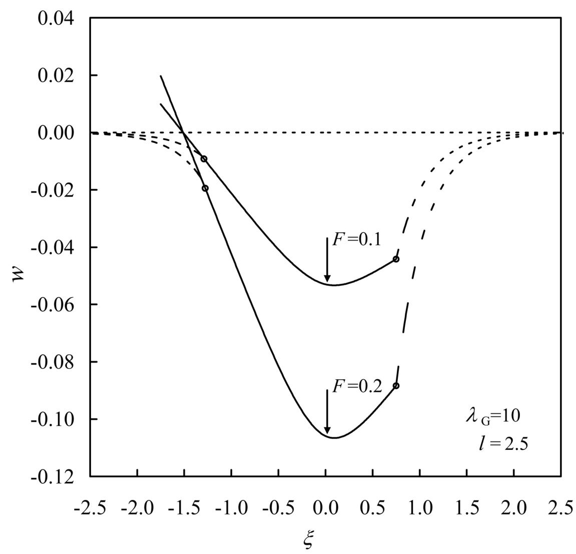

Fig. 5 Deflection curves for the free-free beam under two different asymmetric loads with l = 2.5 and λ_G = 10

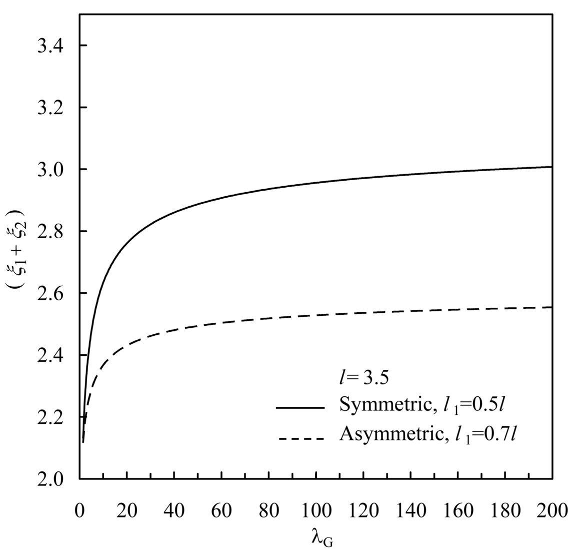

Fig. 6 Variation of total contact length (ξ_1 + ξ_2) with the shear foundation parameter λ_G for F = 0.1 and l = 3.5, for the symmetric and asymmetric case in the free-free beam

case. In contrast, for short beams, the right side contact length increases linearly as $\xi_2 = 0.3l$, which is slower than the symmetric case ($\xi_2 = 0.5l$). When the total contact length (ξ_1 + ξ_2) is considered, this behaviour (i.e., ξ_2 approaches 1.3165 more slowly than ξ_1), which accounts for why the net contact length for the asymmetric case is below the symmetric case (see Fig. 2). At a beam length of l = 2.5, Fig. 4(a) suggests that the left side contact length should be ξ_1 = 1.3165 and that left side lift-off should occur. Also, as seen in Fig. 4(b), the entire right side should be in contact with the foundation and the contact length should be $\xi_2 = 0.3 \times 2.5 = 0.75$. These behaviours agree with Fig. 5, which represents the deflection curves of the beam and the foundation surface for the asymmetric loading case. From Fig. 5, it is seen that the slope of the deflection of the foundation is not continuous at the right end of the beam. This is, of course, due to the second of the boundary conditions (18), which indicates the slope discontinuity of the foundation at the beam end.

The variation of the total contact length with respect to the shear foundation parameter λ_G is shown in Fig. 6 for $F = 0.1$. The beam length in the figure is taken to be $l = 3.5$ in order to clearly represent the behaviour in both the symmetric and asymmetric cases. From Fig. 6, it can be seen that the contact length increases as λ_G increases for both the symmetric and asymmetric cases. This is expected as the foundation becomes softer with the increase of λ_G (with the decrease of G, in dimensional terms).

4.2 The pinned-pinned beam

Fig. 7 shows the variation of the total contact length ($\xi_1 + \xi_2$) with respect to the beam length l for the symmetric case ($l_1 = 0.5l$) with $F = 0.1$. For short beam lengths, the contact lengths increase with the increase in beam length in both the Winkler and Pasternak foundations. In the case of the Winkler foundation, as in the symmetric free-free beam case, the contact curve has a slope of one. Again, this indicates that the entire beam is in contact with the foundation. However, in the Pasternak foundation case, the contact curves do not have a slope of one: the beam separates from the foundation. A comparison of Fig. 2 and Fig. 7 for short beam lengths shows that, at a fixed foundation rigidity and beam length, lift-off occurs only in the pinned-pinned beam case. This is because the system is stiffer than the free-free beam case. As seen in Fig. 7, the increase in contact length with beam length persists until $l = 6.187$ for the Winkler foundation. This beam length,

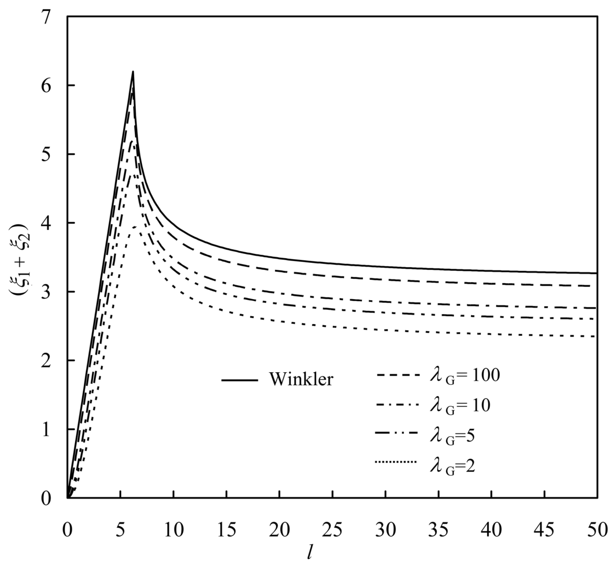

Fig. 7 Variation of total contact length ($\xi_1 + \xi_2$) with beam length l for $F = 0.1$ and some values of λ_G, for the symmetric loading case in the pinned-pinned beam

separating the regions of increasing versus shrinking contact length, is the critical length for the pinned-pinned beams on the tensionless Winkler foundation (Zhang and Murphy 2004). The corresponding critical lengths for the Pasternak foundation are: $l = 6.386$ for $\lambda_G = 2$, $l = 6.245$ for $\lambda_G = 100$ and $l = 6.196$ for $\lambda_G = 1000$. As is expected, the critical lengths decrease with the increase of the shear foundation parameter and approach the value obtained for the Winkler foundation case. After the critical values, the contact lengths begin to shrink with an increase in beam length. As the beam length is increased further, the contact lengths decrease and asymptotically approach the values obtained for the free-free case; e.g., 2.221 for $\lambda_G = 2$, and π for the Winkler foundation.

Fig. 8 shows the contact behaviour of the beam for $\lambda_G = 10$ both in the symmetric and asymmetric loading cases. In the asymmetric case, as in the symmetric case, the contact length initially increases with the beam length. However, as the beam length is increased, the contact curve shows two peaks: the contact length increases, shrinks, increases again, and then shrinks again as it asymptotically approaches the value 2.633, which is obtained for the free-free case. To explain this behaviour, consider Figs. 9(a) and 9(b), which show, respectively, the variation of the left and right contact lengths with respect to the beam length. The symmetric case is also shown for comparison. As seen in Fig. 9(a), the left side contact length initially increases as $\xi_1 = 0.7l$, peaks before the symmetric case, and then shrinks gradually to its asymptotic value of 1.3165. In contrast, the right side contact initially increases more slowly as $\xi_2 = 0.3l$ and peaks after the symmetric case, Fig. 9(b). The summation of the left and right contact lengths gives the total contact length for the asymmetric case, as shown in Fig. 8. The first peak in the figure is due to ξ_1 as it increases faster than ξ_2. After this peak, a drop takes place in the contact length because of the decrease in ξ_1; ξ_2 is still increasing but not fast enough to offset the decrease in ξ_1. The second peak, as expected, is due to ξ_2. Fig. 10 shows the deflection curves for a beam length of $l = 20$ with two different loads, $F = 0.1$ and 0.2. As it is seen, both sides have clearly lifted-off. Moreover, an increase in load results in an increase in displacement without changing the extent of the contact region.

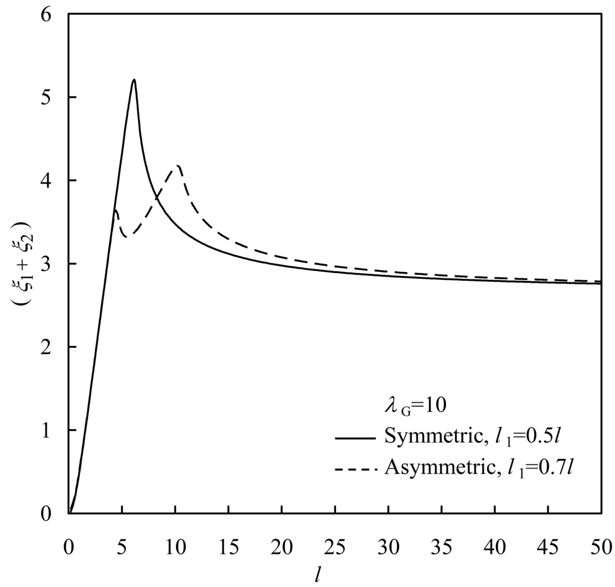

Fig. 8 Variation of total contact length ($\xi_1 + \xi_2$) with beam length l for $F = 0.1$ and $\lambda_G = 10$, for the symmetric and asymmetric case in the pinned-pinned beam

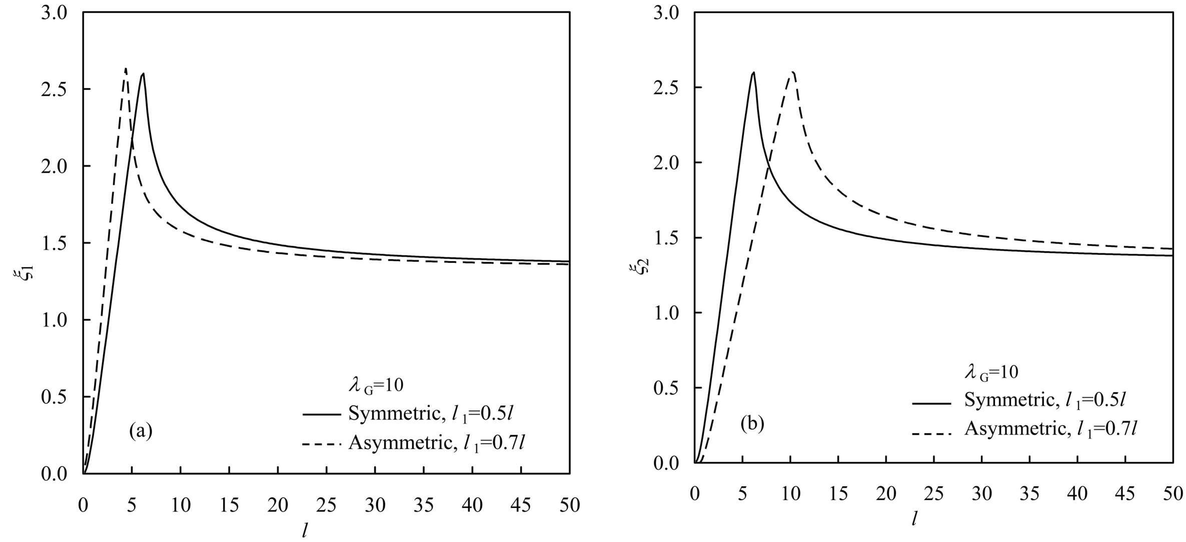

Fig. 9 Contact lengths versus beam length for F = 0.1 and λ_G = 10, for the symmetric and asymmetric case in the pinned-pinned beam. (a) Left side (ξ_1), (b) Right side (ξ_2)

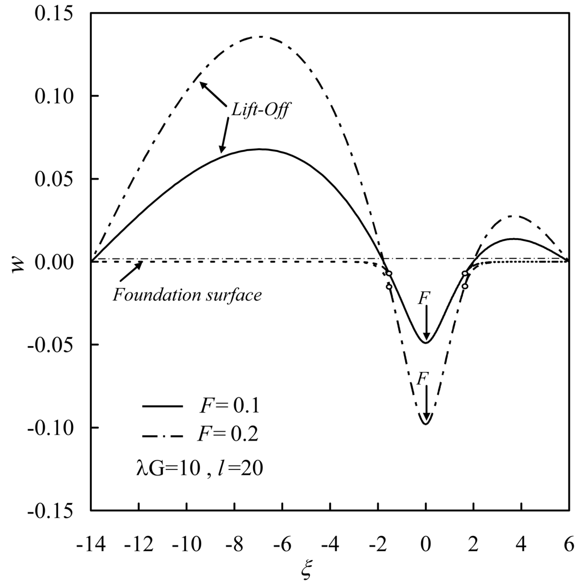

Fig. 10 Deflection curves for the pinned-pinned beam under two different loads with l = 20 and λ_G = 10

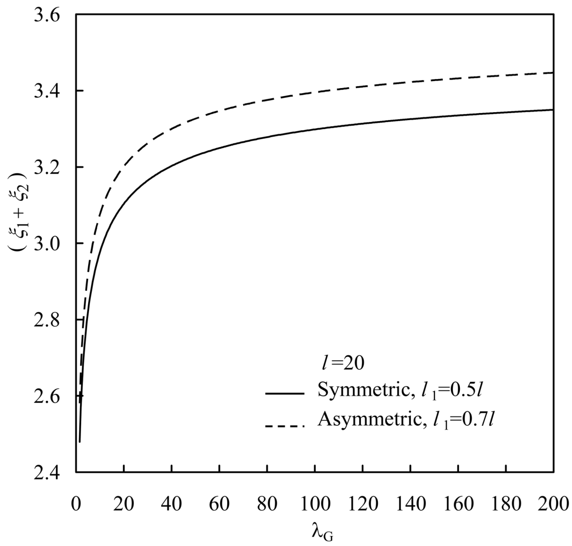

Fig. 11 Variation of total contact length with the shear foundation parameter λ_G for F = 0.1 and l = 20, for the symmetric and asymmetric case in the pinned-pinned beam

Finally, the variation of the total contact length with respect to the shear foundation parameter is shown in Fig. 11 for a beam length of l = 20 with F = 0.1. It is seen that the increase in the shear foundation parameter (λ_G) increases the contact lengths, which is the same as in the free-free beam case.

5. Conclusions

The static displacement responses of free and simply supported finite beams resting on a tensionless Pasternak foundation under symmetric and asymmetric loadings have been analyzed. The influences of the shear foundation parameter, beam length and symmetric and asymmetric loadings on the response have also been studied. The results can be summarized as follows:

1. The problem shows a nonlinear character due to the foundation lift-off in both the symmetric and asymmetric loading cases. This fact holds for the free-free as well as for the pinned-pinned beam cases. Other boundary conditions are expected to only impact the results in a quantitative manner, without changing the nonlinear behaviour.
2. The contact length is independent of the amplitude of the load, whereas the deflection profile is directly proportional to it. However, the contact length depends on the shear foundation parameter (λ_G). The increase in the value of this parameter considerably increases the contact length.
3. In the symmetric loading case, separation does not occur in the system for a Winkler foundation if the beam length is less than the critical lengths. However, lift-off takes place below these values in the Pasternak foundation case. The contact lengths are always smaller in this case as the foundation is stiffer. When the beam length increases beyond the critical values, constant contact length behaviour occurs in the free-free beam case. However, for the pinned-pinned case, contact lengths decrease beyond the critical values with the increase of beam length and asymptotically approach the values in the free-free beam case (i.e., the infinite beam contact lengths).
4. In the asymmetric loading case, the response for the short beam lengths is similar to the symmetric case. However, one sided lift-off behaviour develops for the free-free beam case as the beam gets longer. In this case, contact lengths become smaller than those for the symmetric case. In the pinned-pinned beam case, a double peak behaviour, which is caused by the sequential increase and decrease of contact lengths in both sides of the beam, occurs in the system as the beam length increases. It may be noted here that there is not a physical mechanism for a third peak to appear in the system. Also, it is found that the effect of the asymmetry diminishes with the increase of beam length and the system approaches the symmetric case for both the free-free and pinned-pinned beams.

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