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Direct integration method for stochastic finite element analysis of nonlinear dynamic response

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Abstract. Stochastic response of systems to random excitation can be estimated by direct integration methods in the time domain such as the stochastic central difference method (SCDM). In this paper, the SCDM is applied to compute the variance and covariance in response of linear and nonlinear structures subjected to random excitation. The accuracy of the SCDM is assessed using two-DOF systems with both deterministic and random material properties excited by white noise. For the former case, closed-form solutions can be obtained. Numerical results also are presented for a simply supported geometrically nonlinear beam. The stiffness of this beam is modeled as a random field, and the beam is idealized by the stochastic finite element method. A perturbation technique is applied to formulate the equations of motion of the system, and the dynamic structural response statistics are obtained in a time domain analysis. The effect of variations in structural parameters and the numerical stability of the SCDM also are examined.

Key words: computational mechanics; dynamics; probability; random vibration; statistics; structural engineering.

1. Introduction

Uncertainty in the response of a structure depends on randomness in material properties, geometric parameters and boundary conditions, and in the excitation. The significance of each source of uncertainty depends on the structural behavior and the response quantity sought.

Material uncertainties may impact the response of a structure to deterministic or random loads. The variation of eigenvalues and eigenvectors of linear vibrating beams due to uncertainties in stiffness and damping was studied by Fox and Kapoor (1968) and Shinozuka and Astill (1972). Vanmarcke (1977, 1983a, 1983b) developed a local averaging technique, which subsequently was combined with the finite element method to evaluate the second-order statistics of the deflection of a beam with random rigidity. The development of the stochastic finite element method was described in a review article by Vanmarcke, *et al.* (1986) and later by Ghanem and Spanos (1991). Liu, *et al.* (1985, 1986a, 1986b) implemented the stochastic finite element method using perturbation analysis, which can be used to solve linear as well mildly nonlinear problems. Chang and Yang (1991) used a modal expansion technique combined with the equivalent linearization method to obtain the dynamic response of nonlinear beams and frames.

The response of linear structures to random excitation forces can be analyzed by classical techniques (Crandall and Mark 1963, Lin 1976). However, the vibration of structures often is

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nonlinear in nature. Flexible structures, such as beams, plates and shells may vibrate with large amplitudes, leading to a nonlinear strain-displacement relation. Exact solutions, even for a single degree of freedom system with nonlinearities, have been obtained only for a few idealized cases; approximate approaches include the perturbation method, the equivalent linearization method and non-Gaussian closure. (e.g., Crandall and Zhu 1983, To 1984, 1987, Roberts 1984, Lin, *et al.* 1986, Roberts and Spanos 1990). While solutions generally can be obtained by Monte Carlo simulation, such solutions can be very costly. More efficient methods are required for the analysis of nonlinear systems.

The stochastic central difference method (SCDM) is a method for direct integration in the time domain to obtain the covariance of response of a nonlinear system subjected to random excitation (Zhang and Zhao 1992, To 1986). The SCDM can be used to estimate the random response of structures idealized by the finite element method and subjected to either stationary or nonstationary excitations without recourse to more sophisticated mathematical concepts, analog or digital simulation techniques. In this paper, it is used to investigate the dynamic response in the time domain of multidegree of freedom systems with random stiffness subjected to random excitation. The numerical stability of the solutions and the effects of variations in the structural parameters and the time interval of integration on the response statistics are investigated.

2. Perturbation formulation of the equations of motion

The equation of motion of a linear (or equivalent linear) system is,

$$M\ddot{X}+C\dot{X}+KX=F(t) \tag{1}$$

where M, C, K are mass, damping and stiffness matrices respectively, $F(t)$ is excitation, and $\ddot{X}$, $\dot{X}$, X are acceleration, velocity and displacement vectors of the system.

Let us assume that $b(u)$ is a random material property that varies with spatial coordinate u. The random function $b(u)$ can be approximated using deterministic shape functions $\psi_i(u)$ (Liu, *et al.* 1986b);

$$b(u)=\sum_{i=1}^{q}\psi_i(u)b_i \tag{2}$$

where b_i are the random nodal values of $b(u)$ at u_i, $i=1,\cdots,q$. The mean and variance of $b(u)$ defined in Eq. (2) are:

$$\bar{b}(u)=\sum_{i=1}^{q}\psi_i(u)\bar{b}_i \tag{3}$$

$$Var(b(u))=\sum_{i,j=1}^{q}\psi_i(u)\psi_j(u)Cov(b_i,\ b_j) \tag{4}$$

in which $\bar{b}_i$, and $Cov(b_ib_j)$=mean of b_i and covariance of b_i and b_j.

To derive the equations of motion using the perturbation method, the following notations are used for a given function $g(b)$ and a small perturbation $(b_i-\bar{b}_i)$ from the mean $\bar{b}_i$:

$$db_i=(b_i-\bar{b}_i) \tag{5}$$

$$db_idb_j=(b_i-\bar{b}_i)(b_j-\bar{b}_j) \tag{6}$$

$$\bar{g}(X)=g(X,\ \bar{b}(X)) \tag{7}$$

$$\bar{g}_{b_i} = \frac{\partial g(\bar{b}_i)}{\partial b_i} \tag{8}$$

$$\bar{g}_{b_i b_j} = \frac{\partial^2 g(\bar{b}_i, \bar{b}_j)}{\partial b_i \partial b_j} \tag{9}$$

$$\Delta \bar{g}_1 = \sum_{i=1}^{q} \bar{g}_{b_i} db_i \tag{10}$$

$$\Delta \bar{g}_2 = \frac{1}{2} \sum_{i,j=1}^{q} \bar{g}_{b_i b_j} db_i db_j \tag{11}$$

Using the first and second order terms of a Taylor series to expand the vectors and matrices of Eq. (1):

$$X \simeq \bar{X} + \Delta \bar{X}_1 + \Delta \bar{X}_2 \tag{12}$$

$$\dot{X} \simeq \dot{\bar{X}} + \Delta \dot{\bar{X}}_1 + \Delta \dot{\bar{X}}_2 \tag{13}$$

$$\ddot{X} \simeq \ddot{\bar{X}} + \Delta \ddot{\bar{X}}_1 + \Delta \ddot{\bar{X}}_2 \tag{14}$$

$$K \simeq \bar{K} + \Delta \bar{K}_1 + \Delta \bar{K}_2 \tag{15}$$

$$C \simeq \bar{C} + \Delta \bar{C}_1 + \Delta \bar{C}_2 \tag{16}$$

$$F \simeq \bar{F} + \Delta \bar{F}_1 + \Delta \bar{F}_2 \tag{17}$$

It is assumed that $F = F(X, t)$ is independent of material property; then $\Delta \bar{F}_1 = \Delta \bar{F}_2 = 0$. Note that $d\bar{b}_i = 0$ because $db_i = (b_i - \bar{b}_i)$.

Eq. (1) describing the motion of the system can be separated into the zero, first and second order perturbation equations, respectively:

$$M\ddot{\bar{X}} + \bar{C}\dot{\bar{X}} + \bar{K}\bar{X} = \bar{F} \tag{18}$$

$$M\Delta\ddot{\bar{X}}_1 + \bar{C}\Delta\dot{\bar{X}}_1 + \bar{K}\Delta\bar{X}_1 = -\Delta\bar{C}_1\dot{\bar{X}} - \Delta\bar{K}_1\bar{X} \tag{19}$$

$$M\Delta\ddot{\bar{X}}_2 + \bar{C}\Delta\dot{\bar{X}}_2 + \bar{K}\Delta\bar{X}_2 = -\Delta\bar{C}_1\Delta\dot{\bar{X}}_1 - \Delta\bar{K}_1\Delta\bar{X}_1 \tag{20}$$

Eq. (19) also can be written in the form:

$$M\frac{\partial \ddot{\bar{X}}}{\partial b_i} + \bar{C}\frac{\partial \dot{\bar{X}}}{\partial b_i} + \bar{K}\frac{\partial \bar{X}}{\partial b_i} = \hat{F} \tag{21}$$

where $i = 1, \cdots, q$, and

$$\hat{F} = -\frac{\partial \bar{C}}{\partial b_i}\dot{\bar{X}} - \frac{\partial \bar{K}}{\partial b_i}\bar{X} \tag{22}$$

Assuming that the responses are zero-mean, i.e., $\langle \bar{X} \rangle = \langle \dot{\bar{X}} \rangle = 0$, and ignoring the terms of 3rd order and higher in expansions of $\langle XX^T \rangle$ and $\langle \dot{X}\dot{X}^T \rangle$, the variances and covariances of response of the system subjected to random excitations become:

$$\langle XX^T \rangle = \langle \bar{X}\bar{X}^T \rangle + \sum_{i,j=1}^{q} \left[\left(\frac{\partial \bar{X}}{\partial b_i} \right) \left(\frac{\partial \bar{X}}{\partial b_j} \right)^T \right] Cov(b_i, b_j) \tag{23}$$

$$\langle \dot{X}\dot{X}^T \rangle = \langle \bar{\dot{X}}\bar{\dot{X}}^T \rangle + \sum_{i,j=1}^{q} \left[\left(\frac{\partial \bar{\dot{X}}}{\partial b_i} \right) \left(\frac{\partial \bar{\dot{X}}}{\partial b_j} \right)^T \right] Cov(b_i, b_j) \tag{24}$$

in which $\langle\rangle$=expectation in the time domain.

3. The SCDM method for nonlinear systems

The equations of motion of a nonlinear system are,

$$M\ddot{X} + g(X, \dot{X}) = F(t) \tag{25}$$

Using the equivalent linearization technique (Roberts 1984, Roberts and Spanos 1990), Eq. (25) can be written in discrete forms:

$$M\ddot{X}_s + C_s\dot{X}_s + K_sX_s = P_s \tag{26}$$

in which the subscript s denotes the s_{th} time step of the integration and C_s and K_s are equivalent linear damping and stiffness matrices. If it is assumed that the response is Gaussian, $C_s = E_s[(\partial/\partial\dot{X})g(X, \dot{X})]$ and $K_s = E_s[(\partial/\partial X)g(X, \dot{X})]$; P_s denotes the continuous force. Since Eq. (26) is linear, its solution can be cast in exactly the same form as Eqs. (18)-(22).

Integrating Eq. (26) in the time domain using the SCDM (To 1986, Zhang and Zhao 1992, Zhang, *et al.* 1994), the covariance of the displacement response of the system can be expressed as,

$$\begin{aligned} R_{s+1} &= \langle X_{s+1}X_{s+1}^T \rangle \\ &= N_2R_sN_2^T + N_3R_{s-1}N_3^T + N_2D_sN_3D_sN_3^T \\ &+ N_3D_sN_2^T + \Delta t^4 N_1B_sN_1^T \end{aligned} \tag{27}$$

where Δt=time interval used in the integration and

$$D_s = \langle X_sX_{s-1}^T \rangle \tag{28}$$

$$D_s^T = \langle X_{s-1}X_s^T \rangle \tag{29}$$

$$B_s = \langle P_sP_s^T \rangle \tag{30}$$

$$D_s = N_2R_{s-1} + N_3D_{s-1}^T \tag{31}$$

Each element of R_{s+1} defines the covariance (or variance) in response at two different points in the system as the $(s+1)_{th}$ time step. The covariance of velocity response is:

$$\begin{aligned} \dot{R}_{s+1} &= \langle \dot{X}_{s+1}\dot{X}_{s+1}^T \rangle \\ &= (1/4\Delta t^2)(N_2R_{s+1}N_2^T + 4N_4R_sN_4^T \\ &- 2N_2D_{s+1}N_4^T - 2N_4D_{s+1}^TN_2^T + \Delta t^4 N_1B_{s+1}N_1^T \end{aligned} \tag{32}$$

In Eqs. (27)-(32), constants $N_1 - N_4$ are defined as

$$N_1 = (M + C_s\Delta t/2)^{-1} \tag{33}$$

$$N_2 = N_1(2M - K_s\Delta t^2) \tag{34}$$

$$N_3 = N_1(C_s\Delta t/2 - M) \tag{35}$$

$$N_4 = N_1 M \tag{36}$$

Let us now assume that $F(t)$ is a continuous white noise $W(t)$. White noise is difficult to handle in the SCDM because it has an infinite mean-square and the continuous white noise must be replaced by an equivalent discrete random process for numerical integration purposes. A binary noise process is used to replace the white noise (Zhang and Zhao 1992). The spectral density function of this binary noise is:

$$\phi(\omega) = \frac{\sigma^2 \Delta t \sin^2(\omega \Delta t/2)}{2\pi(\omega \Delta t/2)^2} \tag{37}$$

When $\Delta t \to 0$, $\sigma^2 \to \infty$ and $\sigma^2 \Delta t = constant$. Considering $lim_{\Delta t \to 0} \phi(\omega) = \sigma^2 \Delta t/2\pi = S_0$, the binary noise process becomes a white noise and $\langle P_s^2 \rangle = \langle W(t)^2 \rangle = \sigma^2$. For a SDOF system, Eq. (30) becomes,

$$B_s = \langle P_s^2 \rangle = 2\pi S_0/\Delta t \tag{38}$$

For a MDOF system, the diagonal elements of the matrix B_s in Eq. (30) should be written:

$$B_{ii} = 2\pi S_{ii}/\Delta t \tag{39}$$

where S_{ii} is the spectral density function of random process i. Because the mass, damping and stiffness matrices are the same in Eqs. (18) and (19), the same SCDM formulae can be used to obtain the response covariances from both equations. Note from Eq. (22) that the covariance of the excitation is:

$$(\hat{F}\hat{F}^T) = \left(\frac{\partial \bar{C}_s}{\partial b_i} \right) \langle \dot{X}_s \dot{X}_s^T \rangle \left(\frac{\partial \bar{C}_s}{\partial b_i} \right)^T + \left(\frac{\partial \bar{K}_s}{\partial b_i} \right) \langle X_s X_s^T \rangle \left(\frac{\partial \bar{K}_s}{\partial b_i} \right)^T \tag{40}$$

The assumption that the excitation is stationary white noise is not necessary for the formulation of the SCDM, and more general excitation processes can be considered (Zhang and Zhao 1992).

4. Numerical examples

4.1. Two-DOF linear system

The equation of motion of the two degree of freedom system illustrated in Fig. 1 is given by Eq. (1), in which

$$M = \begin{bmatrix} m_1 & 0.0 \\ 0.0 & m_2 \end{bmatrix} \tag{41}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \tag{42}$$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \tag{43}$$

and $F(t) = (F_1, F_2)^T$. Proportional damping is assumed in all cases.

We consider three separate cases in the following to check the accuracy of the SCDM method.

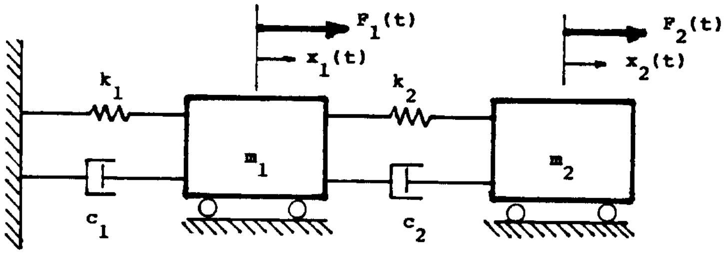


Fig. 1 Two-DOF linear system.

4.1.1. Deterministic material property

Assume that $m_1=m_2=m=1.0$ kg, and $k_1=k_2=k=100$ N/m; the natural frequencies are p_1 and p_2 (rad/s), and $p_1^2=0.382$ k/m, and $p_2^2=2.618$ k/m. The normal modal matrix is

$$X_N=\begin{bmatrix} 0.526 & -0.851 \\ 0.851 & 0.526 \end{bmatrix} \tag{44}$$

Expressing X in terms of normal modal coordinates Q, $X=X_N Q$, and assuming proportional damping, Eq. (1) can be written:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ddot{Q}+\begin{bmatrix} 2\zeta_1 p_1 & 0.0 \\ 0.0 & 2\zeta_2 p_2 \end{bmatrix}\dot{Q}+\begin{bmatrix} p_1^2 & 0.0 \\ 0.0 & p_2^2 \end{bmatrix}Q=X_N^T F(t) \tag{45}$$

where,

$$X_N^T F(t)=\begin{pmatrix} 0.526F_1+0.851F_2 \\ -0.851F_1+0.526F_2 \end{pmatrix} \tag{46}$$

If the excitations $\langle F_1, F_2\rangle^T$ are white noise with $\langle F_1^2\rangle=2\pi S_{10}\delta(0)$, and $\langle F_2^2\rangle=2\pi S_{20}\delta(0)$, the exact solution for mean-square response is (Roberts and Spanos 1990):

$$\langle Q_1^2\rangle=\pi S_1/2\zeta_1 p_1^3 m_1^2 \tag{47}$$

$$\langle Q_2^2\rangle=2\pi S_2/2\zeta_2 p_2^3 m_2^2 \tag{48}$$

where $S_1=0.276S_{10}+0.724S_{20}$, and $S_2=0.724S_{10}+0.276S_{20}$. The exact solution of the displacement response statistics of the system then can be obtained from $X=X_N Q$.

Using the SCDM solutions in Eqs. (27) and (32), and assuming that $S_{10}=1$ $N^2\cdot$sec and $S_{20}=0.5$ $N^2\cdot$sec, we obtain the numerical results given in Table 1 and Table 2. The exact solutions are given in parentheses. In these tables, $\zeta_1=\zeta_2=\zeta$ and $\Delta t=0.01$ sec. The SCDM results are in very close agreement with the exact solutions of the system.

4.1.2. Random material property

First, assume that $k_1=k_2=k$, and that k is the only random variable, i.e., $b=k$. The equations of motion of the system are separated into the zero- and first-order perturbation equations, Eqs. (18), (21) and (22) in which (cf Eq. (22))

Table 1 Displacement response (deterministic material)

ζ	$\langle x_1^2 \rangle$	$\langle x_2^2 \rangle$	$\langle x_1 x_2 \rangle$
0.01	0.2813(0.2816)	0.6324(0.6334)	0.3511(0.3519)
0.02	0.1407(0.1408)	0.3162(0.3167)	0.0175(0.0175)
0.05	0.0563(0.0563)	0.1264(0.1267)	0.0702(0.0703)
0.10	0.0283(0.0281)	0.0630(0.0633)	0.0352(0.0351)

Table 2 Velocity response (deterministic material)

ζ	$\langle v_1^2 \rangle$	$\langle v_2^2 \rangle$	$\langle v_1 v_2 \rangle$
0.01	21.2755(21.1120)	28.0580(28.1490)	7.0204(7.0386)
0.02	10.5015(10.5560)	14.0178(14.0745)	3.5119(3.5193)
0.05	4.1912(4.2224)	5.5884(5.6298)	1.4094(1.4077)
0.10	2.0962(2.1112)	2.7704(2.8149)	0.7126(0.7038)

$$\hat{F} = -\frac{\partial \bar{C}}{\partial k}\dot{\bar{X}} - \frac{\partial \bar{K}}{\partial k}\bar{X} \tag{49}$$

$$\frac{\partial \bar{K}}{\partial k} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \tag{50}$$

$$\frac{\partial \bar{C}}{\partial k} = 0.0895\,\zeta \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \tag{51}$$

Using the perturbation technique to analyze the system with random k,

$$\langle Q_i^2 \rangle = \langle \bar{Q}_i^2 \rangle + \frac{\partial \langle \bar{Q}_i^2 \rangle}{\partial k}\Delta k + \frac{1}{2}\frac{\partial^2 \langle \bar{Q}_i^2 \rangle}{\partial k^2}(\Delta k)^2 \tag{52}$$

and from Eqs. (47) and (48):

$$\langle Q_i^2 \rangle = \langle \bar{Q}_i^2 \rangle (1 - 0.015\,\Delta k + 0.0001875\,\Delta k^2) \tag{53}$$

When $\Delta k = -10$, i.e., the random variable is changed by 10 percent, $\langle Q_i^2 \rangle = 1.16 \langle \bar{Q}_i^2 \rangle$, meaning that the variance of displacement response is changed by 16.9 percent. Similarly, from the stationary velocity response $\langle \dot{Q}_i^2 \rangle = \langle Q_i^2 \rangle p_i^2$, we find that the variance of velocity response is changed by about 28 percent as the material property is changed by 10 percent.

Tables 3 and 4 present the numerical results obtained using the SCDM; perturbation solutions are presented in parentheses for comparison, since exact solutions are unavailable. Damping is assumed to be 0.10. The first row of these Tables 3 and 4 presents results for a deterministic system (cf Tables 1 and 2). The second row indicates the impact of randomness in k. The results obtained by the perturbation analysis and by the SCDM are in very close agreement. Figs. 2(a)-2(d) compare the results for different time steps Δt.

Next, assume that $k_1 = b_1$ and $k_2 = b_2$ are uncorrelated random variables. The equations of motion of the system are separated into equations (cf Eqs. (18) and (21)):

$$M\ddot{\bar{X}} + \bar{C}\dot{\bar{X}} + \bar{K}\bar{X} = \bar{F}(t) \tag{54}$$

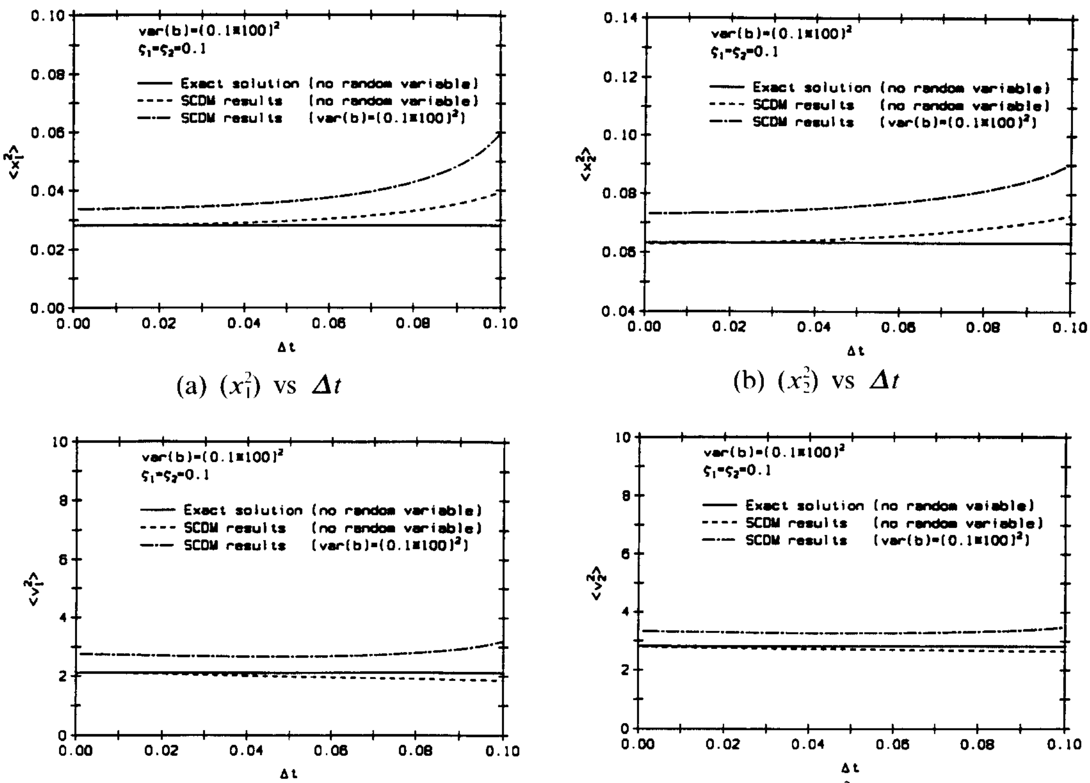
$$M\frac{\partial \ddot{\bar{X}}}{\partial k_1} + \bar{C}\frac{\partial \dot{\bar{X}}}{\partial k_1} + \bar{K}\frac{\partial \bar{X}}{\partial k_1} = \hat{F}_1 \tag{55}$$

Table 3 Displacement response (random material)

No. R.V.	$Var(b)$	$\langle x_1^2 \rangle$	$\langle x_2^2 \rangle$	$\langle x_1 x_2 \rangle$
0	0.	0.0283(0.0281)	0.0630(0.0633)	0.0352(0.0351)
1	$(0.1\times 100)^2$	0.0339(0.0323)	0.0732(0.0725)	0.0399(0.0405)
2	$(0.1\times 100)^2$	0.0339(0.0323)	0.0717(0.0725)	0.0384(0.0405)

Table 4 Velocity response (random material)

No. R.V.	$Var(b)$	$\langle \dot{x}_1^2 \rangle$	$\langle \dot{x}_2^2 \rangle$	$\langle \dot{x}_1 \dot{x}_2 \rangle$
0	0.	2.0963(2.1112)	2.7704(2.8149)	0.7126(0.7039)
1	$(0.1\times 100)^2$	2.7287(2.6993)	3.3204(3.5798)	0.6305(0.7364)
2	$(0.1\times 100)^2$	2.8788(2.6993)	3.3181(3.5798)	0.4822(0.7364)



(a) $\langle x_1^2 \rangle$ vs Δt (b) $\langle x_2^2 \rangle$ vs Δt

(c) $\langle v_1^2 \rangle$ vs Δt (d) $\langle v_1^2 \rangle$ vs Δt

Fig. 2 Variance in response of two-DOF system.

$$M\frac{\partial \ddot{\bar{X}}}{\partial k_2}+\bar{C}\frac{\partial \dot{\bar{X}}}{\partial k_2}+\bar{K}\frac{\partial \bar{X}}{\partial k_2}=\hat{F}_2 \tag{56}$$

where,

$$\hat{F}_1=-\frac{\partial \bar{C}}{\partial k_1}\dot{\bar{X}}-\frac{\partial \bar{K}}{\partial k_1}\bar{X} \tag{57}$$

$$\hat{F}_2 = -\frac{\partial \bar{C}}{\partial k_2}\dot{\bar{X}} - \frac{\partial \bar{K}}{\partial k_2}\bar{X} \tag{58}$$

$$\frac{\partial \bar{K}}{\partial k_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{59}$$

$$\frac{\partial \bar{C}}{\partial k_1} = 0.0895\,\zeta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{60}$$

$$\frac{\partial \bar{K}}{\partial k_2} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{61}$$

$$\frac{\partial \bar{C}}{\partial k_2} = 0.0895\,\zeta \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{62}$$

The SCDM is used to integrate Eqs. (54), (55) and (56) assuming that $\bar{k}_1 = \bar{k}_2 = 100$, and $var(k_1) = var(k_2) = (0.1 \times 100)^2$; the response statistics are given in the third row of Tables 3 and 4. The addition of the second (uncorrelated) random material property has little impact on displacement response statistics, but more of an effect on velocity, particularly on the covariance term. The agreement between the SCDM and perturbation solutions for the displacement terms (Table 3) generally is better than for the velocity terms (Table 4).

4.1.3. Nonstationary and uncorrelated loads

Consider the random loads as two modulated white noise processes:

$$F_1(t) = n_1(t)\,W_1(t) \tag{63}$$

$$F_2(t) = \frac{1}{2}\,n_1(t)\,W_2(t) \tag{64}$$

where $\langle W_1^2(t) \rangle = \langle W_2^2(t) \rangle = 2\pi S_0 \delta(0)$ and the envelope function is (Corotis and Marshall 1977):

$$n_1(t) = 2.32(\exp(-0.09t) - \exp(-1.49t)) \tag{65}$$

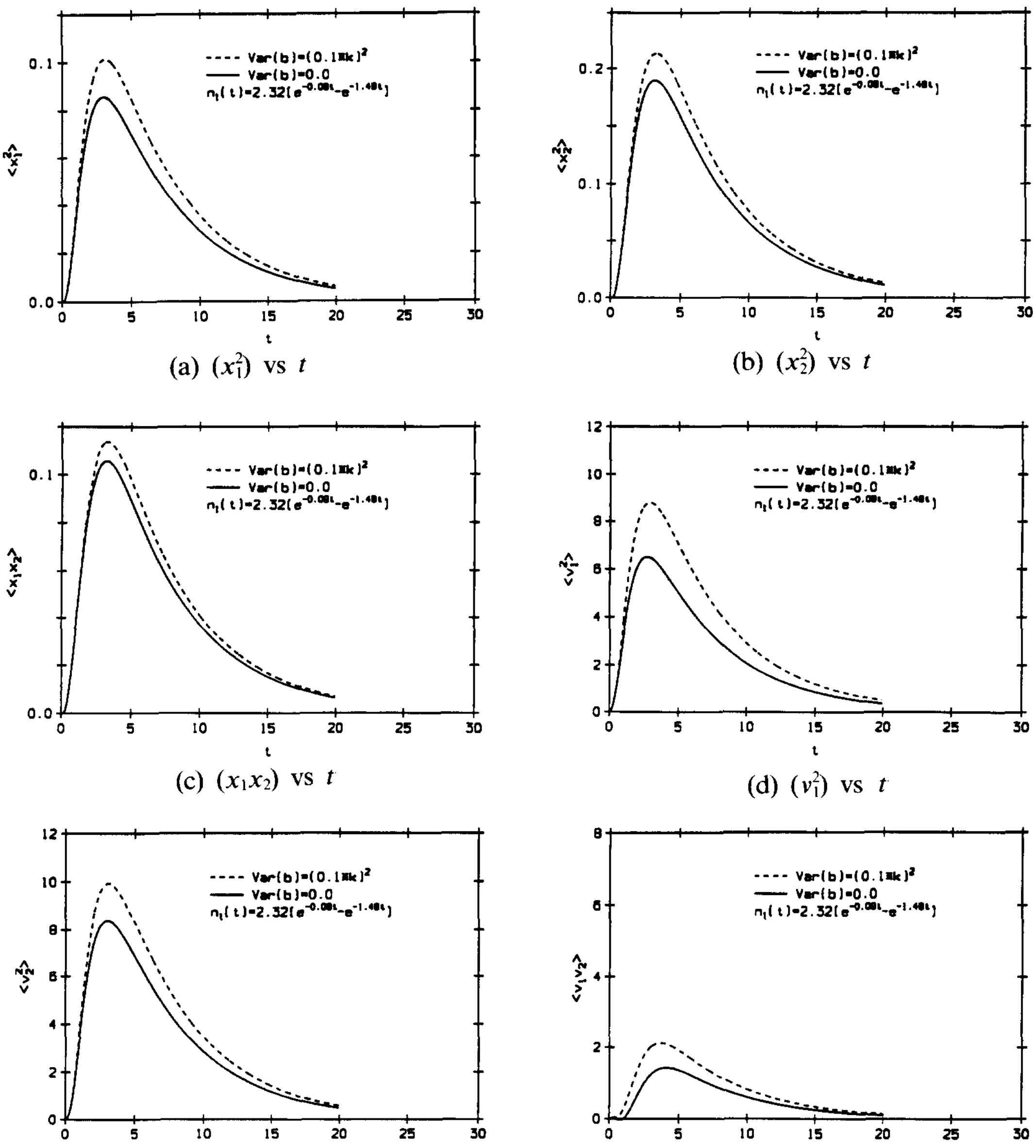
The mean-square displacements and velocities for two cases, one in which the material properties are deterministic and one in which they are described by two random variables, are given in Figs. 3(a)-3(f). The randomness in stiffness appears to have more of an effect on velocity than on displacement; this result is similar to the findings presented in Tables 3 and 4.

4.2. Simply supported beam

The strain energy, kinetic energy and virtual work due to transverse force $f(x, t)$ for a beam can be written in terms of axial and flexural deformations u and w, respectively, as

$$U = \frac{1}{2}\int_0^L \left[EA\,\varepsilon_x^2 + EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \right] dx \tag{66}$$

$$T = \frac{\rho A}{2}\int_0^L (\dot{u}^2 + \dot{w}^2)\,dx \tag{67}$$



(a) $\langle x_1^2 \rangle$ vs t

(b) $\langle x_2^2 \rangle$ vs t

(c) $\langle x_1 x_2 \rangle$ vs t

(d) $\langle v_1^2 \rangle$ vs t

(e) $\langle v_1^2 \rangle$ vs t

(f) $\langle v_1 v_2 \rangle$ vs t

Fig. 3 Nonstationary response of two-DOF system to uncorrelated loading.

$$\delta W = \int_0^L f(x, t)\,\delta w dx \tag{68}$$

where,

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{69}$$

and E, A, I, ρ, L, u and w are modulus of elasticity, cross-sectional area, moment of inertia,

Table 5 Variance and covariance of displacement response of linear beam

Case	No. Ele.	No. R.V.	$Var(b)$	$\langle \theta^2 \rangle$	$\langle v^2 \rangle$	$\langle \theta v \rangle$
1	2	0	0.0	0.00102	0.0386	0.00598
2	4	0	0.0	0.00099	0.0380	0.00581
3	12	0	0.0	0.00098	0.0379	0.00581
4	2	1	$(0.1\times E)^2$	0.00133	0.0417	0.00614
5	4	1	$(0.1\times E)^2$	0.00126	0.0415	0.00589
6	12	1	$(0.1\times E)^2$	0.00123	0.0413	0.00593
7	4	4	$(0.1\times E)^2$	0.00137	0.0395	0.00584
8	4	4	$(0.1\times E)^2$	0.00128	0.0367	0.00530

mass density, length and axial and transverse deflections, respectively.

Applying Lagrange's equation and adopting a two-node, six-degree-of-freedom beam element (three nodal degrees of freedom at each end), the equation of motion for the beam idealized by a system of elements is:

$$M\ddot{X}+C\dot{X}+(K_0+K_g)X=F(X,\ t) \tag{70}$$

where M is the mass matrix, K_0 and K_g are the linear and geometric nonlinear stiffness matrices, respectively, C is the damping matrix, and it is assumed that damping is linearly proportional, i.e., $C=aM+bK_0$. Vectors $\ddot{X}$, $\dot{X}$, X denote acceleration, velocity and displacement at the nodes, while $F(X,\ t)$ is the load vector. Assembling the element matrices and using the statistical linearization technique, we can obtain the equation of motion of the system. The displacement response of the system is assumed to be zero mean.

Response calculations for beams modeled with 2, 4 and 12 elements were performed for several different cases:

4.2.1. Deterministic material properties

The material and geometric properties of the simply supported beam were assumed as: modulus of elasticity $E=3.6\times10^7$ N/m^2, moment of inertia $I=0.05$ m^4, total mass of the beam $M=1000$ kg, and length $L=20.0$ m. A concentrated force is applied at midspan; this force is modeled as a white noise process with spectral density $S_0=100{,}000$ kg$^2-$sec. Damping was assumed to be 0.01. The value of time step Δt in applying the SCDM method was 0.002 sec.

The variances of deflection at midspan and rotation at the support, and their covariance, are given in rows 1-3 of Table 5, developed assuming linear elastic behavior ($K_g=0$ in Eq. (70)). The estimates of mean-square displacement are insensitive to the choice of finite element model. Mean-square velocities also were computed; however, these velocity estimates appeared to be unstable, indicating that additional refinement to the SCDM method may be required if accurate estimates of velocity statistics are required.

4.2.2. Random material properties

Rows 4-6 of Table 5 describe the mean-square midspan displacements and end rotations when the beam stiffness is described by one random variable. Four finite elements appear to be sufficient to model the beam in this example. In row 7, the moduli of elasticity of each of four finite elements used to model the beam were assumed to be statistically independent

Table 6 Variance and covariance of displacement response of nonlinear elastic beam

Case	No. Ele.	No. R.V.	$Var(b)$	$\langle \theta^2 \rangle$	$\langle v^2 \rangle$	$\langle \theta v \rangle$
1	2	0	0.0	0.00096	0.0380	0.00574
2	12	0	0.0	0.00096	0.0338	0.00541
3	2	1	$(0.1 \times E)^2$	0.00127	[illegible]	0.00588
4	12	1	$(0.1 \times E)^2$	0.00121	0.0367	0.00547

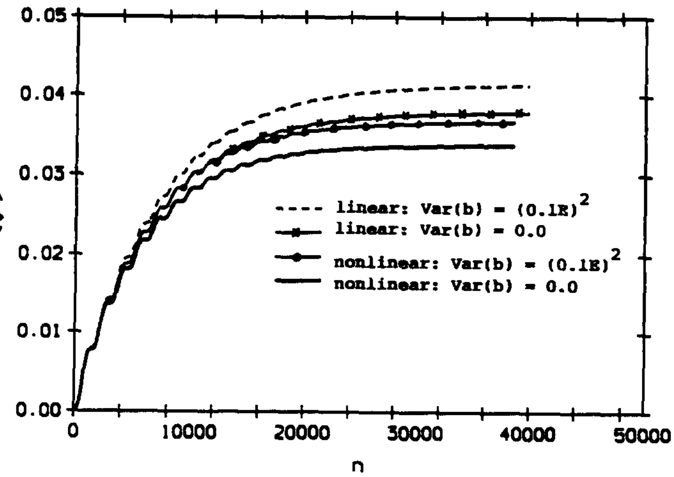


Fig. 4 Variance of center deflection of beam modeled by 12 elements.

Gaussian random variables. More generally, the modulus of elasticity is described by a random field, with covariance function,

$$Cov(b_i, b_j) = \sigma_i \sigma_j (\exp(-4|u_i - u_j|/L) \tag{71}$$

in which $|u_i - u_j|$ = separation distance between the midpoints of two beam finite elements. Row 8 presents the results for correlated stiffness described by Eq. (71), again using four elements to model the beam. Generally, if the number of correlated random variables is m, the number of equations that must be solved by the SCDM is $m(m+1)/2+1$. Table 5 shows that the effect of finite element modeling appears to have at least as significant an effect on the response statistics as correlation in the random field describing the stiffness. The effect of uncertainties in the material properties may be significant for dynamic response of the simply supported beam. The variance of rotation at the support was found to increase by 25 percent as the variance of modulus of elasticity is increased by 10 percent.

4.2.3. The effect of the nonlinearity of the system

Geometric nonlinearity in the beam is reflected in the term K_g in Eq. (70). Comparing results for nonlinear response presented in Table 6 to those in Table 5, the nonlinearity decreases the mean-square response of the system. Fig. 4 compares variances of midspan deflection for four different cases of linear or nonlinear behavior and deterministic or random stiffness, using a finite element model of the beam involving 12 elements and perfectly correlated stiffnesses.

5. Stability of the numerical method

The central difference method is a conditionally stable numerical method. Consider Eq. (1) for arbitrary initial conditions; when no load is specified; the central difference method solution is:

$$\hat{X}_{t+n\Delta t}=A^{n}\hat{X}_{t} \tag{72}$$

From the spectral decomposition of A, we have

$$A^{n}=PJ^{n}P^{-1} \tag{73}$$

where P is the matrix of eigenvectors of A and J is the Jordan form of A with eigenvalues λ_i of A on its diagonal. Let $\rho(A)$ be the spectral radius of A, defined as

$$\rho(A)=max|\lambda_i|;\ i=1,\ 2,\ \cdots \tag{74}$$

The stability criterion is $\rho(A)\leq 1$.

To satisfy $\rho(A)\leq 1$ for the central difference method, the critical time interval in the integration is $\Delta t_{cr}\leq T_n/\pi$ (Bathe and Wilson 1976), in which $T_n=2\pi/w_n$, and w_n is the highest frequency of the system. In the analysis of the beam,

$$w_i=\frac{i^2\pi^2}{L^2}\sqrt{\frac{EI}{\rho A}} \tag{75}$$

where $i=1,\ 2,\ \cdots\infty$. For the beam modeled with 2, 4 and 12 finite elements,

$$\Delta t_2\leq 0.1068\ \text{sec.};\ \Delta t_4\leq 0.0267\ \text{sec.};\ \Delta t_{12}\leq 0.00296\ \text{sec.} \tag{76}$$

in which the subscript denotes the number of the elements used to model the beam. If the number of elements is large, the time step Δt must be very small to obtain a stable result. This places a restriction on the use of the SCDM coupled with finite element analysis.

The SCDM is an explicit numerical method of direct integration in the time domain. The choice of parameter is simple compared with other implicit numerical methods, since only one parameter Δt need be considered to maintain numerical stability and accuracy in the computation. Stable and nearly exact results can be obtained over a wide range of Δt in the linear case (cf Fig. 2). In the nonlinear case, the accuracy of the SCDM is equivalent to that of the equivalent linearization method (Zhang and Zhao 1992). To save computation time, a larger value of Δt should be chosen close to the critical value Δt_{cr}.

6. Conclusions

The SCDM can be applied to compute the covariances of response of MDOF nonlinear systems in which the uncertainty in stiffness is modeled as a random field and the response is determined by the stochastic finite element method. The excitations are modeled as either stationary or nonstationary random process. The accuracy and efficiency of the SCDM are checked using two-DOF linear systems, and the numerical results are compared to exact or perturbation analysis solutions. A simply supported beam example is also given. The stability of the SCDM method is examined.

Comparisons of SCDM solutions to those obtained by perturbation analysis indicated excellent agreement of mean-square displacements in all cases considered. However, the agreement between the estimates of velocity statistics obtained from the SCDM and perturbation analyses were not as close, and discrepancies tended to increase as the number of random variables in the problem increased. This behavior suggests that a different stability criterion may be required in cases where accurate estimates of velocity are needed. An investigation of this issue is in progress.

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