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An explicit time-integration method for damped structural systems

S. Pezeshk† and C.V. Camp†

Department of Civil Engineering, The University of Memphis, Memphis, TN 38152, U.S.A.

Abstract. A damped trapezoidal rule method for numerical time-integration is presented, and its application in analyses of dynamic response of damped structures is discussed. It is shown that the damped trapezoidal rule method has features that make it an attractive approach for applications in dynamic analyses of structures. Accuracy and stability analyses are developed for the damped single-degree-of-freedom systems. Error analyses are also performed for the Newmark beta method and compared with the damped trapezoidal rule method as a basis for discussion of the relative merits of the proposed method. The procedure is fully explicit and easy to implement. However, since the method is an explicit method, it is conditionally stable. The methodology is applied to several example problems to illustrate its strengths, limitations and inherent simplicity.

Key words: numerical time-integration; structural dynamics; stability analysis; explicit method; implicit methods; error analysis; trapezoidal rule method.

1. Introduction

Over the past decade, variable time-integration methods have been quite extensively employed in transient analysis. Each of these methods have different levels of accuracy, stability and computational cost. Each of these numerical approaches employs difference relationships relating displacement, velocity and acceleration in step-by-step computation to obtain the dynamic response of a structure. In general, time-integration methods for structural dynamics may be generally classified as implicit methods and explicit methods. Whereas the commonly advocated explicit methods for computational structural dynamics require less computational effort per time step, they are conditionally stable. On the other hand, implicit methods require much greater computational effort per time step and are unconditionally stable. The critical choice, of course, is the selection of a particular time-integration method that combines accuracy and efficiency. The choice of a method is guided, to an extent, on the specific application. This paper focuses on an explicit time-integration method for computational structural dynamics based on a modified version of the trapezoidal rule method in conjunction with a second-order Taylor series approximation. The damped trapezoidal rule method (DTM) has several features that make it an attractive approach in dynamic analysis of damped structural systems.

The objectives of this paper are to introduce the damped trapezoidal rule method, provide information about the accuracy and stability of the method compared to the well-known Newmark methods (Newmark 1959) and illustrate its application in analyses of the dynamic response of structures. In what follows a formal introduction to the damped modified trapezoidal rule,

† Associate Professor

including error analyses, is presented. The characteristics, strengths and the limitations of the DTM procedure are investigated through several examples.

2. Damped trapezoidal rule method (DTM)

The free vibration of a simple mass-spring system may be described by the following linear homogeneous second order differential equation

$$\ddot{x}+2\xi\omega_d\dot{x}+\omega_d^2x=0 \tag{1}$$

where x is the displacement of the vibrating mass from the equilibrium position, ω_d is the damped circular frequency, ξ is the damping ratio and each dot represents differentiation with respect to time t.

A discrete solution of the governing equation of motion, Eq. (1), may be approximated by defining the displacement x_n, the velocity $\dot{x}_n$ and the acceleration $\ddot{x}_n$ at some time denoted by the subscript n. The time may be expressed as $t=n\Delta t$ where Δt is an arbitrary time step. In terms of the defined discrete time variables, Eq. (1) may be written as

$$\ddot{x}_n=-\omega_d^2x_n-2\xi\,\omega_d\dot{x}_n \tag{2}$$

The displacement of the system x_n is approximated by using a trapezoidal rule for integrating the velocity terms

$$x_{n+1}=x_n+\frac{1}{2}\dot{x}_n\Delta t+\frac{1}{2}\dot{x}_{n+1}\Delta t \tag{3}$$

in which the velocity $\dot{x}_n$ is represented by a second-order Taylor series

$$\dot{x}_{n+1}=\left(1-\frac{1}{2}\gamma^2\right)\dot{x}_n+(1-\xi\gamma)\ddot{x}_n\Delta t \tag{4}$$

where $\gamma=\omega_d\Delta t$.

The damped trapezoidal method (DTM) consists of solving Eqs. (2)-(4) at each time step n. The computation is initiated by defining the initial conditions of the system given in terms of the displacement x_n and the velocity $\dot{x}_n$, at $t=0$, or $n=0$. With the initial conditions prescribed, Eq. (2) may be solved for the initial acceleration $\ddot{x}_n$. The values of the velocity, the displacement and the acceleration at time $n+1$ are determined by solving Eqs. (3), (4) and (2) in sequence. The resulting DTM procedure is quite simple and fully explicit.

2.1. Stability of the damped trapezoidal rule method

The expression for displacement, given in Eq. (3), may be decremented in time to give an expression for x_n.

$$x_n=x_{n-1}+\frac{1}{2}\dot{x}_{n-1}\Delta t+\frac{1}{2}\dot{x}_n\Delta t \tag{5}$$

The resulting relationship, defined in Eq. (5), may be subtracted from Eq. (3) to form a second order difference equation

$$x_{n+1}-2x_n+x_{n-1}=\frac{\Delta t}{2}\left(\dot{x}_n-\dot{x}_{n-1}\right)+\frac{\Delta t}{2}\left(\dot{x}_{n+1}-\dot{x}_n\right) \tag{6}$$

The differences $(\dot{x}_n-\dot{x}_{n-1})$ and $(\dot{x}_{n+1}-\dot{x}_n)$ may be determined by rewriting Eq. (4), for example

$$\dot{x}_{n+1}-\dot{x}_n=-\frac{1}{2}\gamma^2\dot{x}_n+(1-\xi\gamma)\ddot{x}_n\Delta t \tag{7}$$

Substituting Eq. (7) into Eq. (6) gives

$$x_{n+1}-2x_n+x_{n-1}=\frac{\Delta t^2}{2}(1-\xi\gamma)(\ddot{x}_{n-1}+\ddot{x}_n)-\frac{\gamma^2\Delta t}{4}(\dot{x}_{n-1}+\dot{x}_n) \tag{8}$$

The sums $(\dot{x}_{n-1}+\dot{x}_n)$ and $(\ddot{x}_{n-1}+\ddot{x}_n)$ in Eq. (8) may be determined by rewriting Eqs. (3) and (2), respectively.

$$\dot{x}_{n-1}+\dot{x}_n=\frac{2}{\Delta t}(x_n-x_{n-1}) \tag{9}$$

$$\ddot{x}_{n-1}+\ddot{x}_n=-\omega_d^2(x_{n-1}+x_n)-2\xi\omega_d(\dot{x}_{n-1}+\dot{x}_n) \tag{10}$$

Substituting Eqs. (9) and (10) into Eq. (8) results in a linear homogeneous difference equation

$$2x_{n+1}-(\xi\gamma^3+(4\xi^2-2)\gamma^2-4\xi\gamma+4)x_n-(\xi\gamma^3-4\xi^2\gamma^2+4\xi\gamma-2)x_{n-1}=0 \tag{11}$$

which for convenience may rewritten as

$$2x_{n+2}-(\xi\gamma^3+(4\xi^2-2)\gamma^2-4\xi\gamma+4)x_{n+1}-(\xi\gamma^3-4\xi^2\gamma^2+4\xi\gamma-2)x_n=0 \tag{12}$$

$$2x_{n+2}-Ax_{n+1}-Bx_n=0 \tag{13}$$

where

$$A=\xi\gamma^3+(4\xi^2-2)\gamma^2-4\xi\gamma+4 \qquad B=\xi\gamma^3-4\xi^2\gamma^2+4\xi\gamma-2 \tag{14}$$

A characteristic equation is obtained by assuming a general solution of the form

$$x_n=\lambda^n \tag{15}$$

and substituting the expression into Eq. (13) results in

$$2\lambda^2-A\lambda-B=0 \tag{16}$$

The roots of the characteristic expression, Eq. (16), may be determined by the quadratic equation

$$\lambda_{1,2}=\frac{1}{4}A\pm\frac{1}{4}\sqrt{A^2+8B} \tag{17}$$

The stability criterion is dictated by having the spectral radius, given in Eq. (17), to be less than 1 (i.e., $\lambda_{1,2}<1$). In Fig. 1, a plot of $\lambda_{1,2}$ versus γ for two typical critical damping ratios of $\xi=0\%$ and 25% is presented. Spectral radii are within unity provided that $\Delta t/T<2$, 1.912, 1.844, 1.792 and 1.737 for critical damping ratios of $\xi=0.5\%$, 10%, 15% and 25%, respectively. In addition to insure that the roots do not bifurcate, they must remain complex-conjugate. Therefore, the quantity inside the square root should be less than zero. The roots do not bifurcate if $\Delta t/T<2$, 1.903, 1.81, 1.722 and 1.562 for critical damping ratios of $\xi=0.5\%$, 10%, 15% and 25%, respectively.

The roots of the characteristic equation, given in Eq. (17), in polar notation may be written as

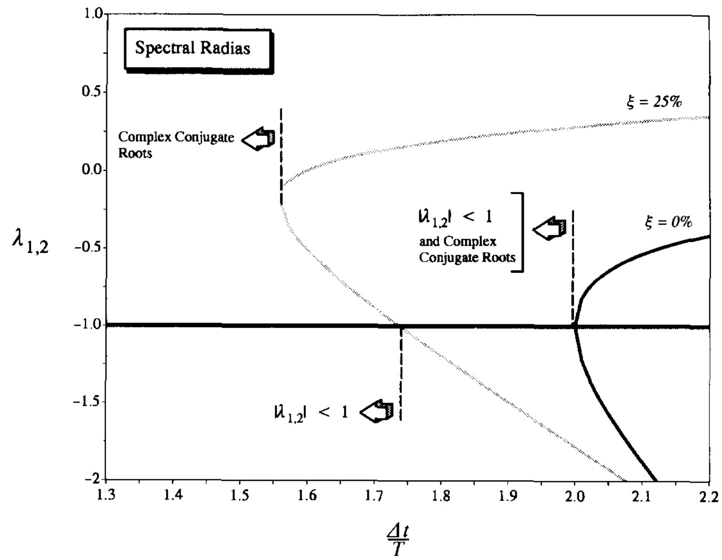

Fig. 1 Spectral radius for $\lambda_{1,2}$ as a function of $\Delta t/T$ in DTM method.

$$\lambda_1 = \Gamma e^{i\mu} \qquad \lambda_2 = \Gamma e^{-i\mu} \tag{18}$$

where $i = \sqrt{-1}$ and μ is defined as

$$\mu = \tan^{-1}\left(\frac{\sqrt{-A^2 - 8B}}{A} \right) \tag{19}$$

and

$$\Gamma^2 = -\frac{1}{2}B = \frac{1}{2}(-\xi\gamma^3 + 4\xi^2\gamma^2 - 4\xi\gamma + 2) \tag{20}$$

By substituting the roots of the characteristic equation, defined in Eq. (18), into Eq. (15) the general solution for the displacement becomes

$$x_n = A\lambda_1^n + B\lambda_2^n = A\Gamma^n e^{in\mu} + B\Gamma^n e^{-in\mu} \tag{21}$$

or, in a more convenient form, the real part of the displacement may be written as

$$x_n = C\Gamma^n \cos(n\mu) + D\Gamma^n \sin(n\mu) \tag{22}$$

where the constants A, B, C and D are determined from the initial conditions of the system.

2.2. Displacement and velocity using DTM

To investigate the accuracy of DTM consider the response of the simple mass-spring system. Expressions for the displacement and the velocity of the system using DTM may be derived from the general solution developed in Eq. (22). The time variable n in the expression for the displacement x_n, given in Eq. (22), may be replaced by $n = t_n/\Delta t$

$$x_n = C\Gamma^n \cos\left(\mu \frac{t_n}{\Delta t} \right) + D\Gamma^n \sin\left(\mu \frac{t_n}{\Delta t} \right) \tag{23}$$

The velocity $\dot{x}_n$ may be derived in terms of the displacement x_n by rewriting Eq. (4) for the velocity $\dot{x}_{n+1}$ and substituting the result into Eq. (9) written in terms of the velocity $\dot{x}_n$

$$\dot{x}_n=\frac{4x_{n+1}+(-2\xi\gamma^3+2\gamma^2-4)x_n}{[(4\xi^2-1)\gamma^2-4\xi\gamma+4]\Delta t} \tag{24}$$

Eq. (24) may be written in a more convenient form as

$$\dot{x}_n=ax_{n+1}+bx_n \tag{25}$$

where

$$a=\frac{4}{[(4\xi^2-1)\gamma^2-4\xi\gamma+4]\Delta t} \qquad b=\frac{-2\xi\gamma^3+2\gamma^2-4}{[(4\xi^2-1)\gamma^2-4\xi\gamma+4]\Delta t} \tag{26}$$

Substituting the solution for the displacement, given in Eq. (23), into the approximation of the velocity $\dot{x}_n$, Eq. (25) results in

$$\begin{aligned}\dot{x}_n=&\Gamma^n\{D(a\Gamma\cos\mu+b)-aC\Gamma\sin\mu\}\sin\left(\mu\frac{t_n}{\Delta t}\right)\\&+\Gamma^n\{C(a\Gamma\cos\mu+b)+a\Gamma D\sin\mu\}\cos\left(\mu\frac{t_n}{\Delta t}\right)\end{aligned} \tag{27}$$

The constants C and D in Eqs. (23) and (27) may be determined by applying the initial conditions for both the displacement $x_n(t=0)=X_0$ and the velocity $\dot{x}_n(t=0)=V_0$. The constants C and D are determined as

$$C=X_0 \qquad D=-\frac{X_0(a\Gamma\cos\mu+b)-V_0}{a\Gamma\sin\mu} \tag{28}$$

In terms of the initial displacement X_0 and the initial velocity V_0 the displacement x_n is

$$x_n=X_0\Gamma^n\cos\left(\mu\frac{t_n}{\Delta t}\right)-\frac{\Gamma^{n-1}[X_0(a\Gamma\cos\mu+b)-V_0]}{a\sin\mu}\sin\left(\mu\frac{t_n}{\Delta t}\right) \tag{29}$$

and the velocity $\dot{x}_n$ is

$$\dot{x}_n=\Gamma^n\left[V_0\cos\left(\mu\frac{t_n}{\Delta t}\right)-\frac{(a^2\Gamma+b^2/\Gamma)X_0+(2bX_0-V_0)a\cos\mu-(b/\Gamma)V_0}{a\sin\mu}\sin\left(\mu\frac{t_n}{\Delta t}\right)\right] \tag{30}$$

3. Review of Newmark Beta method (NBM)

Before discussing the errors associated with the application of DTM, it is of interest to review the features of the NBM and to define the relationship between these two approaches. The implementation of the NBM for the solution of the damped vibrational problem considered is as follows:

(1) With a set of known values of x_n and $\dot{x}_n$, apply Eq. (2) to find $\ddot{x}_n$, then
(2) Estimate $\ddot{\bar{x}}_{n+1}$, the acceleration at the end of the interval.
(3) Compute $\dot{x}_{n+1}$

$$\dot{x}_{n+1}=\dot{x}_n+\frac{1}{2}(\Delta t\ddot{x}_n+\Delta t\ddot{\bar{x}}_{n+1}) \tag{31}$$

(4) Compute the new displacement x_{n+1}

$$x_{n+1}=x_n+\Delta t\dot{x}_n+\frac{\Delta t^2}{2}(1-2\beta)\ddot{x}_n+\Delta t^2\beta\ddot{\bar{x}}_{n+1} \tag{32}$$

(5) Compute the new acceleration $\ddot{x}_{n+1}$

$$\ddot{x}_{n+1}=-\omega_d^2 x_{n+1}-2\xi\omega_d\dot{x}_{n+1} \tag{33}$$

(6) If $\ddot{x}_{n+1}\neq\ddot{\bar{x}}_{n+1}$, return to step (2) and repeat the iteration with an improved estimate of acceleration by using $\ddot{\bar{x}}_{n+1}=\ddot{x}_{n+1}$. If $\ddot{\bar{x}}_{n+1}=\ddot{x}_{n+1}$, the process has converged and the calculations are advanced to the next time interval.

The Newmark Beta method (NBM) has many well-known special cases. For example, NBM with $\beta=1/4$ is referred to as the "average acceleration" method. This approximation is one of the most widely used methods for structural dynamics applications. The NBM with $\beta=0$ is called the "central difference" method and NBM with $\beta=1/6$ is referred to as the "linear acceleration method".

We can avoid the iterations associated with the application of Newmark Beta method by employing an incremental approach such as that implemented in Clough and Penzien (1975), Subbaraj and Dokainish (1989), Dokainish and Subbaraj (1989) and Hughes (1987). However, this incremental technique is based on the use of the concept of the tangent stiffness and may introduce additional errors in the computation of the dynamic response of the nonlinear system. An additional advantage of explicit method, such as DTM, is that they can be easily implemented and used in dynamic analysis of different systems.

3.1. Stability analysis for the NBM

The stability conditions for the NBM is given as (Hughes 1987, 1983)

$$\omega\Delta t\leq\frac{1}{\left(\frac{1}{4}-\beta\right)^{1/2}} \tag{34}$$

or

$$\Delta t\leq\frac{T}{2\pi\left(\frac{1}{4}-\beta\right)^{1/2}} \tag{35}$$

A detailed description of the derivation of the stability criterion for the NBM can be found in Hughes (1993). Eq. (35) shows that the NBM is conditionally stable for $\beta\geq1/4$. It is stable for $\beta=0$, 1/12 and 1/6 (provided that $\Delta t/T<1\pi$, 0.389 and 0.551 respectively). Note that the average acceleration method (NBM with $\beta=1/4$) is unconditionally stable and is an implicit method. The stability analysis given in Eq. (35) for NBM has been based on the assumption that the iterations associated with the application of the method converge. The NBM is unconditionally convergent for $\beta=0$. However, the NBM would converge for $\beta>0$ if

$$\Delta t<\frac{1}{2\pi}\left(\frac{1}{\beta}\right)^{1/2} \tag{36}$$

Thus, for the average acceleration method ($\beta=1/4$), the NBM is unconditionally stable and conditionally convergent, whereas for the explicit method ($\beta=0$), the NBM is conditionally stable and unconditionally convergent.

By performing operations similar to those presented for DTM, Hughes (1983) developed difference equations for a damped-vibrational problem for the NBM methods as:

$$x_{n+1}-2A_1x_n+A_2x_{n-1}=0 \tag{37}$$

where

$$A_1=1-[\xi\Omega+\Omega^2/2]/D \tag{38}$$

$$A_2=1-2\xi\Omega/D \tag{39}$$

$$D=1+\xi\Omega+\beta\Omega^2 \tag{40}$$

$$\Omega=\omega\Delta t \tag{41}$$

Assuming that both the conditions for convergence and the stability are satisfied, the solution by the NBM for the damped-vibrational problem is:

$$x_n=e^{-\xi\omega t}(\bar{X}_0\cos\bar{\omega}_d t_n+\bar{c}\sin\bar{\omega}_d t_n) \tag{42}$$

where

$$\bar{c}=\frac{\frac{1}{2}(A_{11}-A_{22})X_0+A_{12}V_0}{\sqrt{(A_2-A_1^2)}} \tag{43}$$

where A_{11}, A_{12}, A_{21} and A_{22} are the elements of matrix $\boldsymbol{A}$ defined as:

$$\boldsymbol{A}=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}=\boldsymbol{A}_1^{-1}\boldsymbol{A}_2 \tag{44}$$

where the matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are

$$\boldsymbol{A}_1=\begin{bmatrix} 1+\Delta t^2\beta\omega^2 & 2\Delta t^2\beta\xi\omega \\ \Delta t\omega^2/2 & 1+\Delta t\xi\omega \end{bmatrix} \tag{45}$$

$$\boldsymbol{A}_2=\begin{bmatrix} 1-\Delta t^2(1-2\beta)\omega^2 & \Delta t(1-\Delta t(1-2\beta)\xi\omega) \\ -\Delta t\omega^2/2 & 1+\Delta t\xi\omega \end{bmatrix} \tag{46}$$

In addition, the roots of the characteristic Eq. (37) in polar notation can be written as

$$\lambda_1=B_0e^{iv} \qquad \lambda_2=B_0e^{-iv} \tag{47}$$

where v is defined as

$$v=\tan^{-1}\left[\frac{\sqrt{A_2-A_1^2}}{A_1}\right] \qquad B_0=A_2^2 \tag{48}$$

The expression for velocity in the Newmark Beta method can be determined in manner similar to the DTM

$$\dot{x}_n=\bar{a}x_{n+1}+\bar{b}x_n \tag{49}$$

where

$$\bar{a}=\frac{2(\beta\gamma^2+\xi\gamma+1)}{(8\beta-2)\xi^2\gamma^2+2} \qquad \bar{b}=\frac{(1-4\beta)\xi\gamma^3+(1-2\beta)\gamma^2-2\gamma\xi-2}{(8\beta-2)\xi^2\gamma^2+2} \tag{50}$$

4. Comparison of methods

Available direct methods can be subdivided into explicit and implicit methods each with distinct advantages and disadvantages. The proposed DTM and NBM with $\beta=0$ (well-known central difference) are explicit methods. While the NBM with $\beta=1/4$ is an implicit method. Strictly speaking, in the case of multiple-degree-of-freedom the mass matrix $\boldsymbol{M}$ and the damping matrix $\boldsymbol{C}$ need to be diagonal for the central difference method to be explicit. In DTM and NBM ($\beta=0$) the solution at time $t+\Delta t$ is obtained by considering the equilibrium conditions at time t. Such schemes do not require factorization of the stiffness matrix in the step-by-step solution. Hence the method requires no storage of matrices if a diagonal mass matrix is used. In addition, computational cost per time step is much less than implicit methods. However, explicit methods such as DTM are conditionally stable whereas many of the implicit methods are unconditionally stable. The DTM procedure is conditionally stable and requires a time step size to be inversely proportional to the period of the system.

Explicit methods allow the displacements at the current time step to be calculated in terms of known displacements, velocities and accelerations of the previous time step. However, in an implicit procedure the displacement at the current time step must be found in terms of current acceleration which requires an iterative procedure to determine the response.

In the following sections, error analyses are performed for DTM and NBM by considering the free-vibrational response of a linear and damped SDOF system. The accuracy and stability of DTM is investigated and compared with NBM. In later sections, it is shown that DTM can approximate the response of MDOF systems accurately and efficiently.

A comprehensive survey of direct time-integration procedures including both explicit and implicit methods are presented in a set of papers by Subbaraj and Dokainish (1989) and Dokainish and Subbaraj (1989). Interested readers are referred to these papers for more information.

5. Error analysis

In order to measure the effectiveness of the DTM solution procedure, expressions for the error in the displacement, the natural period and the phase angle as a function of the time step are developed. The exact solution for the free and undamped motion of a mass-spring system, defined in Eq. (1), is

$$x(t)=e^{-\xi\omega t}\left[\frac{V_0+X_0\xi\omega}{\omega_d}\sin\left(\frac{2\pi}{T_d}\,t\right)+X_0\cos\left(\frac{2\pi}{T_d}\,t\right)\right] \tag{51}$$

where T_d is the damped natural period of the system. The velocity $\dot{x}$ may be derived by differentiating Eq. (51) with respect to time

$$\dot{x}(t)=e^{-\xi\omega t}\left[(V_0+X_0\xi\omega)\cos\left(\frac{2\pi}{T_d}\,t\right)-\omega_d X_0\sin\left(\frac{2\pi}{T_d}\,t\right)\right]-\xi\omega x(t) \tag{52}$$

5.1. Error in the natural period

A measure of the error introduced by the DTM approximation is estimated by comparing the natural period of vibration for the displacement, Eq. (29), or for the velocity, Eq. (30), with exact solution. In this case, the error is calculated as the ratio of the approximate period to the exact period

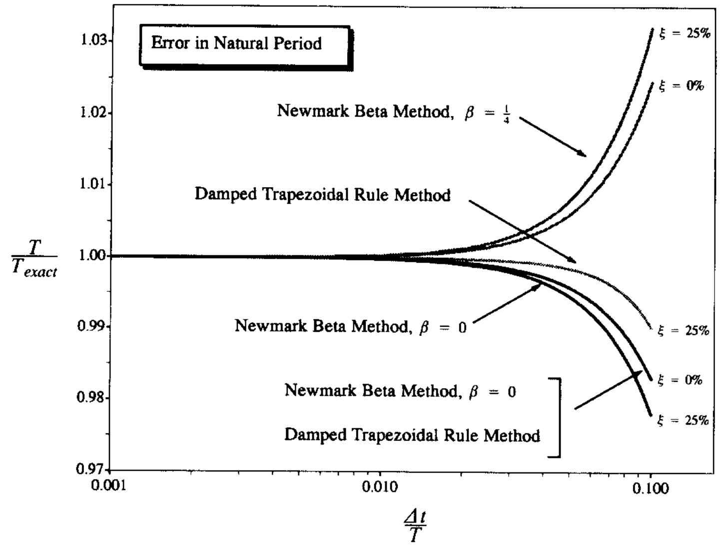

Fig. 2 Error in the natural period for DTM and NBM at various values of $\Delta t/T$.

$$\frac{T}{T_{exact}}=\frac{2\pi}{\mu}\frac{\Delta t}{T_d} \tag{53}$$

Similar equation can be written for Newmark Beta method as

$$\frac{T_{NBM}}{T_{exact}}=\frac{2\pi}{\nu}\frac{\Delta t}{T_d} \tag{54}$$

In Fig. 2, the ratio T/T_{exact} is calculated using the DTM procedure and compared to the NBM for various ratios of $\Delta t/T$. In all cases, the error in the natural period is negligible for $\Delta t/T<0.01$. In the range $0.01<\Delta t/T<0.1$, for the undamped case, DTM introduces less error than NBM with $\beta=1/4$ and is about the same amount of error as the NBM with $\beta=0$. For damped case with $\xi=25\%$, the error associated with DTM decreases and the error generated by NBM increases. The error associated with DTM is smaller than either NBM with $\beta=0$ or $\beta=1/4$. Although the implicit NBM with $\beta=1/4$ is unconditionally stable, the results shown in Fig. 2, indicate that a small time step is needed to avoid period elongation and to improve accuracy.

5.2. Error in the displacement ($V_0=0$)

An estimate for the error introduced in the displacement expression by the DTM approximation is calculated for two cases. In the first case, the displacement developed using DTM is compared to the exact solution for an initial velocity $V_0=0$. In the second case, an error estimate for the displacement is developed for $X_0=0$.

For the case where $V_0=0$, the DTM displacement given in Eq. (29) reduces to

$$x_n=X_0\Gamma^n\cos\left(\mu\frac{t_n}{\Delta t}\right)-\frac{\Gamma^n X_0\,(a\cos\mu+b/\Gamma)}{a\sin\mu}\sin\left(\mu\frac{t_n}{\Delta t}\right) \tag{55}$$

or

$$x_n=\frac{X_0\Gamma^n}{a\sin\mu}\sqrt{a^2+2ab\Gamma^{-1}\cos\mu+b^2\Gamma^{-2}}\cos\left(\mu\frac{t_n}{\Delta t}+\theta\right) \tag{56}$$

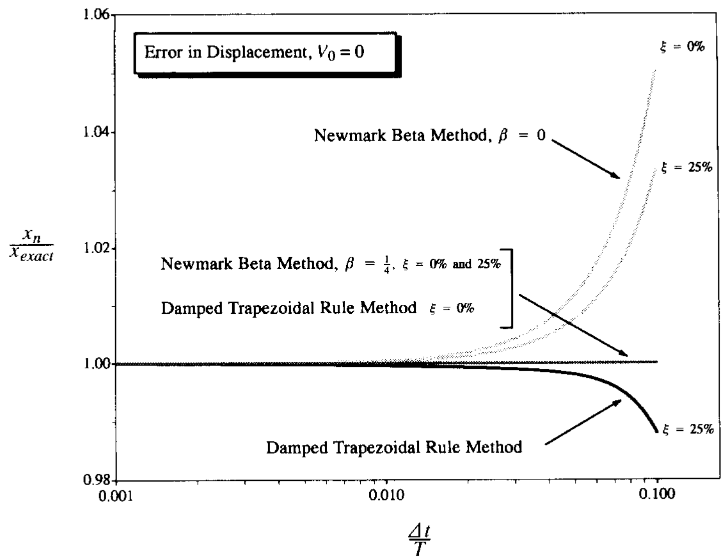

Fig. 3 Error in the displacement when the initial velocity $V_0=0$ for DTM and NBM at various of $\Delta t/T$.

where the phase angle θ is given as

$$\theta=\tan^{-1}\left\{\frac{a\Gamma\cos\mu+b}{a\Gamma\sin\mu}\right\} \tag{57}$$

The exact solution for displacement when $V_0=0$ is

$$x_{exact}=X_0 e^{-\xi\omega t}\cos\left(\frac{2\pi}{T_D}t+\theta_e\right)\sqrt{\frac{\xi^2\omega^2}{\omega_d^2}+1} \tag{58}$$

where the exact phase angle is

$$\theta_{exact}=\tan^{-1}\left\{-\frac{\xi\omega}{\omega_d}\right\} \tag{59}$$

The error introduced in the value of the displacement by DTM is measured by two criteria. First, the error in the displacement is calculated as the ratio of the amplitude obtained from DTM to the exact value

$$\frac{x_n}{x_{exact}}=\frac{\Gamma^n}{ae^{-\xi\omega t}\sin\mu}\sqrt{\frac{a^2+2ab\cos\mu/\Gamma+b^2/\Gamma^2}{\xi^2\omega^2/\omega_d^2+1}} \tag{60}$$

The displacement expression for NBM can easily be determined from Eqs. (42) and (43) by substituting zero for V_0.

In Fig. 3, the ratio defined in Eq. (60) is computed using the DTM procedure for various ratios of $\Delta t/T$. For comparison purposes, in all the error analyses considered, the factor n is set equal to 1. In all cases, the error in the amplitude is negligible for $\Delta t/T<0.01$. In the range $0.01<\Delta t/T<0.1$ the error associated with DTM is much smaller than NBM with $\beta=0$ for both damping ratios of $\xi=0\%$ and 25%. In fact, there is no error generated for DTM for the undamped case ($\xi=0\%$). The NBM with $\beta=1/4$ gives the exact displacement amplitude when $V_0=0$.

A second measure of the error in the displacement introduced by DTM is the phase angle,

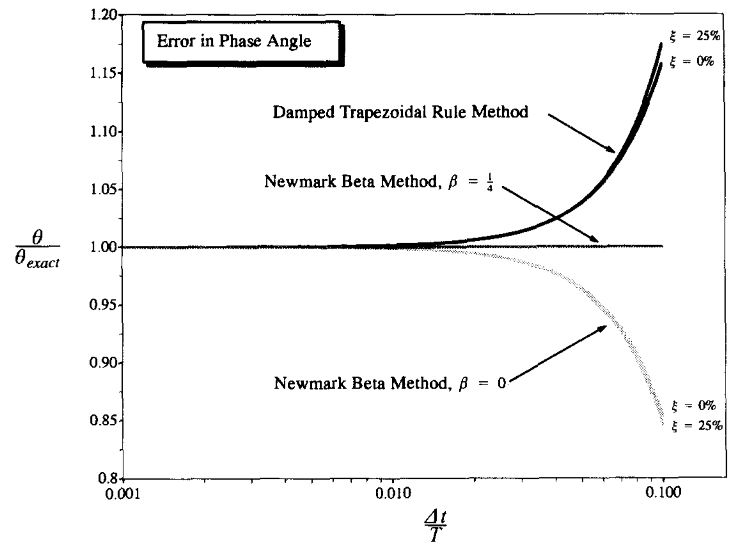

Fig. 4 Phase angle associated with the velocity ($V_0=0$) for DTM and NBM at various values of $\Delta t/T$.

represented by the ratio of the approximate phase angle, Eq. (57) to the exact phase angle, Eq. (59). In Fig. 4, the value of the phase angle is computed using both DTM for various ratios of $\Delta t/T$ and for critical damping of 0% and 25%. Over the range $0.01<\Delta t/T<0.1$ the error in the phase angle using DTM is negligible. Both DTM and NBM with $\beta=0$ generate approximately the same error.

In evaluating the error in displacement for $V_0=0$, the trapezoidal ($\beta=1/4$) form of the Newmark method maintains a displacement amplitude ratio of one; however, different choices of the Newmark parameters can give different results.

5.3. Error in the displacement ($X_0=0$)

For the case where $X_0=0$, the DTM displacement given in Eq. (29) reduces to

$$x_n=\frac{1}{a\sin\mu}\ V_0\Gamma^{n-1}\sin\left(\mu\frac{t_n}{\Delta t}\right) \tag{61}$$

The exact solution for displacement when $X_0=0$ is

$$x_{exact}=\frac{V_0}{\omega_d}\ \sin\left(\frac{2\pi}{T_d}t\right)e^{-\xi\omega t} \tag{62}$$

The ratio of the amplitude obtained from DTM to the exact value is

$$\frac{x_n}{x_{exact}}=\frac{\omega_d\Gamma^{n-1}}{e^{-\xi\omega t}(a\sin\mu)} \tag{63}$$

The displacement expression for NBM can be determined from Eqs. (42) and (43) by substituting zero for X_0.

In Fig. 5, the ratio defined in Eq. (63) is compared for the DTM and the NBM procedures for various ratios of $\Delta t/T$. The error in the amplitude is negligible for $\Delta t/T<0.01$. In the range $0.01<\Delta t/T<0.1$ the DTM is more accurate than NBM with $\beta=0$ for $\xi=0\%$; however, as the

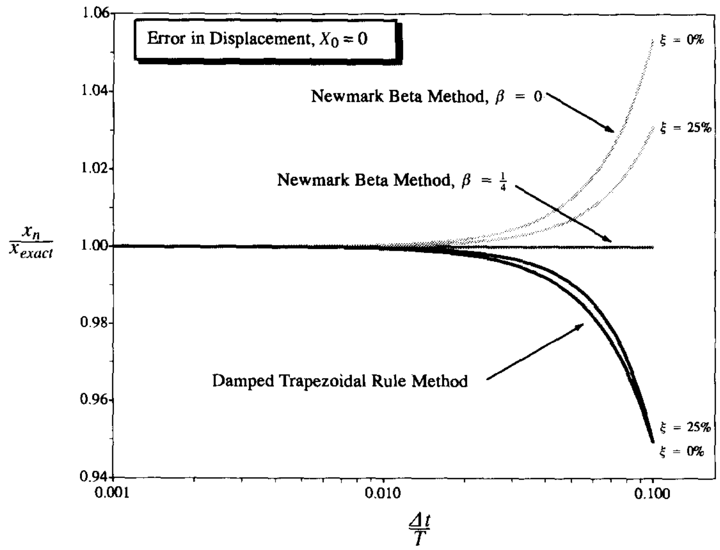

Fig. 5 Error in the displacement when the initial displacement $X_0=0$ for DTM and NBM at various values of $\Delta t/T$.

critical damping increases to $\xi=25\%$, NBM with $\beta=0$ generates slightly less error than DTM. In addition, the NBM with $\beta=1/4$ gives the exact displacement amplitude for both critical damping ratios of $\xi=0\%$, and $\xi=25\%$.

5.4. Error in the velocity ($V_0=0$)

An estimate for the error introduced in the velocity expression by the DTM approximation is calculated for two cases. In the first case, the velocity developed using DTM is compared to the exact solution for an initial velocity $V_0=0$. In the second case, an error estimate for the displacement when $X_0=0$ is calculated.

For the case where $V_0=0$, the DTM velocity given in Eq. (30) reduces to

$$\dot{x}_n = -\frac{X_0 \Gamma^n}{a \sin\mu}\,[a^2\Gamma + 2ab\cos\mu + b^2/\Gamma]\sin\left(\mu \frac{t_n}{\Delta t}\right) \tag{64}$$

The exact value of the velocity is

$$\dot{x}_{exact} = -X_0 \frac{\omega^2}{\omega_d}\, e^{-\xi\omega t}\sin\left(\frac{2\pi}{T}\,t\right) \tag{65}$$

The ratio of the velocity calculated from DTM to the exact value is

$$\frac{\dot{x}_n}{\dot{x}_{exact}} = \frac{\Gamma^n \omega_d [a^2\Gamma + 2ab\cos(\mu) + b^2/\Gamma]}{a\omega^2 \sin(\mu) e^{-\xi\omega t}} \tag{66}$$

Similarly, the ratio of the velocity for NBM to the exact value for the case with $V_o=0$ is

$$\frac{\dot{x}_n^{NBM}}{\dot{x}_{exact}} = \frac{\Gamma^n \omega_d [\bar{a}^2\Gamma + 2\bar{a}\bar{b}\cos(\nu) + \bar{b}^2/\Gamma]}{\bar{a}\omega^2 \sin(\nu) e^{-\xi\omega t}} \tag{67}$$

In Fig. 6, the ratio defined in Eqs. (66) and (67) are computed for various ratios of $\Delta t/T$.

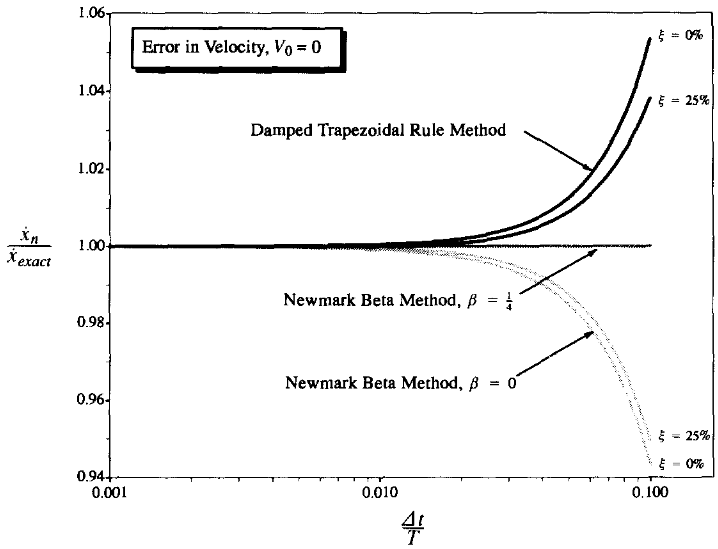

Fig. 6 Error in the velocity when the initial velocity $V_0=0$ for DTM and NBM at various values of $\Delta t/T$.

The results indicate that the DTM procedure is more accurate than NBM with $\beta=0$ for both critical damping ratio of $\xi=0\%$ and $\xi=25\%$. However, the DTM approximation is not as accurate as NBM with $\beta=1/4$. The second measure error in amplitude of velocity for the case $V_0=0$ is the phase angle. None of the procedures generate any error in the phase angle.

5.5. *Error in the velocity* ($X_0=0$)

For the case where $X_0=0$, the DTM velocity given in Eq. (30) reduces to

$$\dot{x}_n=\Gamma^n\left[V_0\cos\left(\mu\frac{t_n}{\Delta t}\right)+\frac{V_0(a\cos\mu+b/\Gamma)}{a\sin\mu}\sin\left(\mu\frac{t_n}{\Delta t}\right)\right] \tag{68}$$

or

$$\dot{x}_n=\frac{V_0\Gamma^n}{a\sin\mu}\sqrt{a^2+2ab\Gamma^{-1}\cos\mu+b^2\Gamma^{-2}}\cos\left(\mu\frac{t_n}{\Delta t}+\theta\right) \tag{69}$$

where the phase angle θ is given as

$$\theta=\tan^{-1}\left\{-\frac{a\Gamma\cos\mu+b}{a\Gamma\sin\mu}\right\} \tag{70}$$

Similarly, the velocity expression for NBM to the exact value is for the case with $X_o=0$ is

$$\dot{x}_n=\frac{V_0\Gamma^n}{\bar{a}\sin v}\sqrt{\bar{a}^2+2\bar{a}\bar{b}\Gamma^{-1}\cos v+\bar{b}^2\Gamma^{-2}}\cos\left(v\frac{t_n}{\Delta t}+\bar{\theta}\right) \tag{71}$$

where the phase angle $\bar{\theta}$ is given as

$$\bar{\theta}=\tan^{-1}\left\{-\frac{\bar{a}\Gamma\cos v+\bar{b}}{\bar{a}\Gamma\sin v}\right\} \tag{72}$$

The exact phase angle is

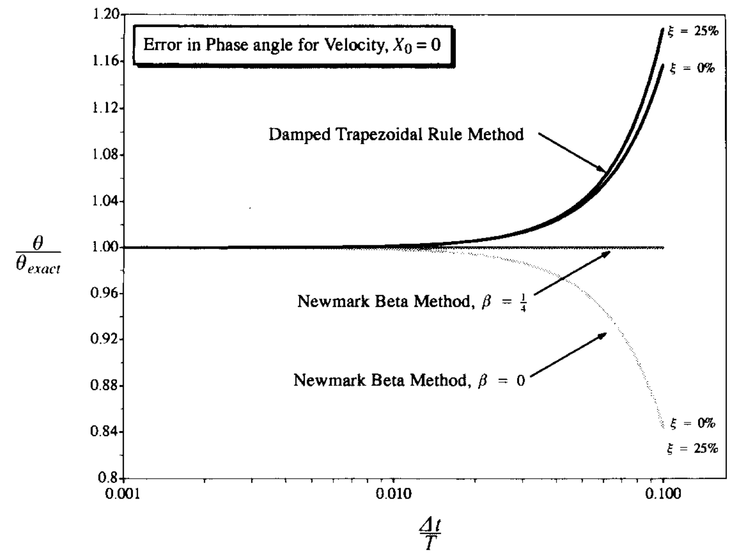

Fig. 7 Error in the phase angle for velocity when the initial displacement $X_0=0$ for DTM and NBM at various values of $\Delta t/T$.

$$\theta_{exact} = \tan^{-1}\left\{-\frac{\xi\omega}{\omega_d}\right\} \tag{73}$$

The exact value for the velocity when $X_0=0$ is

$$\dot{x}_{exact}(t) = e^{-\xi\omega t}\left[V_0\cos\left(\frac{2\pi}{T_d}t\right) - \frac{V_0}{\omega_d}\sin\left(\frac{2\pi}{T_d}t\right)\right] \tag{74}$$

The ratio of the amplitude of velocity calculated from the DTM to the exact value is identical to that given in Eq. (60) which is plotted in Fig. 3.

The ratio of the approximate phase angles, Eq. (72), to the exact phase angle, Eq. (73), is shown in Fig. 7. Both DTM and NBM with $\beta=0$ generate about the same error for critical damping ratio of $\xi=0\%$. The NBM with $\beta=0$ performs slightly better as the critical damping increases to $\xi=25\%$. In addition, the NBM with $\beta=1/4$ predicts the phase angle perfectly.

6. Application of DTM to dynamics analysis

To demonstrate the effectiveness, characteristics and merits of DTM in the analysis of structural dynamics, several application problems are presented. The first example involves a SDOF system and the second example applies DTM to a MDOF system.

6.1. Dynamic response to harmonic loading

The dynamic response of a simple damped mass-spring system subjected to a forced harmonic load is defined as

$$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = \omega^2 (X_{st})_0 \sin\bar{\Omega}t \tag{75}$$

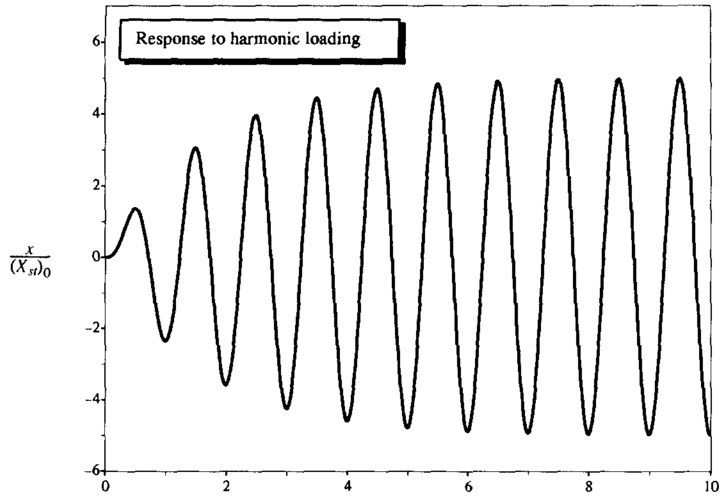

Fig. 8 DTM solution for the response of a free and undamped simple mass-spring system subjected to a harmonic loading with $\Delta t/T=0.01$.

where x is the displacement, ξ is the critical damping ratio, ω is the natural frequency, $(X_{st})_0$ is the displacement associated with the maximum static load and $\bar{\Omega}$ is the frequency of the applied harmonic loading.

The response defined in Eq. (75) is approximated using DTM by the following discrete equations

$$\ddot{x}=-2\xi\omega\dot{x}-\omega^2x+\omega^2(X_{st})_0\sin\bar{\Omega}t \tag{76}$$

$$\dot{x}_{n+1}=\left(1-\frac{1}{2}\gamma^2\right)\dot{x}_n+(1-\xi\gamma)\ddot{x}_n\Delta t \tag{77}$$

$$x_{n+1}=x_n+\frac{1}{2}\dot{x}_n\Delta t+\frac{1}{2}\dot{x}_{n+1}\Delta t \tag{78}$$

Consider a system initially at rest, where the natural frequency $\omega=2\pi$, the frequency of the harmonic loading $\bar{\Omega}=2\pi$ and the damping ratio $\xi=0.1$. The response of the system is approximated using DTM using $\Delta t/T=0.01$. In Fig. 8, the ratio of the displacement x to $(X_{st})_0$ is plotted as a function of time. The results obtained from DTM are in excellent agreement with the exact solution (Humar 1990).

6.2. Application of DTM for direct integration of MDOF systems

The equation of motion for a linear, discrete MDOF system excited by dynamic loads $\boldsymbol{P}(t)$ can be expressed as

$$\boldsymbol{M}\ddot{\boldsymbol{x}}+\boldsymbol{C}\dot{\boldsymbol{x}}+\boldsymbol{K}\boldsymbol{x}=\boldsymbol{P}(t) \tag{79}$$

where $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are the mass, damping and stiffness matrices of the system, respectively and $\boldsymbol{x}$ is the displacement vector.

An analysis procedure for a MDOF system using DTM is accomplished by direct integration,

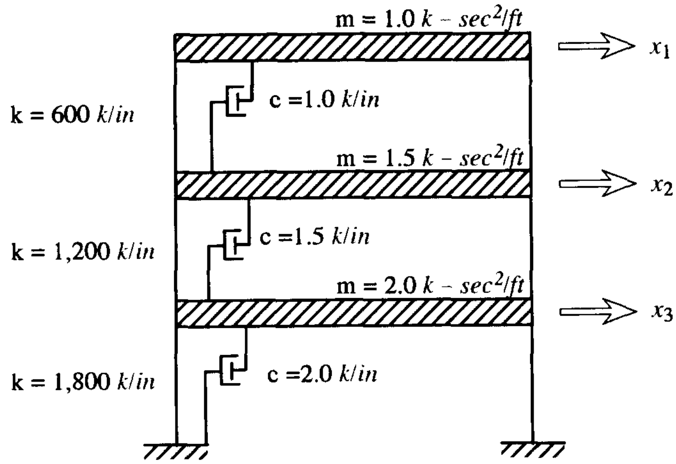

Fig. 9 Idealized properties of the three degree-of-freedom structure.

similar to a SDOF system, as the following

$$\ddot{\boldsymbol{x}}_n = -\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{x}_n - \boldsymbol{M}^{-1}\boldsymbol{C}\dot{\boldsymbol{x}}_n = \boldsymbol{\Psi}_1\boldsymbol{x}_n - \boldsymbol{\Psi}_2\dot{\boldsymbol{x}}_n \tag{80}$$

$$\dot{\boldsymbol{x}}_{n+1} = \left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{\Psi}_1 \Delta t\right)\dot{\boldsymbol{x}}_n + \left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{\Psi}_2 \Delta t\right)\ddot{\boldsymbol{x}}_n \Delta t \tag{81}$$

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \frac{1}{2}\dot{\boldsymbol{x}}_n \Delta t + \frac{1}{2}\dot{\boldsymbol{x}}_{n+1} \Delta t \tag{82}$$

where $\boldsymbol{I}$ = the identity matrix and $\boldsymbol{\Psi}_1 = \boldsymbol{M}^{-1}\boldsymbol{K}$ and $\boldsymbol{\Psi}_2 = \boldsymbol{M}^{-1}\boldsymbol{C}$.

6.3. Dynamic response of a MDOF system

Consider a three-story frame structure, shown in Fig. 9. Various aspects of the DTM direct integration of a linear MDOF system are illustrated in this example. As a convenience, the physical and vibration properties of the structure are summarized as

$$\boldsymbol{M} = \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix} kip\text{-}s^2/in \tag{83}$$

$$\boldsymbol{K} = 600\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix} kip/in \tag{84}$$

$$\boldsymbol{C} = \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix} kip\text{-}s/in \tag{85}$$

$$\{\omega\} = \begin{Bmatrix} 14.5 \\ 31.1 \\ 46.1 \end{Bmatrix} rad/s \tag{86}$$

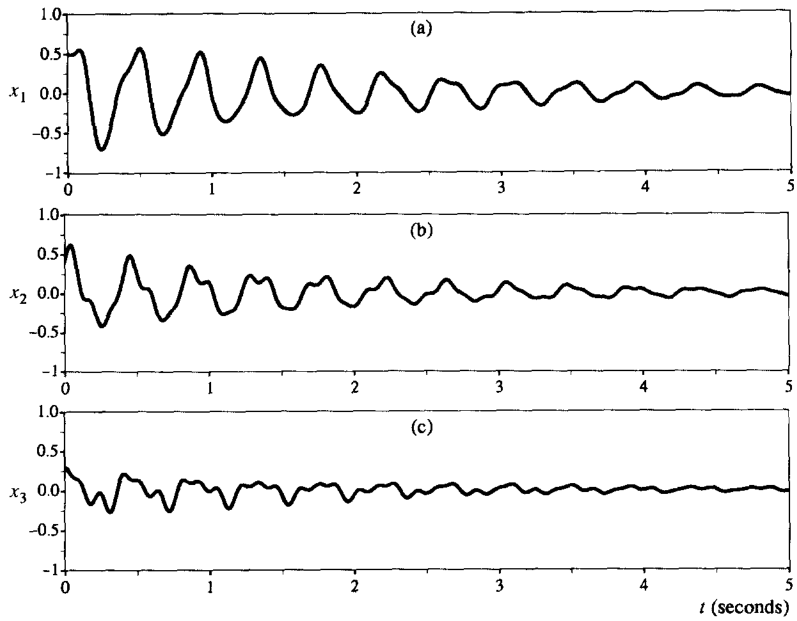

Fig. 10 Displacement history for the three degree-of-freedom structure using DTM: (a) first degree of freedom x_1; (b) second degree of freedom x_2; and (c) third degree of freedom x_3.

The free vibrations which results from the following arbitrary initial conditions are evaluated, assuming the structure is undamped

$$X_0=\begin{Bmatrix} 0.5 \\ 0.4 \\ 0.3 \end{Bmatrix} in \qquad V_0\begin{Bmatrix} 0 \\ 9 \\ 0 \end{Bmatrix} in/s \tag{87}$$

where X_0 and V_0 is the initial displacement and velocity vectors, respectively.

Using Eqs. (80)-(82) the free-vibration motion of each story is obtained. In Fig. 10, the first 5 seconds of the motion for each story is shown and compared to the exact solution. A value of $\Delta t/T_1=0.01$ ($\Delta t=0.004$) is selected based on the SDOF error analysis. In general, the time increment is calculated based on the shortest period of the MDOF system. The DTM approximation follows the exact solution very closely for all the three degrees of freedom for a time increment based on the largest period of the system.

The response of a linear MDOF system may be found by using a mode-superposition procedure, in which DTM is used to solve the uncoupled equations of motion for each mode in terms of the generalized coordinates.

7. Summary and conclusions

A simple and effective procedure based on a damped trapezoidal rule method is introduced. A systematic and fundamental procedure for stability and accuracy analysis for a damped SDOF free-vibrational system using the DTM procedure is presented. Based on an analysis of the error in period, amplitude and phase angle of a free and damped simple mass-spring system it is shown that the DTM produces accurate estimates of the response. Typically, DTM introduces less error than the well-known Newmark Beta method with $\beta=0$. The DTM procedure can be easily implemented and can be an effective and accurate method for dynamic analyses of different structural systems.

The DTM procedure is conditionally stable and requires a time step size inversely proportional to the period of the system. The main advantage of DTM over NBM is its ease of implementation and low computational cost. In addition, from error analyses of the vibration of a free and undamped mass-spring system, the following conclusions may be drawn: (i) the error in period is smaller for DTM compared to NBM for both $\beta=0$ and $\beta=1/4$, (ii) the error in displacement amplitude for the case with an initial velocity $V_0=0$ using DTM is zero for a critical damping ratio of $\xi=0\%$ and is much less than NBM with $\beta=0$ for both critical damping of $\xi=0\%$ and 25%, (iii) the error in phase angle is slightly higher for DTM compared to NBM with $\beta=0$, (iv) the error in displacement for an initial displacement $X_0=0$ for NBM with $\beta=1/4$ is zero, however, there exists some errors in the NBM procedure with $\beta=0$ which is slightly higher than the error generated by DTM. Each error term can be minimized by choosing a smaller $\Delta t/T$ with the penalty of additional computational time.

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