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Reliability analysis by numerical quadrature and maximum entropy method

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Abstract. Since structural systems may fail in any one of several failure modes, computation of system reliability is always difficult. A method using numerical quadrature for computing structural system reliability with either one or more than one failure mode is presented in this paper. Statistically correlated safety margin equations are transformed into a group of uncorrelated variables and the joint density function of these uncorrelated variables can be generated by using the Maximum Entropy Method. Structural system reliability is then obtained by integrating the joint density function with the transformed safety domain enclosed within a set of linear equations. The Gaussian numerical integration method is introduced in order to improve computational accuracy. This method can be used to evaluate structural system reliability for Gaussian or non-Gaussian variables with either linear or nonlinear safety boundaries. It is also valid for implicit safety margins such as computer programs. Both the theory and the examples show that this method is simple in concept and easy to implement.

Key words: structural system reliability; numerical integration; maximum entropy method; computational accuracy.

1. Introduction

The probabilistic approach to structural analysis has witnessed substantial progress during the last two decades. Considerable work in computing failure probability has been published for structural systems with either a single failure mode or with more than one.

For linear safety margin equations and Gaussian basic variables, Ditlevsen's narrow bound method (e.g., Ditlevsen 1979), involving one- and two-order joint probabilities, is the most widely used approach in evaluating structural system reliability. A quite narrow bound of failure probability can be obtained, especially when the correlation coefficients between each of the two failure modes are smaller than 0.6. Feng (1989) presented an improved method involving one-, two- and three-order joint probabilities which yields quite accurate results. In Zhu (1993), a numerical integration format is put forward to compute structural system reliability yielding accuracy as high as the Monte Carlo method. The computer time consumed by this method is much less than that of the Monte Carlo method. For nonlinear safety boundaries, perhaps the best known and most powerful technique is the Monte Carlo simulation method. It can provide a reasonably precise estimate of structural system failure probability. Though a number of techniques that improve its effectiveness are available, the fact that it requires a very large sample and a great amount of computer time is still serious. This is particularly true in case where the dimension

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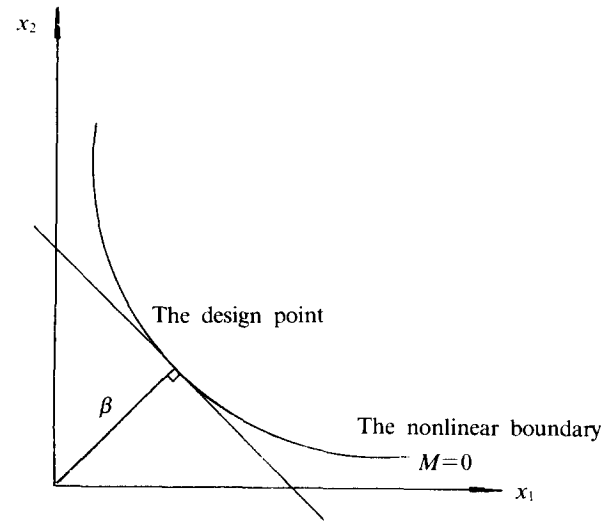

Fig. 1 The Hasofer Lind method

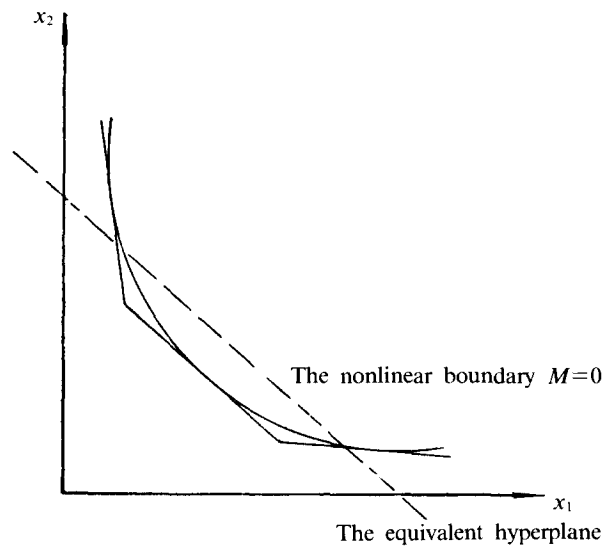

Fig. 2 Feng's equivalent hyperplane method.

of basic variables or the number of failure modes is great. Another method for nonlinear boundaries is to linearize safety margin equations. The most straightforward idea is to linearize at the means of the basic variables. However, such linearization yields unsatisfactory results. Hasofer-Lind (1974) presented a linearization method at the design points (Fig. 1) in order to improve accuracy. Feng (1990) suggested a technique using an equivalent hyperplane to approximate the nonlinear safety boundary (see Fig. 2).

This paper presents a numerical integration method. The maximum entropy method is used to generate the probability density function of the safety margin equation. Reliability is then

obtained by integrating this density function with the safety domain.

2. The reliability corresponding to a single failure mode

2.1. The basic concept

Let X_i, $i=1, 2, \cdots n$ be the n basic variables of an engineering design. Define a safety margin equation of a failure mode as

$$M=G(X_1, X_2,\cdots,X_n) \tag{1}$$

such that safety is defined as $M>0$ and failure is $M<0$. Then the reliability will be given as

$$R=P(M>0)=\underset{G(x_1,x_2,\cdots x_n)>0}{\int\int\cdots\int} f_x(x_1,x_2,\cdots,x_n)dx_1dx_2\cdots dx_n \tag{2}$$

where $f_x(x_1,x_2,\cdots,x_n)$ is the joint probability density function of the basic variables X_i $i=1,2,\cdots,n$. It is very difficult to determine the analytical solution of reliability R from Eq. (2). In some cases, the safety margin equation M is in an implicit form, such as the computer results of the deflection of a structure from Finite Element Methods. Sometimes, only several moments of M are available from a computer program or from an experiment. So, Eq. (2) is seldom used.

However, since the safety margin equation M is also a random variable, the probability $P(M>0)$ can then be expressed as

$$R=P(M>0)=\int_0^\infty f_M(z)dz \tag{3}$$

where $f_m(z)$ is the density function of the random variable M. Eq. (3) can be solved by numerical quadrature. The Gauss-Laguerre method should be used in this case. The reliability is then

$$R=\sum_{k=1}^{N} f_M(z_K)w_k\exp(z_k) \tag{4}$$

in which the point (z_k) is the Gaussian point, W_k is the weighted value corresponding to this Gaussian point, and N is the number of Gaussian points.

2.2. The generation of the density function $f_M(z)$

There are several methods of generating the probability density function $f_M(z)$ from the first several moments of the random variable M. A technique employing curve fitting is widely used (e.g., Elderton 1976). Based on the knowledge of the moments of M, an empirical distribution can be fitted to the distribution of M. Different methods of curve fitting are available in the literature such as The Lambda family of frequency curves (e.g., Elderton 1976 and Shapiro 1981) and The Pearson family of curves (e.g., Li and Lumb 1985).

In this paper, the maximum entropy method is used to generate the probability density function.

The maximum entropy method is based on the principle that the minimally prejudiced probability distribution is that which maximizes the entropy subject by constraints supplied by the

given information.

For a continuous random variable X, entropy is defined as

$$S=-\int_{-\infty}^{\infty} f(x)ln[f(x)]dx \tag{5}$$

and, for a discrete random variable, it is

$$S=-\sum_{i=1}^{n} f(x_i)ln[f(x_i)] \tag{6}$$

where $f(x)$ is the probability density function and $f(x_i)$ is the probability mass function. This principle is a valuable approach to generating density or mass function, in which the information incorporated in the constraints is whatever data the engineer has available.

From the maximum entropy method, the probability density function $f_M(z)$ can be solved from the following equations:

$$S=-\int_{-\infty}^{\infty} f_M(z)ln[f_M(z)]dz=Maximum \tag{7}$$

$$\int_{-\infty}^{\infty} f_M(z)dz=1 \tag{8}$$

$$\int_{-\infty}^{\infty} z^k f_M(z)dz=m_k,\ k=1,2,\cdots,L \tag{9}$$

where S is the entropy of the random variable M, L is the number of moments to be used and m_k is the kth moment about the origin, determined either numerically from the sample or from the following equation:

$$m_k=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}[G(x_1,x_2,\cdots,x_n)]^k f_X(x_1,x_2,\cdots,x_n)dx_1dx_2\cdots dx_n \tag{10}$$

The analytical solution of Eqs. (7), (8) and (9) is given by (e.g., Sidall 1983)

$$f_M(z)=\exp\left(\alpha_0+\sum_{k=1}^{L}\alpha_k z^k\right) \tag{11}$$

where $\alpha_0, \alpha_1, \alpha_2, \cdots \alpha_L$ satisfy:

$$\alpha_0=-ln\left[\int_{-\infty}^{\infty}\exp\left(\sum_{k=1}^{L}\alpha_k z^k\right)dz\right] \tag{12}$$

and

$$m_k=\frac{\int_{-\infty}^{\infty} z^k\exp\left(\sum_{k=1}^{L}\alpha_k z^k\right)dz}{\int_{-\infty}^{\infty}\exp\left(\sum_{k=1}^{L}\alpha_k z^k\right)dz},\qquad k=1,2,\cdots L \tag{13}$$

Eq. (13) represents L simultaneous equations to be solved for the constants α_k, $k=1,2,\cdots L$. Having these, α_0 is obtained from Eq. (12).

The equations above can be turned into more convenient form as

$$1-\frac{\int_{-\infty}^{\infty} z^k \exp\left(\sum_{k=1}^{L}\alpha_k z^k\right)dz}{m_k\cdot\int_{-\infty}^{\infty}\exp\left(\sum_{k=1}^{L}\alpha_k z^k\right)dz}=R_k, \qquad k=1,2,\cdots L \tag{14}$$

which can be numerically solved for α_k, $i=1,2,\cdots L$. In Eq. (14), R_k represents the residuals that are reduced to near zero by a numerical technique. Jacobson and Oksman (e.g., Sidall 1982) proposed a successful nonlinear programing technique to obtain the solutions of α_k, $k=1,2,\cdots,L$ by minimizing the sum of the squares of residuals, that is

$$R=\sum_{k=1}^{L}R_k^2=Minimum \tag{15}$$

Convergence is achieved when $R<\varepsilon$ or all $|R_k|<\varepsilon$, where ε is the specified acceptable error.

3. The reliability of a structural system

Generally, a structural system may fail in more than one possible mode; i.e., it will invariably have a number of potential modes, and the occurence of any one of the possible failure modes may result in failure of the whole structural system. For such a system, failure can be defined by several safety margin equations, such as

$$M_i=G_i(X_1, X_2,\cdots,X_n),\ i=1,\ 2,\cdots,m \tag{16}$$

where M_i is the safety margin corresponding to the ith failure mode and m is the total number of failure modes. Then, the reliability of the structural system is given by

$$R_S=P(M_1>0,M_2>0,\cdots,M_M>0) \tag{17}$$

The individual failure modes are generally correlated or statistically dependent. The correlated variables $\boldsymbol{M}^T=(M_1,M_2,\cdots,M_m)$ can be transformed into the uncorrelated variables $\boldsymbol{Y}^T=(Y_1,Y_2,\cdots,Y_m)$ by the following linear transformation:

$$\boldsymbol{Y}=\boldsymbol{A}^T\boldsymbol{M} \tag{18}$$

where $\boldsymbol{A}$ is an orthogonal matrix with column vectors equal to the orthonomal eigenvectors of the covariance matrix

$$\boldsymbol{C_M}=\begin{bmatrix} Var(\mathrm{M_1}) & Cov(\mathrm{M_1},M_2) & \cdots & Cov(\mathrm{M_1},M_m) \\ Cov(\mathrm{M_2},M_1) & Var(\mathrm{M_2}) & \cdots & Cov(\mathrm{M_2},M_m) \\ \cdots & \cdots & \cdots & \cdots \\ Cov(\mathrm{M_m},M_1) & Cov(\mathrm{M_m},M_2) & \cdots & Var(\mathrm{M_m}) \end{bmatrix} \tag{19}$$

Then, the joint density function of the uncorrelated variables Y_i, $i=1,2,\cdots,m$ can be calculated as

$$f_Y\ (y_1,y_2,\cdots,y_n)=f_{Y_1}(y_1)\cdot f_{Y_2}(y_2)\cdot\cdots\cdot f_{Y_n}(y_n) \tag{20}$$

where f_{Y_i} represents the density of the variables $\boldsymbol{Y}_i$, which can be determined by Eqs. (11), (12) and (13). The moments used in those equations for this case should be solved from Eq. (18).

Therefore, the joint density function of the variables M_i $i=1,2,\cdots,m$ becomes

$$f_M(z_1,z_2,\cdots,z_m)=f_Y\ (y_1,y_2,\cdots,y_m)|\boldsymbol{J}| \tag{21}$$

in which $\boldsymbol{J}$ is the the Jacobian matrix defined as

$$\boldsymbol{J}=\begin{bmatrix} \frac{\partial y_1}{\partial z_1} & \frac{\partial y_1}{\partial z_2} & \cdots & \frac{\partial y_1}{\partial z_m} \\ \frac{\partial y_2}{\partial z_1} & \frac{\partial y_2}{\partial z_2} & \cdots & \frac{\partial y_2}{\partial y_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_m}{\partial z_1} & \frac{\partial y_m}{\partial z_2} & \cdots & \frac{\partial y_m}{\partial z_m} \end{bmatrix} \tag{22}$$

and the transformation between $\boldsymbol{z}^T=(z_1,z_2,\cdots,z_m)$ and $\boldsymbol{y}^T=(y_1,y_2,\cdots,y_m)$ is

$$\boldsymbol{y}=\boldsymbol{A}^T\cdot\boldsymbol{z} \tag{23}$$

From Eq. (23), one has

$$|\boldsymbol{J}|=1, \tag{24}$$

and

$$f_M(z_1,z_2,\cdots,z_m)=f_Y(y_1,y_2,\cdots,y_m) \tag{25}$$

Therefore, the structural system reliability is given by

$$R_s=\int_0^\infty\int_0^\infty\cdots\int_0^\infty f_M(z_1,z_2,\cdots,z_m)dz_1dz_2\cdots dz_m \tag{26}$$

It is not easy to obtain the analytical expression of integration (26), so the numerical method is introduced to compute reliability.

Gaussian numerical integration methods are widely used nowadays due to their high accuracy. For the open domain ($z_1>0$, $z_2>0,\cdots,z_m>0$), the Gaussian-Laguerre (G-L) method is the most suitable and practical one. In this case, results using the G-L method possess very high accuracy. The G-L integration formula is

$$\begin{aligned} R&=\int_0^\infty\int_0^\infty\cdots\int_0^\infty f(z_1,z_2,\cdots,z_m)dz_1dz_2\cdots dz_m \\ &=\sum_{k_1=1}^{L}\sum_{k_2=1}^{L}\cdots\sum_{k_m=1}^{L}A_{K_1}A_{K_2}\cdots A_{K_m}f(z_{K_1},z_{K_2},\cdots,z_{K_m})\ \exp\left\{\sum_{i=1}^{m}z_{K_i}\right\} \end{aligned} \tag{27}$$

in which, the point $(z_{k_1},z_{k_2},\cdots,z_{k_m})$ is the G-L Gaussian point, $A_{K_1},A_{K_2},\cdots,A_{K_m}$ are the weights corresponding to the G-L Gaussian point, and L is the number of Gaussian points corresponding to each z_i axis, respectively.

Eq. (27) is not suitable for computer use. We can use an algorithm which adopts a coded word p varying from 1 to N, the total number of Gaussian points, to identify each Gaussian point. In this case, Eq. (27) becomes

$$R=\sum_{p=1}^{N} W_p f(z_{p1},z_{p2},\cdots,z_{pm}) \exp\left\{\sum_{i=1}^{m} z_{pi}\right\} \tag{28}$$

where

$$N=mL \tag{29}$$

is the total number of Gaussian points, (z_{pi}, $i=1,2,\cdots,m$; $p=1,2,\cdots,N$) is the pth Gaussian point, and

$$W_p=A_{K1}A_{K2}\cdots A_{Km} \tag{30}$$

is the combined weight corresponding to the pth Gaussian point.

The coded word p can be decoded into k_i, the discretization indices of z_i. Sidall (1983) uses a coding system which is suitable for this case. In this system, the coding address p is convenient for computer storage, with corresponding decoding algorithm, which can identify the k_i associated with any p.

The address p is simply coded as a base 10 number ranging from 1 to N. We then conceive of the address as being coded in an m-base number C_p which is

$$C_p=q_n q_{n-1}\cdots q_1 \tag{31}$$

where q_i represents the index value of the discretized Z_i, which is

$$q_i=k_i-1 \tag{32}$$

The conversion (31) is introduced since the digits in C_p must range from 1 to N. The m-based number C_p is never actually recorded. The algorithm for decoding p in order to get the k_i's is:

1. $i=1$, $s=0$
2. do until $i=n$
3. $j=n-i+1$
4. if $j=n$ go to 6
5. $s=s+q_{j+1}L^j$
6. $q_j=(p-s-1)/L^{j-1}$
7. $k_j=q_j+1$
8. $i=i+1$
9. end of do loop.

N is limited by the integer word size of the computer used. This varies from 32767 for an 8-bit computer to 2^{59} for a 64-bit computer. However, it is always possible to use two or more words to represent p.

4. Numerical examples

4.1. Example 1

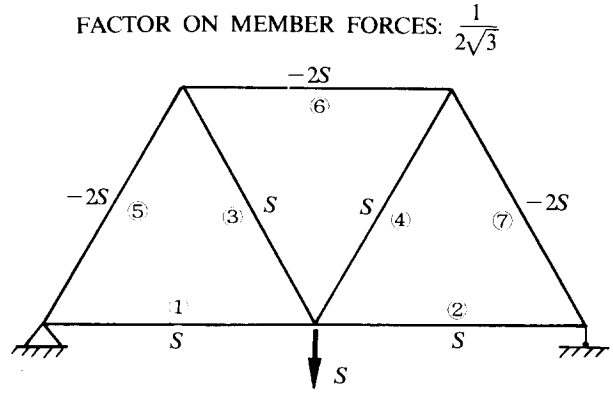

Fig. 3 Example 3.

This example(taken from Sidall 1983) illustrates the application of the method presented in this paper when only one sample from the safety margin M is available. The following data are the observed values of live loads on a warehouse floor.

0.0	7.8	36.2	60.6	64.0	64.2	79.2	88.4	38.0	72.7
72.2	72.6	74.4	21.8	17.1	48.5	16.8	105.9	57.2	75.7
225.7	42.5	59.8	41.7	39.9	55.5	67.2	122.8	45.2	62.9
55.1	55.9	87.7	59.2	63.1	58.8	67.7	90.4	43.3	55.2
36.6	26.0	90.5	23.0	43.5	52.1	102.1	71.7	4.1	37.3
129.4	66.4	138.7	127.9	90.9	46.9	197.5	151.1	157.3	197.0
134.6	73.4	80.9	53.3	80.1	62.9	150.8	102.2	6.4	45.4
121.0	106.2	94.4	139.6	152.5	70.2	111.8	174.1	85.4	83.0
178.8	30.2	44.1	157.0	105.3	87.0	50.1	198.0	86.7	64.6
78.6	37.0	70.7	83.0	179.7	180.2	60.6	212.4	72.2	86.0
94.5	24.1	87.3	80.6	74.8	72.4	131.1	116.1	53.6	99.1
40.2	23.4	8.4	42.6	43.4	27.4	63.8	18.4	16.2	58.7
92.2	49.8	50.9	116.4	122.9	132.3	105.2	160.3	28.7	46.8
99.5	106.9	55.9	136.8	110.4	123.5	92.4	160.9	45.4	96.3
88.5	48.4	62.3	71.3	133.2	92.1	111.7	67.9	53.1	39.7
93.2	55.0	80.8	143.5	122.3	184.2	150.0	57.6	6.8	53.3
96.1	54.8	63.0	228.3	139.3	59.1	112.1	50.9	158.6	139.1
213.7	65.7	90.3	198.4	97.5	155.1	163.4	155.3	229.5	75.0
137.6	62.5	156.5	154.1	134.3	81.6	194.4	155.1	89.3	73.4
79.8	68.7	85.6	141.6	100.7	106.0	131.1	157.4	80.2	65.0
78.5	118.2	126.4	33.8	124.6	78.9	146.0	100.3	97.8	75.3
24.8	55.6	135.6	56.3	66.9	72.2	105.4	98.9	101.7	58.2

The maximum entropy distribution using five moments is then (Sidall 1983)

$$f_m(z)=\exp(-6.7212+0.056423x-0.00019852x^2-0.51147\times10^{-5}x^3 +0.4066\times10^{-7}x^4-0.85672\times10^{-10}x^5)$$

Then, loading reliability less than 200 is $R=0.97086$.

4.2. Example 2

Suppose that the safety margin equation of two standardized independent Gaussian variables,

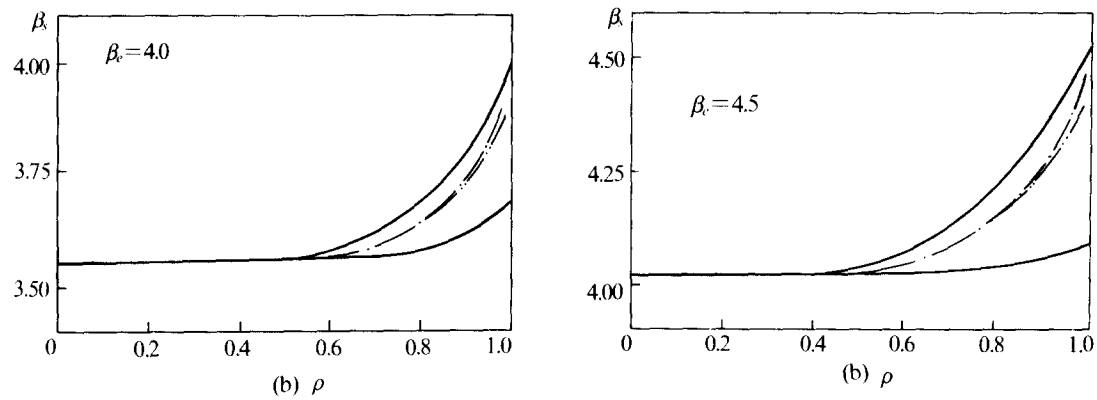

Fig. 4 Reliability index of example 3
—— narrow bounds from Ditlevsen (1979)
-·- reliability index of exact result
-··- *reliability index from the present method*

Table 1 Computational results of example 2

P	P_1	P_2	δ_1	δ_2
0.001819	0.002389	0.001873	31.34%	2.97%

X_1 and X_2, is

$$M=1-4X_1X_2 \qquad (\text{for } X_1>0 \text{ and } X_2>0)$$

The computational results of this example are listed in Table 1, in which P, P_1 and P_2 represent the failure probability calculated, respectively, using the Importance Sampling Method (ISM), the Hasofer-Lind method, and the maximum entropy method, δ_1 and δ_2 denote

$$\delta_i=\frac{|P-P_i|}{P}, \qquad i=1,2$$

By comparison, the maximum entropy method has better accuracy than that of the Hasofer-Lind method and the computer time used by the present method is less than that of the ISM.

4.3. Example 3

Consider the truss in Fig. 3 subjected to a non-random load (Ditlevsen 1979). Member strengths, in compression or in tension, are equi-correlated normally distributed random variables with correlation coefficients $\rho>0$. Failure of any of the seven members constitutes failure of the system. Then the safety margins are of the form

$$M_1=2\sqrt{3}A_1T_1-S$$
$$M_2=2\sqrt{3}A_2T_2-S$$
$$M_3=\sqrt{3}A_3T_3-S$$

$$M_4=\sqrt{3}A_4T_4-S$$
$$M_5=\sqrt{3}A_5C_5-S$$
$$M_6=\sqrt{3}A_6C_6-S$$
$$M_7=\sqrt{3}A_7C_7-S$$

Here $T_1,\cdots,T_4$ are tension yield strengths, $C_5,\cdots,C_7$ are compression yield strengths and $A_1,A_2,\cdots,A_7$ are cross-sectional areas. The safety margins are equi-correlated with correlation coefficient ρ. In this example $A_1,A_2,\cdots,A_7$ are selected such that all modes have the identical reliability index β.

The computational results from Ditlevsen (1979), exact solution and the present method are shown in Fig. 4. From Fig. 4, we can find that the present method is quite precise.

5. Conclusions

1. A method using the maximum entropy method and numerical integration is presented in order to evaluate the reliability of engineering designs. This method is applicable to Gaussian or non-Gaussian variables with linear or nonlinear safety margin equations. It can also apply to implicit safety margin equations.
2. This method possesses reasonable accuracy.
3. In the case of implicit safety margin equations, the sample size is of great importance. For a small sample, using maximum entropy method with moments has dubious validity, since the higher moments become meaningless. A similar approach can be used in which ranks are substituted for the moments used in the maximum entropy method (e.g., Sidall 1983). However, the results will not always be satisfactory.

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