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Transition membrane elements with drilling freedom for local mesh refinements

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Abstract. A transition membrane element designated as CLM which has variable mid-side nodes with drilling freedoms has been presented in this paper. The functional for the linear problem, in which the drilling rotations are introduced as independent variables, has been formulated. The transition elements with variable side nodes can be efficiently used in the local mesh refinement for the in-plane structures, which have stress concentrations. A modified Gaussian quadrature is needed to be adopted to evaluate the stiffness matrices of these transition elements mainly due to the slope discontinuity of displacement within the elements. Detailed numerical studies show the excellent performance of the new transition elements developed in this study.

Key words: finite element; membrane transition element; drilling D.O.F.; irregular node; variable-node.

1. Introduction

In many engineering practices, stress concentration phenomena occur at the areas where abrupt geometrical changes exist, or at the points under concentrated loading. For such problems, a relatively finer grid is used in the areas of high stress gradients and a rather coarser grid where the stress distribution is relatively uniform. There are several possible ways to generate a locally refined finite element grid by using quadrilateral elements.

Bathe(1982) and Hughes(1987) used a rectangular shaped element to generate locally refined mesh in a way that the two layers of subdivided rectangular element are connected to an undivided larger element. In this case, some nodes of subdivided element which cannot be connected to the sides of the larger element where the physical nodes do not exist may have to be generated. These nodes are termed irregular nodes(Choi and Park 1992). To preserve the compatibility between the refined and unrefined meshes in this case, linear dependencies between the unknown nodal displacements should be introduced by means of application of constraints to the force-displacement equation.

Zienkiewicz, et al.(1991) used the meshing techniques which use some distorted isoparametric elements in the transition zone without any constraints. However, it should be noted that the performance of the isoparametric element is generally at its best when used without shape distortion. Although the effect of distorted elements on the accuracy of the solution depends to a large degree on the problem considered and the elements used, it is desired

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A mixed use of the elements of different types for local mesh refinement, for example, quadrilateral elements with triangular elements in the transition zone, may be another possibility for mesh gradation as practiced by many investigators for the mesh transition(Yunus, et al. 1990, Evans, et al. 1991). The triangular elements, however, show worse results than the quadrilaterals in general and the introduction of triangular elements to the mesh that consist of quadrilateral elements can degrade the solution even if the quadrilaterals work well.

Choi and Park(1992) effectively used the quadrilateral transition elements which had a variable number of additional nodes on edges of a basic 4-node plate bending element to connect directly to different layer patterns.

The need for membrane elements with drilling freedom has arised in many practical engineering problems. A drilling rotation is defined as an in-plane rotation about an axis normal to the plane of the element. When combined with a plate bending element, this membrane element with drilling freedom provides a simple yet versatile tool to analyze shell structures. The previous efforts to construct this type of element including the independent approaches of Allman(1984) and Bergan and Fellipa(1985) were not quite successful but opened some prospectives for a future success. As a result, the revived interest of the engineering community in the membrane elements with drilling degrees of freedom was manifested by a series of papers on the subject(Allman 1988, Cook 1986, 1987, Lee and Yoo 1988).

The main advantages of developing the membrane element with drilling degrees of freedom are:

(1) To improve the element performance while avoiding the use of higher order elements with midpoint nodes which have lower valency than corner nodes and demand extra effort in mesh definition and generation.

(2) To simplify the modeling of connection between plates(or shells) and beams, which have rotational degrees of freedom as well as the treatment of junctures in shells and folded plates.

(3) To solve the normal rotation problem of smooth shells analyzed through finite elements programs that carry six degrees of freedom per node.

Lots of efforts have been made to define a real rotational stiffness about an axis normal to the plane of plate/shell, in which the coupling between the drilling rotation and the displacements is taken into account. The existing approaches to develop a shell element with a real rotational stiffness may be placed into three categories: (1) to derive a displacement function with a corner rotation taken as an independent degree of freedom(MacNeal and Harder 1988, Yunus 1988, 1989, Aminpour 1992, Hughes and Brezzi 1989, Ibrahimbegovic, et al. 1990, Ibrahimbegovic 1990) in which the rotational degrees of freedom actually induce inplane deformation, (2) to derive a functional with drilling rotations as independent variables(Naghdi 1964, Reissner 1965, de Veubeke 1972, Amara and Thomas 1979, Atluri 1980) in which the rotational degrees of freedom are independent of the inplane deformation, and (3) to utilize the so-called higher-order theory(Herrmann 1983) in which the anti-symmetric components of strain is represented as a rotation.

Ressiner(1965) was the first to suggest a variational formulation utilizing the skew-symmetric part of the stress tensor as a Lagrange multiplier to enforce the equality of independent rotations with the skew-symmetric part of the displacement gradient. A similar formulation was also given by de Veubeke(1972). Hughes and Brezzi(1989) have extended Reissner's formulation by recognizing the instability of discrete approximations and suggested a way in which the discrete approximation could be stabilized. They assumed in their work that the variational formulation employs an indenpendent rotation field, based on the separate kinematic variables of displacement and rotation. In developing 4-node membrane element, Ibrahimbegovic, et al.(1990) extended the applications of Hughes and Brezzi's work(1989) to combine an Allman-type interpolation for the displacement field with an independently interpolated rotation field.

In this paper, variable-node membrane elements of Type I and Type II with drilling degress fo freedom have been developed by extending the application of the Allman-type interpolation. A mixed-type variational formulation was presented with the skew-symmetric part of the stress tensor which is chosen as a constant over each element. The developed transition membrane elements can be easily combined with the quadrilateral transition plate bending element(Choi and Park 1992) to establish the flat quadrilateral shell element with variable selective mid-side nodes. The shell element possesses six degrees of freedom per node, which allow an easy modelling of complex shell surface intersections and the compatibility with other types of elements with rotational degrees of freedom.

Several numerical examples are carried out to demonstrate the validity and applicability of the present work. The Type II element which gives slightly better result than Type I is designated as CLM element(Choi and Lee Membrane element) in this study. the merits of the newly devised transition membrane element can be viewed from examples with transition zone.

2. Formulation

In this section, the discussion starts with following the works of Hughes and Brezzi(1989) and Ibrahimbegovic, et al.(1990). For the sake of brevity, only the quasi-static Dirichlet boundary value problem with homogeneous boundary conditions is considered and the discussion is limited to linear elastostatic problems.

Let Ω be a region occupied by a body. The boundary value problem under consideration is:

For all $x \in \Omega$

$$div\ \sigma + \boldsymbol{f} = 0;\ skew\ \sigma = 0;\ \psi = \nabla \boldsymbol{u};\ symm\ \sigma = \boldsymbol{C} \cdot symm\ \nabla \boldsymbol{u} \tag{1}$$

where we give the equilibrium equations, the symmetry conditions for stress σ, the definition of rotation ψ in terms of the displacement gradient $\nabla \boldsymbol{u}$ and the constitutive equations. When the Euclidean decompositions of second-rank tensor is employed for σ in Eq. (1).

$$\sigma = symm\ \sigma + skew\ \sigma, \tag{2}$$

where

$$symm\ \sigma = \frac{1}{2}(\sigma + \sigma^T);\ skew\ \sigma = \frac{1}{2}(\sigma - \sigma^T). \tag{3}$$

Reissner's(1965) principle for the boundary value problem in Eq. (1) leads to a formulation which is inappropriate for numerical applications and to convenient interpolation fields. Hughes and Brezzi(1989) modified the variational problem of Reissner by subtracting the term $1/2\gamma^{-1} \int_{\Omega} |skew\ \tau|^2 d\Omega$ in order to preserve the stability of the discrete problem. This mo dification preserves Eq. (1) as the Euler-Lagrange equations. In addition, the symmetrical components of stress are eliminated by using the constitutive equations in Eq. (1) to give:

$$\Pi_\gamma(\boldsymbol{v},\ \omega,\ skew\ \tau)=\frac{1}{2}\int_\Omega(symm\ \nabla\boldsymbol{v})\cdot\boldsymbol{C}\cdot(symm\ \nabla\boldsymbol{v})d\Omega$$
$$+\int_\Omega skew\ \tau^T\cdot(skew\ \nabla\boldsymbol{v}-\omega)d\Omega \tag{4}$$
$$-\frac{1}{2}\gamma^{-1}\int_\Omega|skew\ \tau|^2 d\Omega-\int_\Omega\boldsymbol{v}\cdot\boldsymbol{f}d\Omega,$$

where $\boldsymbol{v}\in\mathbf{V}$, $\omega\in\mathbf{W}$, $\tau\in\mathbf{T}$ are spaces of trial displacement, rotations and stresses, respectively. This variational formulation requires that the rotations ω and stresses τ together with the displacement gradient belong to the space of square-integrable functions over the region $\boldsymbol{\Omega}$.

The variational equation of a mixed-type discrete formulation resulted from variations on Eq.(4)

$$0=\int_{\Omega^h}(symm\nabla\boldsymbol{v}^h)^T\boldsymbol{C}(symm\nabla\boldsymbol{u}^h)d\Omega$$
$$+\int_{\Omega^h}skew\ \tau^{hT}(skew\nabla\boldsymbol{u}^h-\psi^h)d\Omega+\int_{\Omega^h}(skew\nabla\boldsymbol{v}^{hT}-\omega^{hT})\ skew\sigma^h d\Omega \tag{5}$$
$$-\gamma^{-1}\int_{\Omega^h}skew\tau^{hT}skew\sigma^h d\Omega-\int_{\Omega^h}\boldsymbol{v}^{hT}\boldsymbol{f}d\Omega.$$

where superscript h denotes and distinguishes the matrix quantities.

The parameter γ which appears in the formulation is problem dependent(Hughes and Brezzi 1989). For isotropic elasticity and the Dirichlet boundary value problem, γ can be taken to be equal to the shear modulus. However, the formulation is rather insensitive to the value used for γ(at least for several orders of magnitude which bound the shear modulus).

3. Finite element interpolation

We consider a variable-node quadrilateral element with degrees of freedom shown in Fig. 1. When the mixed-type variational equation of Eq. (5) is applied to the finite dimensional

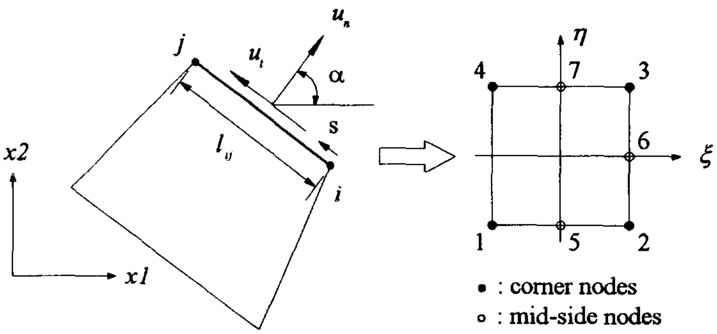


Fig. 1 Configuration of variable-node element.

space $\mathbf{V}^h$, $\mathbf{W}^h$, $\mathbf{T}^h$(subspace of $\mathbf{V}$, $\mathbf{W}$, $\mathbf{T}$, respectively), the skew-symmetric stress field is chosen as a constant over the element, i.e.

$$skew\ \tau^h = \sum_e \tau_o^e \tag{6}$$

The independent rotation field of variable-node element is interpolated as the discontinuous field over the element.

$$\psi^h = \sum_{i=1}^{n} N_i(\xi,\ \eta)\psi_i \tag{7}$$

where

$$N_i = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta) - \frac{1}{2}(N_j+N_k), \tag{8a}$$

$$N_k = \frac{1}{2}(1+|\xi_k|\xi_k\xi - |\eta_k||\xi|)(1+|\eta_k|\eta_k\eta - |\xi_k||\eta|), \tag{8b}$$

$$i=1,2,3,4\ ;\ j=\mathrm{aint}(1/i)\times 4+i+3;\ k=i+4 \tag{8c}$$

If the mid-side nodes do not exist, the corresponding shape functions become zeros($N_5=0$, $N_6=0$, $N_7=0$).

For the transition elements with drilling degrees of freedom, the Allman-type shape functions are used for the basic behavior of the element. The two types of in-plane displacement approximations are considered in this study. The Type II displacement field has the incompatible mode to the tangential direction of element sides in addition to an Allman type interpolation while the Type I displacement field does not have the incompatible mode:

(1) Type I

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \boldsymbol{u}^h = \sum_{i=1}^{n} N_i\,(\xi,\eta)\boldsymbol{u}_i + N_0\boldsymbol{\Delta u}_0$$
$$+\sum_{i=1}^{4}\frac{1}{8}(\boldsymbol{l}_{ki}NIS_i\boldsymbol{n}_{ki} - l_{ij}NFS_i\boldsymbol{n}_{ij})\psi_i + \sum_{i=5}^{n}\frac{1}{8}l_{op}NS_i\boldsymbol{n}_{op}\psi_i \tag{9}$$

(2) Type II

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \boldsymbol{u}^h = \sum_{i=1}^{n} N_i(\xi,\ \eta)\boldsymbol{u}_i + \sum_{i=1}^{n} NIS_i \triangle\boldsymbol{u}_{ti}\boldsymbol{t}_{ki} + N_0\triangle\boldsymbol{u}_0$$
$$+\sum_{i=1}^{4}\frac{1}{8}(\boldsymbol{l}_{ki}NIS_i\boldsymbol{n}_{ki} - l_{ij}NFS_i\boldsymbol{n}_{ij})\psi_i + \sum_{i=5}^{n}\frac{1}{8}l_{op}NS_i\boldsymbol{n}_{op}\psi_i \tag{10}$$

where l_{ij} is the length, $\boldsymbol{n}_{ij} = \langle\cos\alpha\ \sin\alpha\rangle^T$ is the outward unit normal vector and $\boldsymbol{t}_{ij} = \langle -\sin\alpha\ \cos\alpha\rangle^T$ is a unit tangent vector on the element side associated with the corner nodes i and j, i.e.

$$\boldsymbol{n}_{ij} = \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} \cos\alpha_{ij} \\ \sin\alpha_{ij} \end{Bmatrix},\ \boldsymbol{t}_{ij} = \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} = \begin{Bmatrix} -\sin\alpha_{ij} \\ \cos\alpha_{ij} \end{Bmatrix},\ \boldsymbol{l}_{ij} = ((x_{j1}-x_{i1})^2 + (x_{j2}-x_{i2})^2)^{1/2} \tag{11}$$

and a FORTRAN-like definition of adjacent corner nodes

$$j=\mathrm{mod}(i,4)+1;\ k=i-1+4\times\mathrm{aint}(1/i);\ o=i-4;\ p=i-3;\ 0\leq n\leq 7 \tag{12}$$

In Eqs.(9) and (10), NIS_i, NFS_i and NS_i are serendipity shape functions. The first two shape functions are associated with the element side between corner node k and i, i and j and the last is associate with the element side with variable-node of node i. The serendipity shape functions are defined by;

(1) 4-node element

$$\begin{aligned} &NIS_i=NFS_i=\frac{1}{2}(1+\xi_i\xi)(1-\eta^2);\ i=1,3 \\ &NIS_i=NFS_i=\frac{1}{2}(1-\xi^2)(1-\eta_i\eta);\ i=2,4 \\ &NS_5=NS_6=NS_7=0 \end{aligned} \tag{13}$$

(2) 5-node element - NFS_1, NIS_2 and NS_5 of the 4-node element are replaced by

$$\begin{aligned} &NFS_1=\frac{1}{2}(\,|\xi|-\xi)(1+\xi)(1-\eta) \\ &NIS_2=\frac{1}{2}(\,|\xi|+\xi)(1-\xi)(1-\eta) \\ &NS_5=-\xi(1-|\xi|)(1-\eta) \end{aligned} \tag{14}$$

(3) 6-node element - NFS_2, NIS_3 and NS_6 of the 5-node case are replaced by

$$\begin{aligned} &NFS_2=\frac{1}{2}(1+\xi)(\,|\eta|-\eta)(1+\eta) \\ &NIS_3=\frac{1}{2}(1+\xi)(\,|\eta|+\eta)(1-\eta) \\ &NS_6=-(1+\xi)\eta(1-|\eta|) \end{aligned} \tag{15}$$

(4) 7-node element - NFS_3, NIS_4 and NS_7 of the 6-node case are replaced by

$$\begin{aligned} &NFS_3=\frac{1}{2}(\,|\xi|+\xi)(1-\xi)(1+\eta) \\ &\mathrm{NIS}_4=\frac{1}{2}(\,|\xi|-\xi)(1+\xi)(1+\eta) \\ &NS_7=\xi(1-|\xi|)(1+\eta) \end{aligned} \tag{16}$$

A hierarchical bubble shape interpolation is added to Eqs. (9) and (10) where the shape function is given as

$$N_0=(1-\xi^2)(1-\eta^2) \tag{17}$$

We also define matrix notation

$$symm\nabla \boldsymbol{u}=\boldsymbol{B}_i^e \boldsymbol{u}_i+\boldsymbol{G}_i^e \psi_i+\boldsymbol{R}_i^e \triangle \boldsymbol{u}_{ti} \tag{18}$$

where $\boldsymbol{u}_i$, ψ_i and $\Delta \boldsymbol{u}_{ti}$ are the nodal values of the displacement, the rotation field and mid-side incompatible displacement parameters, respectively. The $\boldsymbol{B}_i^e$ matrix in Eq. (18) has the form

$$\boldsymbol{B}_i^e=\begin{bmatrix} \mathrm{N}_{i,x1} & 0 \\ 0 & \mathrm{N}_{i,x2} \\ \mathrm{N}_{i,x2} & \mathrm{N}_{i,x1} \end{bmatrix} \qquad 1\leq i\leq 7 \tag{19}$$

The part of the displacement interpolation associated with the rotation defines

$$\boldsymbol{G}_i^e=\frac{1}{8}\begin{bmatrix} (l_{ki}\cos\alpha_{ki}NIS_{i,x1}-l_{ij}\cos\alpha_{ij}NFS_{i,x1}) \\ (l_{ki}\sin\alpha_{ki}NIS_{i,x2}-l_{ij}\sin\alpha_{ij}NFS_{i,x2}) \\ (l_{ki}\cos\alpha_{ki}NIS_{i,x2}-l_{ij}\cos\alpha_{ij}NFS_{i,x2}) \\ \qquad +(l_{ki}\sin\alpha_{ki}NIS_{i,x1}-l_{ij}\sin\alpha_{ij}NFS_{i,x1}) \end{bmatrix} \qquad i=1,2,3,4 \tag{20a}$$

$$\boldsymbol{G}_i^e=\frac{1}{8}\begin{bmatrix} l_{op}\cos\alpha_{op}NS_{i,x1} \\ l_{op}\sin\alpha_{op}NS_{i,x2} \\ (l_{op}\cos\alpha_{op}NIS_{i,x2}-l_{op}\sin\alpha_{op}NS_{i,x1}) \end{bmatrix} \qquad i=5,6,7 \tag{20b}$$

and the displacement interpolation from the tangential incompatible mid-side displacement gives

$$\boldsymbol{R}_i^e=\begin{bmatrix} -\sin\alpha_{ki}NIS_{ki,x1} \\ \cos\alpha_{ki}NIS_{ki,x2} \\ -\sin\alpha_{ki}NIS_{ki,x2}+\cos\alpha_{ki}NIS_{ki,x1} \end{bmatrix} \qquad i=1,\cdots,7 \tag{21}$$

In Eqs. (20) to (21) and also Eqs. (25) to (26) below, the same FORTRAN-like definition of indices is used as the Eq. (12).

Wilson and Ibrahimbegovic(1990) imposed the requirement that both incompatible modes and nodes associated with drilling degrees of freedom be orthogonal to the constant strain field. This will ensure convergence of the analysis in the spirit of the patch test. The modification fits into the framework of well-known B-bar methods(1989) and reduces to changing strain-displacement matrices into

$$\bar{\boldsymbol{G}}^e=\boldsymbol{G}^e-\frac{1}{\Omega^e}\int \boldsymbol{G}^e d\Omega, \qquad \bar{\boldsymbol{R}}^e=\boldsymbol{R}^e-\frac{1}{\Omega^e}\int \boldsymbol{R}^e d\Omega \tag{22}$$

Furthermore, we introduce the matrix notation for the infinitesimal rotation fields as

$$skew\nabla \boldsymbol{u}-\psi=\boldsymbol{b}_i^e \boldsymbol{u}_i+\boldsymbol{g}_i^e \psi_i+\boldsymbol{r}_i^e \triangle \boldsymbol{u}_{ti} \tag{23}$$

where

$$\boldsymbol{b}_i^e = \langle -\frac{1}{2} N_{i,x2}\,;\ \frac{1}{2} N_{i,x1} \rangle \qquad 1 \le i \le 7 \tag{24}$$

and

$$\boldsymbol{g}_i^e = [-\frac{1}{16}(l_{ki}\cos\alpha_{ki} NIS_{i,x2} - l_{ij}\cos\alpha_{ij} NFS_{i,x2}) + \frac{1}{16}(l_{ki}\sin\alpha_{ki} NIS_{i,x1} - l_{ij}\sin\alpha_{ij} NFS_{i,x1}) - N_i] \quad i=1,2,3,4 \tag{25a}$$

$$\boldsymbol{g}_i^e = [-\frac{1}{16} l_{op}\cos\alpha_{op} NS_{i,x2} - \frac{1}{16} l_{op}\sin\alpha_{op} NS_{i,x1} - N_i] \quad i = 5,6,7 \tag{25b}$$

while

$$\boldsymbol{r}_i^e = [-\sin\alpha_{ki} NIS_{ki,x2} + \cos\alpha_{ki} NIS_{ki,x1}] \quad i=1,\cdots,7 \tag{26}$$

The first term in the discrete formulation in Eq. (5) gives the variable-node membrane element stiffness matrix

$$\boldsymbol{K}^e = \int_{\Omega^e} [\boldsymbol{B}^e \overline{\boldsymbol{G}}^e \overline{\boldsymbol{R}}^e]^T \boldsymbol{C} [\boldsymbol{B}^e \overline{\boldsymbol{G}}^e \overline{\boldsymbol{R}}^e] d\Omega \tag{27}$$

And the second term in Eq. (5) is denoted as

$$\boldsymbol{h}^e = \int_{\Omega^e} \langle \boldsymbol{b}^e ; \boldsymbol{g}^e ; \boldsymbol{r}^e \rangle^T \, d\Omega \tag{28}$$

With this notation at hand, the discrete mixed-type formulation can be rewritten as

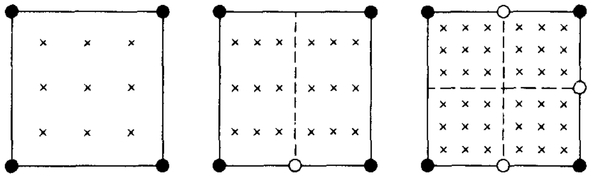
$$\begin{bmatrix} \boldsymbol{K}^e & \boldsymbol{h}^e \\ \boldsymbol{h}^{eT} & \gamma^{-1}\Omega^e \end{bmatrix} \begin{Bmatrix} \boldsymbol{a} \\ \tau_0^e \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{Bmatrix} ; \ \mathbf{a} = \begin{Bmatrix} \boldsymbol{u} \\ \psi \\ \triangle \boldsymbol{u} \end{Bmatrix} \tag{29}$$

Since the skew-symmetric part of the stress is interpolated independently in each element, the corresponding part of the stiffness matrix in Eq. (29) may be eliminated at the element level to yield a rank-one update to the element stiffness:

$$\hat{\boldsymbol{K}}^e \boldsymbol{a} = \boldsymbol{f}; \quad \hat{\boldsymbol{K}}^e = \boldsymbol{K}^e + \frac{\gamma}{\Omega^e} \boldsymbol{h}^e \boldsymbol{h}^{eT} \tag{30}$$

Static condensation(Wilson 1974) on the element stiffness matrix $\hat{\boldsymbol{K}}^e$ is then used to eliminate the relative displacement $\Delta \boldsymbol{u}$ at the element level.

The parts of the 4-node membrane element stiffness matrix $\boldsymbol{K}^e$ and $\boldsymbol{h}^e$ in Eq. (29) are computed using 3×3 Gaussian quadrature for Allman-type shape function and the variable-node membrane stiffness matrices are evaluated by the modified Gaussian quadrature(Gupta 1978). And if the displacement field do not have the incompatible mode to the tangential direction of element sides such as Type I, the terms of $\boldsymbol{R}^e$, $\overline{\boldsymbol{R}}^e$ and $\boldsymbol{r}^e$ in Eqs.(18),(21), (22),(23),(26),(27)



(a) 4 node (b) 5 node (c) 6-7 node

Fig. 2 Gaussian Point

Table 1 Modified Gaussian quadrature for transition element.

point	ξ_i, η_i	Weight
1	−0.887298335	0.277777778
2	−0.500000000	0.444444444
3	−0.112701665	0.277777778
4	+0.112701665	0.277777778
5	+0.500000000	0.444444444
6	+0.887298335	0.277777778

and (28) are disregarded.

4. Numerical integration

In evaluating the stiffness matrix of these types of elements, a normal(usual) numerical integration may not be applied directly over the entire element domain($-1 \leq \xi, \eta \leq +1$) because the slope discontinuity of displacement assumed in the elements may cause a singular integral. Therefore, the Gaussian quadrature needs to be modified to be carried out over each sub-domains and assembled to form the entire stiffness matrix. In case of the 5-node element, the element is divided into two subdomains and the 6-node and 7-node element into four subdomains as shown in Fig. 2(b) and (c) with dashed lines as boundaries of subdomains. For the sake of simple programming, a 6×6 modified Gaussian quadrature is used for the 5-node element like the case of 6-node and 7-node element in this study. The coordinates and corresponding weight coefficients for the modified Gaussian quadrature points are listed in Table 1(Gupta 1978).

The load vector for the transition elements should also be evaluated in accordance with the modified Gaussian quadrature as discussed above.

5. Numerical analysis

Some simple example problems are solved to evaluate the performance of the new transition elements, i.e., both Type I and Type II elements. The eigen-value analysis is carried to check if the stiffness matrices of transition elements have a correct rank. The results obtained are compared with the exact solutions or the numerical results from other elements with drilling freedom(Ibrahimbegovic 1990, Iura and Atluri 1992) and those from element of Q4(Zienkiewicz

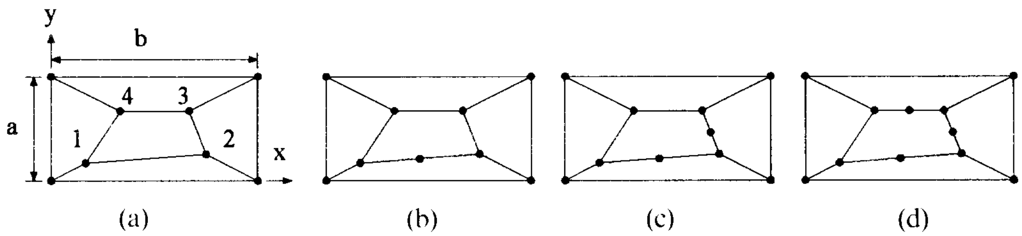


Fig. 3 Patch test model. a=0.12; b=0.24; $E=1.0\times10^{6}$; $\nu=0.25$;

Location of inner nodes

Node	x	y
1	0.04	0.02
2	0.18	0.03
3	0.16	0.08
4	0.08	0.08

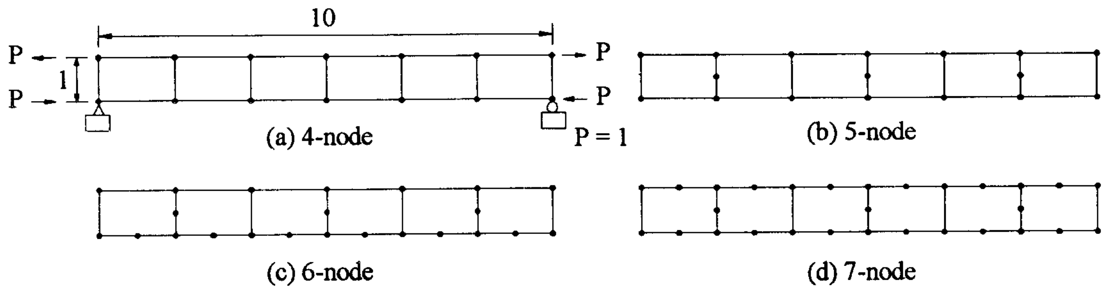


Fig. 4 A simple beam.

and Taylor 1989) and CP4(Choi and Paik 1994) without drilling freedom. The Q4 is standard bilinear 4-node membrane element. Only the membrane behavior of the CP4 which is a 4-node degenerated shell element was considered in this study.

5.1. *Patch test*

In order to check if the transition elements have the capability of representing constant strain states, the patch test has been undertaken. The patch test model composed of 4-node elements and 5-node to 7-node transition elements with boundary conditions of $u=10^{-3}(x+y/2)$, $v=10^{-3}(y+x/2)$ in Fig. 3.

The solutions obtained by using both Type I and Type II transition elements are exactly coincide with the theoretical solutions of $\varepsilon_x=\varepsilon_y=\gamma=10^{-3}$; $\sigma_x=\sigma_y=1333$; $\tau_{xy}=400$. The different positions of variable nodes(Fig. 3(b),(c) and(d)) did not affect the results at all. Thus both Type I and Type II elements passed the patch test.

5.2. *A simple beam*

A simple beam with a length to height ratio of 10 is subjected to a pure bending state.

Table 2 Displacements and rotations of simply supported beam

Element	Nodes/element	Displacement	Rotation
Q4	4	1.5	–
CP4	4	1.5	–
Ibrahimbegovic	4	1.5	0.6
Iura	4	1.5	0.6
Type I	4	1.5	0.6
	5	1.5	0.6
	6	1.5	0.6
	7	1.5	0.6
Type II	4	1.5	0.6
	5	1.5	0.6
	6	1.5	0.6
	7	1.5	0.6
Beam theory		1.5	0.6

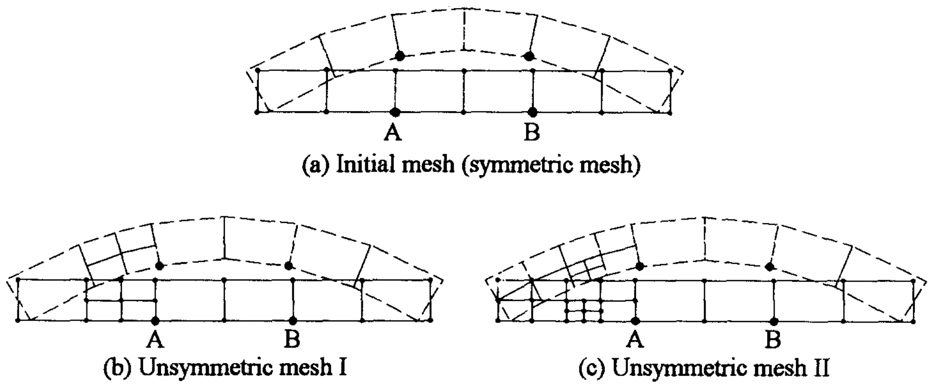


Fig. 5 Deformed shapes of the simple beam

The beam is modelled by one row of six membrane elements with drilling degrees of freedom, as shown in Fig. 4. With the material properties of $E=100$ and $\nu=0$, the numerical results obtained are given in Table 2. Both types of elements with different number of nodes gave the exact solutions by the beam theory i.e., 1.5 for vertical displacement and 0.6 for end rotation.

When the transition elements are tested in the unsymmetrical model as shown in Fig. 5, the symmetry of deformed shapes is maintained and the displacement at points A and B in Fig. 5(a), (b) and (c) are virtually identical. It may imply that the use of transition elements with other elements in a mixed modelling produces no abrupt or abnormal changes in the stress and strain distribution.

5.3. *A cantilever beam*

A shear-loaded cantilever beam has been frequently tested to verify the behavior of newly developed elements by many authors(Bergan and Fellipa 1985, Allman 1988, Ibrahimbegovic, et al. 1990, Ibrahimbegovic 1990, Iura 1992). The elastic solution(Timoshenko and Goodier 1951) for the tip displacement is 0.3553 for the properties selected(see Fig. 6 for details). The fixed boundary condition is idealized by constraining the corresponding displacement com-

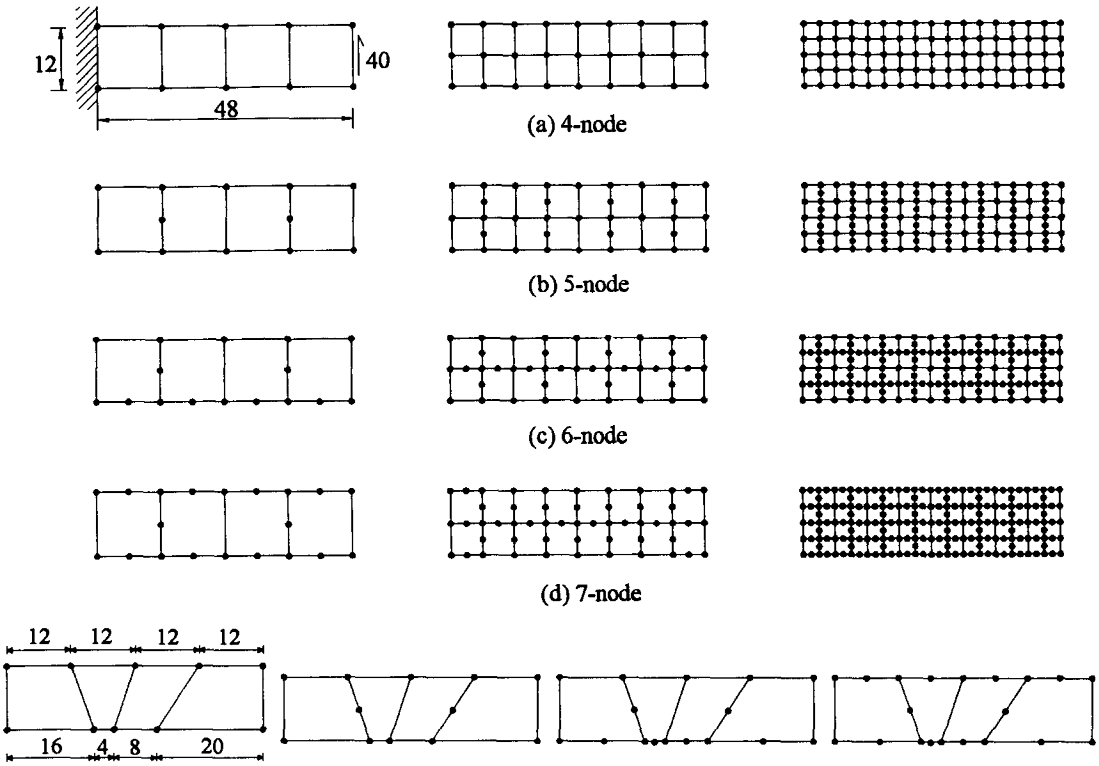


Fig. 6 Short cantiever beam.($E=30000$, $\nu=0.25$).

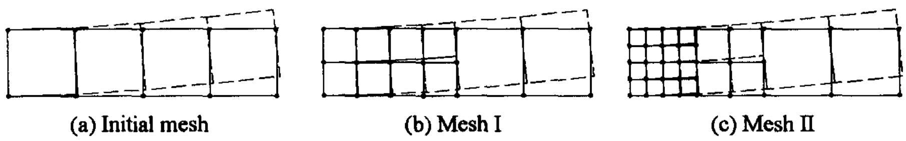
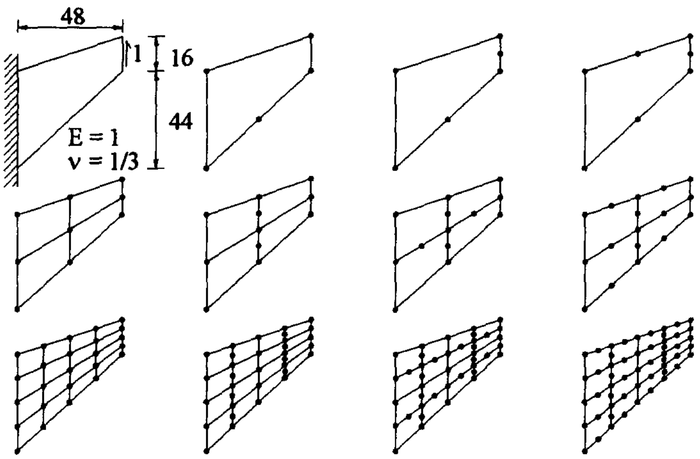


Fig. 7 Deformed shapes of the cantilever beam

Table 3 Displacements of the cantilever beam

Element	Nodes/elelment	Distorted	4×1 mesh	8×2 mesh	16×4 mesh	Mesh I	Mesh II
Q4	4	0.2131	0.2424	0.3162	0.3447	–	–
CP4	4	0.3283	0.3283	0.3460	0.3530	–	–
Ibrahimbegovic	4	0.3066	0.3445	0.3504	0.3543	–	–
Iura	4	0.3252	0.3494	0.3515	0.3543	–	–
Type I	4	0.3042	0.3415	0.3495	0.3538	0.3491	0.3520
	5	0.3148	0.3453	0.3519	0.3548		
	6	0.3357	0.3428	0.3533	0.3548		
	7	0.3321	0.3502	0.3534	0.3550		
Type II	4	0.3122	0.3493	0.3516	0.3543	0.3527	0.3551
	5	0.3236	0.3507	0.3536	0.3550		
	6	0.3488	0.3462	0.3548	0.3552		
	7	0.3463	0.3547	0.3549	0.3554		
Exact				0.3553			



(a) 4-node (b) 5-node (c) 6-node (d) 7-node

Fig. 8 Cook's membrane

ponents. The results obtained are compared with other results available in the literature and those of the cantilever beam with locally refined non-uniform meshes(Fig. 7) in Table 3.

Both Type I and Type II elements produced reasonably good results as presented in Table 3. It is noted in particular that the results by Type II are better than those by Type I in all the cases tested as expected. However, the differences are insignificant.

5.4. Cook's problem

A trapezoidal membrane(Fig. 8) suggested by Cook(1974) is another test problem frequently

Table 4 Tip displacements of Cook's problem

Element	Nodes/element	1×1 mesh	2×2 mesh	4×4 mesh
Q4	4	5.969	11.845	18.299
CP4	4	10.050	17.310	21.650
Ibrahimbegovic	4	14.065	20.682	22.984
Iura	4	17.928	21.916	23.360
Type I	4	12.777	19.442	22.734
	5	18.707	19.669	22.863
	6	18.876	20.932	23.133
	7	23.146	22.967	23.577
Type II	4	13.638	19.689	22.816
	5	19.633	19.877	22.953
	6	20.051	21.120	23.205
	7	24.020	23.334	23.694
Ref.	23.91			

used(Bergan and Fellipa 1985, Allman 1988, Ibrahimbegovic, et al. 1990, Ibrahimbegovic 1990, Iura 1992). Besides the shear dominant behaviour, it also displays the effects of mesh distortion. The results for the tip deflection can be compared to the reference value of 23.91 obtained by numerical analysis for a refined model. The results from Type II which has additional non-conforming modes are better than the Type I as previous examples.

6. Conclusions

A transition membrane element with drilling freedom based on a mixed-type variational formulation has been presented. This type of element can provide a consistency in combining the element into the general finite element mesh for in-filled frames, folded plates or similar complex structural systems which normally have the rotational degree of freedom at each node. In its formulation, the skew-symmetric part of the stress tensor was utilized as a Lagrange multiplier to enforce the equality of independent rotations with the skew-symmetric part of the displacement gradient. The behavior of two different elements, namely Type I and Type II, were verified through a series of basic tested problems. Also a significantly better behavior of elements with drilling freedom has been observed comparing with those of elements without drilling freedom(Q4 and CP4). The Type II element which contains the incompatible mode in element sides gives slightly better result than the Type I, in particular where the mesh contains distorted elements. Thus the Type II element is designated as CLM-element in this study will be more extensively used in the future study. This membrane element can also be combined with a transition plate bending element(for example, Choi and Park 1992) to form a transition flat shell element which has 6 degrees of freedom at each node.

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