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Finite element fracture reliability of stochastic structures

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Abstract. This study presents a methodology for the system reliability analysis of cracked structures with random material properties, which are modeled as random fields, and crack geometry under random static loads. The finite element method provides the computational framework to obtain the stress intensity solutions, and the first-order reliability method provides the basis for modeling and analysis of uncertainties. The ultimate structural system reliability is effectively evaluated by the stable configuration approach. Numerical examples are given for the case of random fracture toughness and load.

Key words: brittle fracture; stochastic finite element; random field; systems reliability; stable configuration approach.

1. Introduction

A problem of considerable and increasing importance within the fields of mechanical, aeronautical, nuclear, marine and military engineering is the predominantly brittle fracture of structures. The failure process initiates with the presence of small cracks which can cause catastrophic fracture. The fracture behavior of a linear elastic structure can be inferred by comparing the applied stress intensity factor with the fracture toughness of the material. In real situations, there are usually some degrees of uncertainty associated with the flaw sizes and material properties including fracture toughness. Extraordinary loads can result in stresses significantly above the intended design level. Because of these complexities, fracture should be viewed probabilistically rather than deterministically. In general, the solution of the response field in other than very simple structures can not be obtained in closed form, but must be computed approximately. In structural reliability analysis, the finite element method is well suited for dealing with random spatial variabilities in the material properties due to the segmentation of the structure into elements, each of which can be represented by its own properties. Previous works (Besterfield, *et al.* 1990, Der Kiureghian and Ke 1988, Mahadevan and Haldar 1991) in structural reliability analysis are limited to the evaluation of the probability of initial damage of the component or structure using given failure criteria on deflection or strength. In this study, the system reliability of structures with multiple cracks will be determined through the finite element method employing the first-order reliability method. The uncertainties in load, crack geometry and material properties including fracture toughness are taken into consideration. These variables are modeled as random fields on the entire domain of a structure. In order to model singularities at the crack tips,

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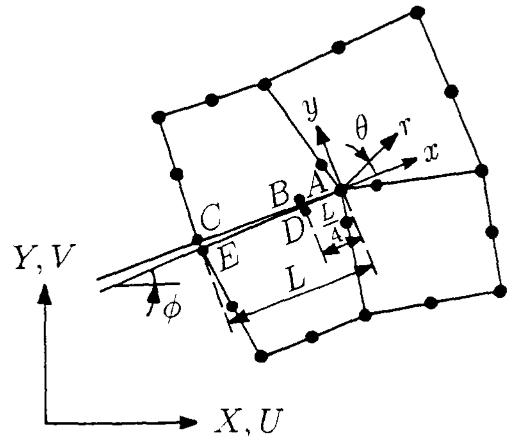

Fig. 1 Degenerate isoparametric elements at a crack tip.

appropriate crack tip elements are employed. In the presence of multiple cracks in a structure, there are multiple failure modes, which are correlated. The system failure probability, that is, the collapse probability, is effectively computed by the stable configuration approach. The branch event corresponding to each crack is defined by appropriate fracture criteria. The system failure event is derived as the intersection of the unions of branch events by the stable configuration approach.

2. Finite element modeling of crack tip singularity

The use of finite elements in fracture predictions requires two distinct considerations: (i) crack tip singularity modeling, and (ii) interpretation of the finite element results. In this study the degenerate isoparametric quadrilateral elements (Barsoum 1976) are utilized since these elements are simple to implement without any change in a standard finite element program and can give accurate results for mixed-mode fracture. The stress intensity factors are easily computed from the displacements along the element edge of the degenerate element using the displacement matching method. For plane crack problems, the finite element mesh idealization dictates the free surfaces of the crack to be the most convenient choices for the evaluation of the stress intensity factors. By combining the stress intensity factor expressions along the $\theta = \pm 180°$ rays emanating from the crack tip as shown in Fig. 1, the resulting expression is obtained as follows.

$$\begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix} = \frac{\mu}{\kappa+1}\sqrt{\frac{2\pi}{L}}\begin{bmatrix} -\sin\phi & \cos\phi \\ \cos\phi & \sin\phi \end{bmatrix}\begin{bmatrix} 4 & 0 & -1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 4 & 0 & -1 & 0 & -4 & 0 & 1 \end{bmatrix} \cdot \{U_B \; V_B \; U_C \; V_C \; U_D \; V_D \; U_E \; V_E\}^T \tag{1}$$

where μ is the shear modulus, and κ is $(3-\nu)/(1+\nu)$ for plane stress, $3-4\nu$ for plane strain. U_I and V_I represent the nodal displacements of the node I in the X and Y directions, respectively.

3. Performance function

3.1. Mode I fracture

Mode I loading has the most practical importance. For mode I fracture, the fracture criterion which is commonly used states that crack propagation will occur when the stress intensity factor K_I reaches a critical value K_{Ic}, termed the fracture toughness which is a mechanical property of the material. Thus, the performance function for the mode I fracture can be expressed as

$$g=K_{Ic}-K_I \tag{2}$$

Accordingly, the failure state is defined as $g<0$. When only mode I fracture is present, the direction of crack propagation measured from the current crack orientation θ is equal to 0; that is, the crack extends along a straight path.

3.2. Mixed-mode fracture

Practical structures are not only subjected to tension but may also experience shear and torsional loadings. Cracks may therefore be exposed to tension and shear, which lead to mixed mode cracking. There are currently several fracture criteria available, which include the maximum tangential stress criterion, the maximum energy release rate criterion, the strain energy density criterion, and the elliptic rule criterion. The elliptic rule criterion (Yishu 1990) is the general criterion superseding all the mixed-mode fracture criteria. This criterion describes the loci of critical points by the fracture envelope as

$$\left(\frac{K_I}{K_{Ic}}\right)^2+A\left(\frac{K_{II}}{K_{Ic}}\right)^2=1 \tag{3}$$

where the material constant $A=(K_{Ic}/K_{IIc})^2$. The elliptic rule criterion is employed in the present study because it can be easily formulated. The performance function for the mixed mode *I-II* fracture is then expressed as

$$g=K_{Ic}^2-(K_I^2+\mathrm{A}K_{II}^2) \tag{4}$$

Accordingly, $g<0$ defines the failure state which represents that the crack extension occurs in one direction. It is noted that the fracture angles predicted by various criteria are basically in close agreement with the measured one (Yishu 1990). Thus, the maximum tangential criterion (Erdogan and Sih 1963) is employed to calculate the fracture angle for the next configuration as follows.

$$\theta_o=2\tan^{-1}\left[\frac{1}{4}\frac{K_I}{K_{II}}\pm\frac{1}{4}\sqrt{\left(\frac{K_I}{K_{II}}\right)^2+8}\,\right] \tag{5}$$

and

$$K_I(1-3\cos\theta_o)\cos\frac{\theta_o}{2}+K_{II}(5+9\cos\theta_o)\sin\frac{\theta_o}{2}<0 \tag{6}$$

4. Representation of random field

Material properties, structural geometry and external loads have random spatial variabilities and are modeled by random fields rather than random variables. For finite element reliability analysis, it is necessary that such random fields be represented in terms of random variables. The midpoint method is used in this study. The field value for an element is assumed to be constant as the value at the centroid and represented by a random variable. The correlation coefficient between any two random variables is directly defined in terms of the autocorrelation function of the random field. One important consideration in this representation is the size of the random field element. This size is controlled by the correlation measure. In the first-order reliability method, the basic random variables V are transformed into a set of statistically independent, standard normal variables:

$$Y = Y(V) \tag{7}$$

Let Z_i and Z_j be a pair of standard normal variates obtained by marginal transformation of V_i and V_j. The correlation coefficient ρ'_{ij} between Z_i and Z_j can be expressed in terms of the correlation coefficient ρ_{ij} of V_i and V_j (Ang and Tang 1984, Der Kiureghian and Liu 1986), and the transformation to the standard normal space is then given by

$$y = T'z = T' \begin{Bmatrix} \Phi^{-1}[F_{V_1}(v_1)] \\ \vdots \\ \Phi^{-1}[F_{V_n}(v_n)] \end{Bmatrix} \tag{8}$$

in which $T' = (L')^{-1}$, where L' is the lower triangular matrix obtained from the Cholesky decomposition of the correlation matrix R' of Z. The preceding transformation of Eq. (8) is unique for an arbitrary number of variables with arbitrary marginal distributions and correlation coefficients and is computationally simpler than the Rosenblatt transformation.

5. Finite element reliability implementation

System reliability problems can be solved by replacing each individual limit-state surface by a first-order approximate surface at the corresponding minimum-distance point. Efficient solution methods for the optimization problem require the gradient vector of the limit-state function with respect to the basic variables. The gradient vector can be determined using the element partial stiffness matrices and load vectors, each of which is established analytically or numerically. By the finite element formulation, the nodal equilibrium equation for the whole structure is obtained as

$$KU = R \tag{9}$$

where K is the stiffness matrix, U the nodal displacement vector, and R the nodal load vector for the whole structure. The structural response S which are the stress intensity factors at the crack tip can be expressed in terms of U as

$$S = QU \tag{10}$$

where Q is the displacement-response transformation matrix which is obtained from Eq. (1). The basic random variables which are discretized from the corresponding random fields can be represented by a vector V. For convenience, the basic random variables can be divided into three groups: (i) material and geometry variables V_M such as Young's moduli, Poisson's ratios, coordinates of crack tips, (ii) load variables V_L such as distributed or concentrated loads and (iii) resistance variables V_R such as fracture toughness. The limit-state function can be expressed as an explicit function of resistance variables V_R and response quantities S, *i.e.*,

$$g(V) = g(V_R, S) \tag{11}$$

in which the structural response S is a function of the basic random variables. Using the chain rule of differentiation, the gradient vector of the limit-state function with respect to the basic random variable vector V is

$$\nabla_V g = \nabla_{V_R} g J_{V_R,V} + \nabla_S g J_{S,V} \tag{12}$$

where the computation of $\nabla_{V_R} g$, $\nabla_S g$ and $J_{V_R,V}$ are easily carried out in closed form. Since the response S is a function of material and geometry variables V_M and load variables V_L only, $J_{S,V}$ can be expressed with submatrices:

$$\begin{aligned} J_{S,V} &= [J_{S,V_M} \; J_{S,V_L} \; \mathbf{0}] \\ &= [C_Q \; \mathbf{0} \; \mathbf{0}] + QK^{-1}[-C_K \; J_{R,V_L} \; \mathbf{0}] \end{aligned} \tag{13}$$

where $C_Q = \left[\frac{\partial Q}{\partial V_{M_1}} U \cdots \frac{\partial Q}{\partial V_{M_m}} U\right]$ and $C_K = \left[\frac{\partial K}{\partial V_{M_1}} U \cdots \frac{\partial K}{\partial V_{M_m}} U\right]$, in which m is the size of the vector V_M. The matrices K, Q, C_K, C_Q and $J_{V_R,V}$ are first set up for each element and then assembled in global sense.

6. Probability of system failure

6.1. The first-order approximation

The limit-state function for a crack tip i in a given configuration j can be denoted as

$$G_{ij}(Y) = g_{ij}(V(Y)) = 0 \tag{14}$$

in which Y denotes the independent standard normal variables. The limit-state surface in the independent standard normal space may be replaced by its tangent hyperplane at the point nearest to the origin. This point is denoted by y^*. By expanding the limit-state function $G_{ij}(Y)$ in a Taylor series at the point y^*, the first-order approximation of the function $G_{ij}(Y)$ is as follows:

$$G_{ij}(Y) \cong \nabla G_{ij}(y^*)(Y - y^*) \tag{15}$$

where $\nabla G_{ij}(y^*)$ is the gradient of $G_{ij}(Y)$ computed at y^*. The minimum distance point can be obtained by the following iteration scheme (Rackwitz and Fiessler 1978):

$$y_{k+1} = \left[a_k^T y_k + \frac{G(y_k)}{|\nabla G(y_k)|}\right] a_k \tag{16}$$

where $a_k = -\nabla^T G(y_k)/|\nabla G(y_k)|$, and the gradient vector is obtained as

$$\nabla G = \nabla_V g J_{V,Y} = \nabla_V g J_{Y,V}^{-1} \tag{17}$$

6.2. Formulation of the stable configuration approach

In the stable configuration approach, the failure of the system can be defined as

$$E = \bigcap_{i=1}^{n} \bar{C}_i = \bigcap_{i=1}^{n} \left(\bigcup_{j=1}^{k_i} B_{ij} \right) \tag{18}$$

where $\bar{C}_i$ refers to cut-set i not being realized and B_{ij} is the failure of component j in cut-set i. For each B_{ij}, a performance function $G_j(Y)$ is defined such that $G_j(Y)<0$ and $G_j(Y)>0$ imply the failure and survival of component j, respectively. For structures exhibiting brittle behavior, the event $\bar{C}_i$ can be simplified as the event corresponding to the further damage of the configuration i (Quek and Ang 1990). For practical purposes, only a limited number of configurations

Table 1 Statistics for the example plates

Variable	Unit	Mean	c.o.v.	Distribution
Crack length a	in.	a	0.0	–
Applied load w	lb/in.	$\bar{w}$	0.15	normal
Modulus E	ksi	3×10^4	0.0	–
Poisson ratio ν	–	0.3	0.0	–
Toughness K_{Ic}	ksi$\sqrt{\text{in.}}$	43.0	0.15	normal (field)

can be considered. In addition to using only the stochastically dominant stable configurations, the number of essential configurations may be further reduced, because the configurations with low damage levels are more stable than those with high damage. This is particularly true for brittle structures. In mode I fracture, a crack propagates along the straight line, i.e., the crack extension angle measured from the initial crack line is 0°. The dominant stable configurations are naturally taken as the configurations which have the straight line cracks with appropriate crack extension. However, in the mixed-mode fracture the crack propagates along a curved path. The dominant stable configuration for the mixed-mode fracture can be taken as the configuration that has the crack extended from the current configuration in the direction determined by the fracture criteria using the basic random quantities corresponding to the minimum distance point of the limit-state surface in the independent normal space.

7. Examples

Three simple examples are presented to illustrate the reliability analysis of cracked structures. The fracture toughness and load are modeled as a random field and random variable, respectively. These examples can be solved using a conventional finite element program. The statistics of the variables are summarized in Table 1.

7.1. A plate with a single edge crack under tension

The example structure is a 1 in. thick, 5×20 in.^{2} rectangular plate with a 0.5 in. single edge crack as shown in Fig. 2. The autocorrelation coefficient function for the fracture toughness K_{Ic} is specified as

$$\rho(\Delta x,\ \Delta y)=\exp\left[-\frac{\Delta x^2+\Delta y^2}{(cL)^2}\right] \tag{19}$$

where $\Delta x^2+\Delta y^2$ is the square of the distance between any two points on the plate and L is taken as the plate width and c is a dimensionless measure of the correlation length. The correlation length, which is a measure of the fluctuation rate of a random field, may be defined as cL in Eq. (19). This example involves only mode I fracture where the crack extends straightly along the crack line. For the reliability analysis by the stable configuration approach, several finite element meshes for different crack length including the initial crack length are used. For the sake of simplicity, one half the plate is modeled in finite element analysis. In each mesh, two degenerate isoparametric elements are employed for the singularity at the crack tip and the remainder of the plate is modeled by regular 8-node isoparametric elements. Let Δa denote the difference between crack lengths in adjacent configurations. Thus, Δa also denotes the segment

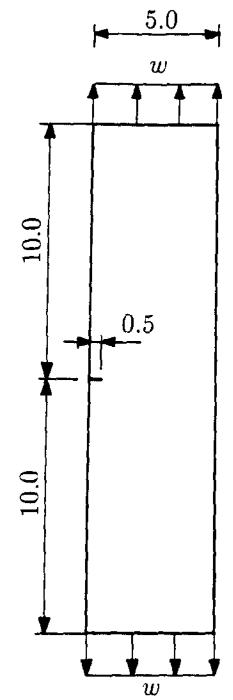

Fig. 2 A stochastic plate with a single edge crack under tensile load.

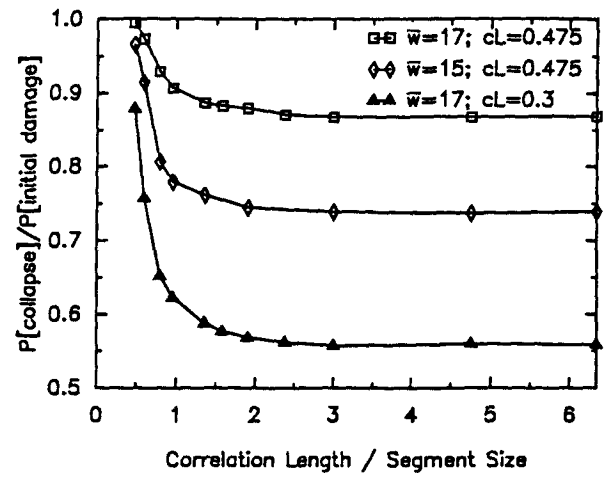

Fig. 3 The effect of the segment size upon the probability of collapse.

size of the fracture toughness field. For a short correlation length for the random field of fracture toughness, the rate of fluctuation is high and, thus, a small value of Δa is required in evaluation of the system reliability. To investigate the effect of Δa for a given correlation length cL, the probability of system failure, i.e., the probability of collapse is computed by simulation for different values of Δa as shown in Fig. 3, where three different cases are considered. From the results in Fig. 3 it is apparent that the convergence in the probability of collapse is effectively achieved when Δa is one third of the correlation length. Thus, the appropriate value of Δa is given as

$$\Delta a = \frac{cL}{3} \tag{20}$$

When $cL=0.475$ in. and $\bar{w}=15$ lb/in., the probability of collapse is obtained as 7.45×10^{-4} with seven dominant configurations, whereas the probability of fracture initiation is 1.01×10^{-3}.

7.2. A plate with two single edge cracks

The example structure is a 1 in. thick, 5×20 in.^{2} rectangular plate with two single edge cracks as shown in Fig. 4. Each crack length is 1.0 in. and the mean of the distributed load is 8.5 lb/in. This example also involves only mode I fracture. However, this example, in the presence of two cracks, has many failure modes which are correlated. The system failure probability can be effectively evaluated by the stable configuration approach. Using Eq. (20), Δa is taken as 0.2 in. when $cL=0.6$ in. By simulation, the probability of collapse is obtained as 4.7×10^{-4} with 19 configurations, whereas the probability of fracture initiation is 8.5×10^{-4}. To investigate

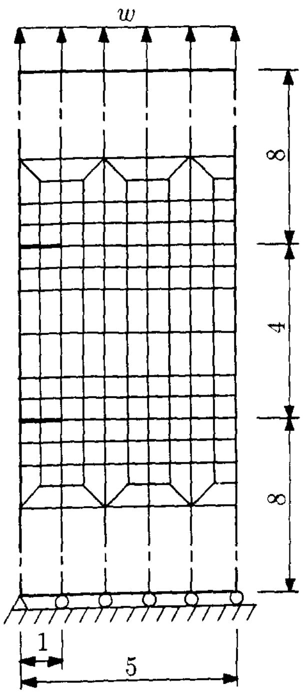

Fig. 4 Finite element mesh for the initial configuration of the plate with two single edge cracks.

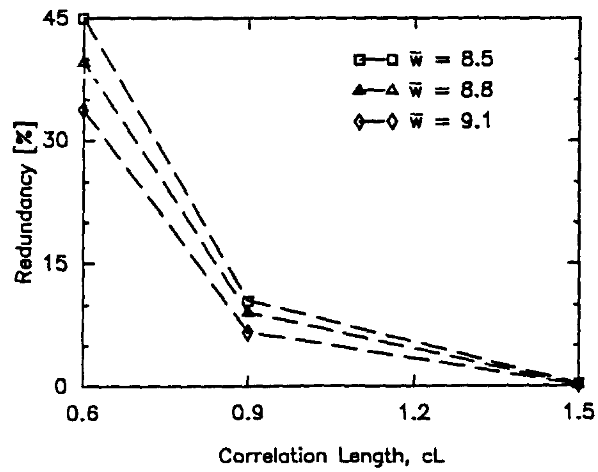

Fig. 5 Redundancies for different correlation lengths.

the effect of the correlation length of the fracture toughness on the probability of collapse, three different values of the correlation length, 0.6, 0.9 and 1.5 in., are considered. As a measure of redundancy in the structure, the percentage of redundancy is defined as

$$R=\left[1-\frac{P(\text{collapse})}{P(\text{initial damage})}\right]\times 100\% \tag{21}$$

Fig. 5 shows the redundancies for the different correlation lengths. When $cL=1.5$ in., the redundancy is zero, which means that collapse is imminent once any initial damage occurs.

7.3. A plate with a slant edge crack

The example structure is a 1 in. thick, 2.5×5 in.^{2} rectangular plate with a 45-degree slant edge crack as shown in Fig. 6. The crack length is 1.0 in. and the mean of the distributed load is 11.0 lb/in. This example involves modes I and II with the crack extending along a curved path. Available data suggest that $K_{IIc}=0.75\ K_{Ic}$. Thus, A in Eq. (3) is assumed to be 1.78. Fig. 7 shows the crack propagation path from which the possible configurations are determined. For $cL=0.225$ in., the probability of collapse is obtained as 5.44×10^{-4} with seven configurations, whereas the probability of fracture initiation is 7.08×10^{-4}.

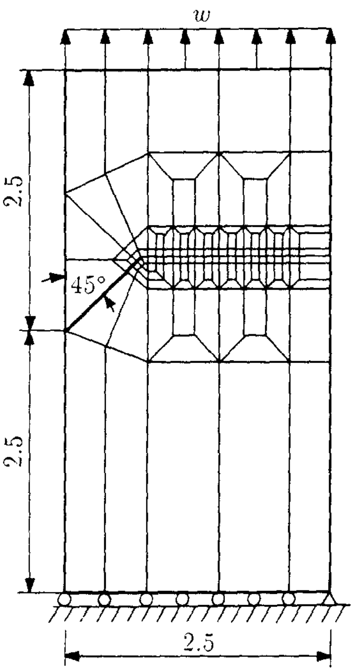

Fig. 6 Finite element mesh for the initial configuration of the plate with a slant edge crack.

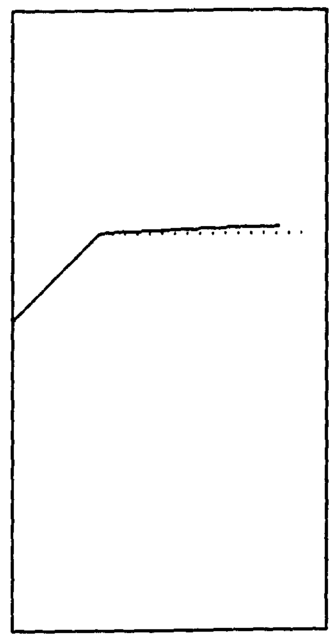

Fig. 7 Crack propagation path for the plate with a slant edge crack under tension.

8. Conclusions

The ultimate structural reliability of cracked structures is evaluated on the basis of the stable configuration approach and FEM. For random fracture toughness, the appropriate segment size is found as one third of the correlation length. The results show that for the small correlation length of the fracture toughness, the probability of ultimate system failure is smaller than the corresponding probability of fracture initiation, which is due to the reserved safety margin of the structure. The collapse probability appears to depend on the correlation length of the toughness.

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