Structural Engineering and Mechanics, Vol. 29, No. 5 (2008) 551-565
DOI: http://dx.doi.org/10.12989/sem.2008.29.5.551

Exact solution for asymmetric transient thermal and mechanical stresses in FGM hollow cylinders with heat source

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(Received August 10, 2006, Accepted February 25, 2008)

Abstract. Transient solution of asymmetric mechanical and thermal stresses for hollow cylinders made of functionally graded material is presented. Temperature distribution, as function of radial and circumferential directions and time, is analytically obtained, using the method of separation of variables and generalized Bessel function. A direct method is used to solve the Navier equations, using the Euler equation and complex Fourier series.

Keywords: transient; thermoelasticity; functionally graded material; hollow cylinder.

1. Introduction

Functionally graded materials are heterogeneous materials which consist of graded material variation from one surface to the other. These materials are useful to withstand high thermal stresses in applications where high heat fluxes and large temperature gradients exist. A ceramic rich region of a functionally graded material is exposed to hot temperature, while a metal rich region, providing the necessary flexibility, is exposed to the cold temperature.

In 2002, Yee and Moon (2002) presented plane thermal stress analysis of an orthotropic cylinder subjected to an arbitrary transient asymmetric temperature distribution. The thermoelastic solution was obtained by the stress function approach. The problem of transient thermal stresses in a solid

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elastic homogeneous and isotropic sphere was solved for uniform and nonuniform local surface heating by Cheung *et al*. (1974). In 1987, Chu and Tzou (1987) presented the transient response of a composite finite hollow cylinder heated by a moving line source on its inner boundary and cooled convectively on the exterior boundary using eigen function expansion method and the Fourier series. Yang *et al.* (1986) studied a transient response of one-dimensional axisymmetric quasistatic coupled thermoelastic problems. A numerical technique to analyze the one-dimensional transient temperature distributions in a hollow circular cylinder of functionally graded ceramic-metal based materials is presented by Awaji and Sivakumar (2002). Jane and Lee (1999) presented transient thermoelastic response of an infinitely long annular multilayered cylinder. They employed a numerical method to solve the thermoelastic response of infinite length cylinders subjected to known temperatures at traction-free inner and outer surfaces. Ashida and Noda (1995) studied the transient thermoelasticity in a transversely isotropic infinite cylinder containing a flat circular rigid inclusion. The numerical method of successive approximation, as well as the Fourier integral and the Bessel series, are used to satisfy the boundary conditions of displacement and stresses. Kim and Noda (2001) presented the Green's function approach to obtain the solution for transient thermal stresses of functionally graded material mediums. In this paper, transient temperature solution for a general heat conduction equation with a source and nonhomogeneous boundary conditions is obtained using the Green's function, where the solution is expressed by eigenvalues and corresponding eigen functions. Sugano *et al*. (1993) analyzed the transient thermal stresses in a hollow circular cylinder of functionally graded material with temperature-dependent material properties. The formulation is established by deriving the conditions necessary to assure the single valuedness of rotation and displacements in a hollow circular cylinder with arbitrary nonhomogeneous and temperature-dependent properties of material. Awaji and Sivakumar (2001) presented a numerical technique for the analysis of one-dimensional transient temperature distribution in a circular hollow cylinder that is made of functionally graded ceramic-metal based materials. The transient temperature and related thermal stresses in the FGM cylinder were analyzed numerically for a model of the mullite-molybdenum FGM system. Chen and Awaji (2003) studied the one-dimensional transient and residual stress fields in a hollow cylinder of functionally graded material for two models of ceramic-metal systems subjected to severe thermal shock. Bahtui and Eslami (2005) presented the coupled thermoelastic response of a functionally graded circular cylindrical shell. The Galerkin finite element approach was utilized to study the effect of functionally graded properties on the coupled stress fields. In 2002, Kim and Noda (2002) studied the Green's function approach to obtain the unsteady thermal stresses in an infinite hollow cylinder of functionally graded material. A Green's function approach based on the laminated theory was adopted for solving the two-dimensional unsteady temperature field.

This paper present an analytical method to obtain the transient thermal and mechanical stresses in a functionally graded hollow cylinder subjected to the two-dimensional asymmetric loads. Temperature distribution is assumed to be a function of radial and circumferential directions and time. The Navier equations are solved analytically using a direct method of series expansion. Material properties are assumed to be expressed by power functions in radial direction.

2. Transient temperature solution

Consider a functionally graded cylinder of inner radius r_i and outer radius r_o. The cylinder's material is graded through the r-direction. Heat conduction equation for the functionally graded cylinder is

$$\frac{1}{r}[rk(r)T_{,r}]_{,r} + \frac{1}{r^2}[k(r)T_{,\theta}]_{,\theta} = c(r)\rho(r)\dot{T} - R(r,\theta,t), \qquad -\pi \leq \theta \leq \pi \tag{1}$$

where $T(r,\theta,t)$ is the temperature distribution, $k(r)$ is the thermal conductivity, $c(r)$ is specific heat capacity, $\rho(r)$ is mass density, and $R(r,\theta,t)$ is the energy source. A comma denotes partial differentiation with respect to the space variable. The symbol dot $(\cdot)$ denotes derivative with respect to time. The initial condition and the Robin-type boundary conditions are assumed as

$$T(r,\theta,0) = f_1(r,\theta) \tag{2}$$

$$C_{11}T(r_i,\theta,t) + C_{12}T_{,r}(r_i,\theta,t) = 0 \tag{3}$$

$$C_{21}T(r_o,\theta,t) + C_{22}T_{,r}(r_o,\theta,t) = 0 \tag{4}$$

where C_{ij}, $i,j = 1,2$ are the constants related to the thermal boundary condition parameters, and $f_1(r,\theta)$ is the known initial condition. The thermal material properties are assumed to be described with the power law functions as

$$k(r) = k_0 r^{m_3}, \quad \rho(r) = \rho_0 r^{m_4}, \quad c(r) = c_0 r^{m_5} \tag{5}$$

where k_0, ρ_0, c_0 and m_3, m_4, m_5 are the material parameters. Using the definition for the material properties, the heat conduction equation becomes

$$\frac{\partial^2 T}{\partial r^2} + (m_3+1)\frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = \frac{c_0\rho_0}{k_0} r^{m_4+m_5-m_3}\frac{\partial T}{\partial t} - \frac{R}{k_0} r^{-m_3} \tag{6}$$

The solution of heat conduction equation may be obtained by the method of separation of variables, generalized Bessel function (Rice and Do 1995), and the eigen-function method as

$$T(r,\theta,t) = \sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty} r^{-\frac{m_3}{2}} C_p\left(\xi_{mn}\frac{r^f}{f}\right) G_{mn}(t) e^{in\theta} \tag{7}$$

Substituting Eq. (7) into the heat conduction equation yields

$$G_{mn}(t) = e^{-\frac{k_0}{c_0\rho_0}\xi_{mn}^2 t}\left[\frac{k_0}{c_0\rho_0}\int a_{mn}(t) e^{\frac{k_0}{c_0\rho_0}\xi_{mn}^2 t} dt + b_{mn}\right] \tag{8}$$

inwhich $a_{mn}(t)$ is the coefficient of complex Fourier series of heat source $R(r,\theta,t)$ as

$$a_{mn}(t) = \frac{1}{2\pi\|F_{mn}(r)\|^2}\int_{r_i}^{r_o}\int_{-\pi}^{\pi}\frac{R(r,\theta,t)}{k_0 r^{m_4+m_5}} W(r) F_{mn}(r) e^{-in\theta} d\theta dr \tag{9}$$

$$F_{mn}(r) = r^{-\frac{m_3}{2}} C_p\left(\xi_{mn}\frac{r^f}{f}\right) \tag{10}$$

where $F_{mn}(r)$ is derived from the general solution of energy equation without heat source. According to the mathematical Sturm-Liouville theorem, the function $F_{mn}(r)$ is an orthogonal function and $\|F_{mn}(r)\|$ is the norm of this function as

$$\|F_{mn}(r)\|^2 = \int_{r_i}^{r_o} W(r)F_{mn}^2(r)dr \tag{11}$$

where $W(r)$ is the weight function of $F_{mn}(r)$ as

$$W(r) = r^{m_3+2f-1} \tag{12}$$

and b_{mn} is derived from the initial thermal boundary condition defined by Eq. (2) as

$$b_{mn} = \frac{1}{2\pi\|F_{mn}(r)\|^2}\int_{r_i}^{r_o}\int_{-\pi}^{\pi} f_1(r,\theta)W(r)F_{mn}(r)e^{-in\theta}d\theta dr - \frac{k_0}{c_0\rho_0}\left[\int a_{mn}(t)e^{\frac{k_0}{c_0\rho_0}\xi_{mn}^2 t}dt\right]_{t=0} \tag{13}$$

and C_p is the mathematical Cylindrical Function given by

$$C_p\left(\xi_{mn}\frac{r^f}{f}\right) = J_p\left(\xi_{mn}\frac{r^f}{f}\right) + c_{mn}J_{-p}\left(\xi_{mn}\frac{r^f}{f}\right) \tag{14}$$

$$c_{mn} = -\frac{\left(C_{11}-\frac{C_{12}m_3}{2r_i}\right)J_p\left(\xi_{mn}\frac{r_i^f}{f}\right) + C_{12}\left[J_p'\left(\xi_{mn}\frac{r^f}{f}\right)\right]_{r=r_i}}{\left(C_{11}-\frac{C_{12}m_3}{2r_i}\right)J_{-p}\left(\xi_{mn}\frac{r_i^f}{f}\right) + C_{12}\left[J_{-p}'\left(\xi_{mn}\frac{r^f}{f}\right)\right]_{r=r_i}} \tag{15}$$

$$p = \frac{\sqrt{m_3^2+4n^2}}{m_4+m_5-m_3+2},\quad f=\frac{1}{2}(m_4+m_5-m_3+2) \tag{16}$$

Here, J_p is the Bessel function of the first kind of order p, the symbol ($'$) denotes derivative with respect to r, and the eigenvalues ξ_{mn} are the positive roots of (Cheung *et al.* 1974)

$$\begin{aligned}
&\left(C_{11}-\frac{C_{12}m_3}{2r_i}\right)J_p\left(\xi_{mn}\frac{r_i^f}{f}\right) + C_{12}\left[J_p'\left(\xi_{mn}\frac{r^f}{f}\right)\right]_{r=r_i}\times\\
&\left(C_{21}-\frac{C_{22}m_3}{2r_o}\right)J_{-p}\left(\xi_{mn}\frac{r_o^f}{f}\right) + C_{22}\left[J_{-p}'\left(\xi_{mn}\frac{r^f}{f}\right)\right]_{r=r_o}-\\
&\left(C_{11}-\frac{C_{12}m_3}{2r_i}\right)J_{-p}\left(\xi_{mn}\frac{r_i^f}{f}\right) + C_{12}\left[J_{-p}'\left(\xi_{mn}\frac{r^f}{f}\right)\right]_{r=r_i}\times\\
&\left(C_{21}-\frac{C_{22}m_3}{2r_o}\right)J_p\left(\xi_{mn}\frac{r_o^f}{f}\right) + C_{22}\left[J_p'\left(\xi_{mn}\frac{r^f}{f}\right)\right]_{r=r_o} = 0\\
&m = 1,2,3,\ldots;\quad n=-\infty,\ldots,\infty
\end{aligned} \tag{17}$$

3. Stress distribution

The governing two-dimensional strain-displacement relations in cylindrical coordinates are

$$\varepsilon_{rr} = u_{,r},\quad \varepsilon_{\theta\theta} = \frac{1}{r}(v_{,\theta}+u),\quad \varepsilon_{r\theta} = \frac{1}{2}\left(v_{,r}+\frac{1}{r}u_{,\theta}-\frac{v}{r}\right) \tag{18}$$

inwhich u and v are the displacement components along the radial and circumferential directions, respectively. The asymmetric stress-strain relations of a Hookean material are

$$\sigma_{rr} = \frac{E(r)}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}] - \frac{E(r)\alpha(r)}{1-2\nu}T(r,\theta,t)$$
$$\sigma_{\theta\theta} = \frac{E(r)}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_{\theta\theta} + \nu\varepsilon_{rr}] - \frac{E(r)\alpha(r)}{1-2\nu}T(r,\theta,t)$$
$$\sigma_{zz} = \frac{E(r)}{(1+\nu)(1-2\nu)}[\nu(\varepsilon_{rr} + \varepsilon_{\theta\theta})] - \frac{E(r)\alpha(r)}{1-2\nu}T(r,\theta,t)$$
$$\sigma_{r\theta} = \frac{E(r)}{1+\nu}\varepsilon_{r\theta} \tag{19}$$

where σ_{ij} and ε_{ij} $(i,j = r, \theta)$ are the stress and strain tensors, respectively, and

$$E(r) = E_0 r^{m_1} \tag{20}$$

$$\alpha(r) = \alpha_0 r^{m_2} \tag{21}$$

Here, $\alpha(r)$ is the coefficient of thermal expansion, $E(r)$ is the Young's modules, ν is the Poisson's ratio assumed to be a constant, and E_0, α_0, m_1, and m_2 are the constant material parameters. The two-dimensional equilibrium equations, disregarding the body forces and inertia terms, are

$$\sigma_{rr,r} + \frac{1}{r}\sigma_{r\theta,\theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$
$$\sigma_{r\theta,r} + \frac{\sigma_{\theta\theta,\theta}}{r} + \frac{2}{r}\sigma_{r\theta} = 0 \tag{22}$$

Appropriate formulations for the separation of variables to solve the equations of equilibrium are

$$u(r,\theta,t) = \sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty} u_{mn}(r,t)e^{in\theta} \tag{23}$$

$$v(r,\theta,t) = \sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty} v_{mn}(r,t)e^{in\theta} \tag{24}$$

Using Eq. (18) to Eq. (24) yield the Navier equations in the form

$$u''_{mn} + (m_1+1)\frac{1}{r}u'_{mn} + \left(\frac{\nu m_1}{1-\nu} - 1 - \frac{(1-2\nu)n^2}{2(1-\nu)}\right)\frac{1}{r^2}u_{mn} + \frac{in}{2(1-\nu)}\frac{1}{r}v'_{mn}$$
$$+ in\frac{(4+2m_1)\nu - 3}{2(1-\nu)}\frac{1}{r^2}v_{mn} = \frac{(1+\nu)\alpha_0}{1-\nu}\left[\left(m_1 + m_2 - \frac{m_3}{2}\right)r^{m_2 - \frac{m_3}{2} - 1}C_p\left(\xi_{mn}\frac{r^f}{f}\right)\right.$$
$$\left. + r^{m_2 - \frac{m_3}{2}}C'_p\left(\xi_{mn}\frac{r^f}{f}\right)\right]G_{mn}(t) \tag{25}$$

$$v''_{mn} + (m_1 + 1)\frac{1}{r}v'_{mn} - \left(m_1 + 1 + \frac{2(1-2\nu)n^2}{1-2\nu}\right)\frac{1}{r^2}v_{mn} + \frac{in}{1-2\nu}\frac{1}{r}u'_{mn}$$

$$+ in\left(\frac{3-4\nu}{1-2\nu} + m_1\right)\frac{1}{r^2}u_{mn} = in\frac{2(1+\nu)\alpha_0}{1-2\nu}\left[r^{m_2 - \frac{m_3}{2} - 1}C_p\left(\xi_{mn}\frac{r^f}{f}\right)\right]G_{mn}(t) \tag{26}$$

Eqs. (25) and (26) are a system of ordinary differential equations with non-constant coefficients, having general and particular solutions in the form

$$u_{mn}(r,t) = u_n^g(r) + u_{mn}^p(r,t)$$

$$v_{mn}(r,t) = v_n^g(r) + v_{mn}^p(r,t) \tag{27}$$

The general solution is assumed to be $u_n^g(r) = dr^{\zeta}$ and $v_n^g(r) = Cr^{\zeta}$. Substituting these assumptions into Eqs. (25) and (26) yield $C_{nj} = M_{nj}d_{nj}$. Finally the general solution is

$$u_n^g(r) = \sum_{j=1}^{4} d_{nj}r^{\zeta_{nj}}$$

$$v_n^g(r) = \sum_{j=1}^{4} M_{nj}d_{nj}r^{\zeta_{nj}}, \quad j = 1,2,3,4; \quad n \neq 0 \tag{28}$$

$$M_{nj} = -\frac{\left[\zeta_{nj}(\zeta_{nj}-1) + (m_1+1)\zeta_{nj} + \frac{\nu m_1}{1-\nu} - 1 - \frac{(1-2\nu)n^2}{2(1-\nu)}\right]}{ni\left[\frac{\zeta_{nj}}{2(1-\nu)} + \frac{(4+2m_1)\nu - 3}{2(1-\nu)}\right]}, \quad j = 1,2,3,4; \quad n \neq 0 \tag{29}$$

and ζ_{nj}, $n \neq 0$ are the eigenvalues of the eigen function

$$\left[\zeta_{nj}(\zeta_{nj}-1) + (m_1+1)\zeta_{nj} + \frac{\nu m_1}{1-\nu} - 1 - \frac{(1-2\nu)n^2}{2(1-\nu)}\right] \times$$

$$\left[\zeta_{nj}(\zeta_{nj}-1) + (m_1+1)\zeta_{nj} - m_1 - 1 - \frac{2(1-\nu)n^2}{1-2\nu}\right]$$

$$+ n^2\left[\frac{\zeta_{nj}}{2(1-\nu)} + \frac{(4+2m_1)\nu - 3}{2(1-\nu)}\right]\left[\frac{\zeta_{nj}}{1-2\nu} + \frac{3-4\nu}{1-2\nu} + m_1\right] = 0 \tag{30}$$

For the particular solution of the Navier equations, we consider the definition of the Bessel function as

$$J_p\left(\frac{\xi_{mn}}{f}r^f\right) = \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{1}{2f}\xi_{mn}r^f\right)^{2k+p}}{k!\,\Gamma(k+p+1)} \tag{31}$$

Substituting Eqs. (14) and (31) into Eqs. (25) and (26) yield

$$u''_{mn} + (m_1 + 1)\frac{1}{r}u'_{mn} + \left(\frac{\nu m_1}{1-\nu} - 1 - \frac{(1-2\nu)n^2}{2(1-\nu)}\right)\frac{1}{r^2}u_{mn} + \frac{in}{2(1-\nu)}\frac{1}{r}v'_{mn}$$

$$+ in\frac{(4+2m_1)\nu - 3}{2(1-\nu)}\frac{1}{r^2}v_{mn} = \sum_{k=0}^{\infty}(e_1 r^{\beta_{pk1}} + e_2 r^{\beta_{pk2}})G_{mn}(t) \tag{32}$$

$$v''_{mn} + (m_1 + 1)\frac{1}{r}v'_{mn} - \left(m_1 + 1 + \frac{2(1-2\nu)n^2}{1-2\nu}\right)\frac{1}{r^2}v_{mn} + \frac{in}{1-2\nu}\frac{1}{r}u'_{mn}$$

$$+ in\left(\frac{3-4\nu}{1-2\nu} + m_1\right)\frac{1}{r^2}u_{mn} = \sum_{k=0}^{\infty}(e_3 r^{\beta_{pk1}} + e_4 r^{\beta_{pk2}})G_{mn}(t) \tag{33}$$

where

$$\beta_{pk1} = (2k+p)f + m_2 - \frac{m_3}{2} - 1 \tag{34}$$

$$\beta_{pk2} = (2k-p)f + m_2 - \frac{m_3}{2} - 1 \tag{35}$$

The particular solution $u^p_{mn}(r)$ and $v^p_{mn}(r)$ are assumed to be

$$u^p_{mn}(r,t) = \sum_{k=0}^{\infty}(A_{mnk1} r^{\beta_{pk1}+2} + A_{mnk2} r^{\beta_{pk2}+2})G_{mn}(t)$$

$$v^p_{mn}(r,t) = \sum_{k=0}^{\infty}(A_{mnk3} r^{\beta_{pk1}+2} + A_{mnk4} r^{\beta_{pk2}+2})G_{mn}(t) \tag{36}$$

Substituting Eq. (36) into Eqs. (32) and (33) yield

$$\sum_{k=0}^{\infty}\{[e_5 A_{mnk1} + e_6 A_{mnk3}]r^{\beta_{pk1}} + [e_7 A_{mnk2} + e_8 A_{mnk4}]r^{\beta_{pk2}}\}G_{mn}(t)$$

$$= \sum_{k=0}^{\infty}[e_1 r^{\beta_{pk1}} + e_2 r^{\beta_{pk2}}]G_{mn}(t) \tag{37}$$

$$\sum_{k=0}^{\infty}\{[e_9 A_{mnk3} + e_{10} A_{mnk1}]r^{\beta_{pk1}} + [e_{11} A_{mnk4} + e_{12} A_{mnk2}]r^{\beta_{pk2}}\}G_{mn}(t)$$

$$= \sum_{k=0}^{\infty}[e_3 r^{\beta_{pk1}} + e_4 r^{\beta_{pk2}}]G_{mn}(t) \tag{38}$$

Equating the coefficients of identical power indexes, yield

$$A_{mnk1} = \frac{e_1 e_9 - e_3 e_6}{e_5 e_9 - e_6 e_{10}}, \quad A_{mnk3} = \frac{e_3 e_5 - e_1 e_{10}}{e_5 e_9 - e_6 e_{10}}$$

$$A_{mnk2} = \frac{e_4 e_8 - e_2 e_{11}}{e_8 e_{12} - e_7 e_{11}}, \quad A_{mnk4} = \frac{e_2 e_{12} - e_4 e_7}{e_8 e_{12} - e_7 e_{11}}, \quad n \neq 0 \tag{39}$$

The constants e_5 to e_{12} are given in the Appendix. For $n = 0$, Eqs. (32) and (33) are uncoupled into two ordinary differential equations, which yield

$$u_0(r) = \sum_{j=1}^{2}(d_{0j}r^{\zeta_{0j}}) + \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}(A_{m0k1}r^{\beta_{pok1}+2} + A_{m0k2}r^{\beta_{pok2}+2})G_{m0}(t)$$

$$v_0(r) = \sum_{j=3}^{4}(d_{0j}r^{\zeta_{0j}}) \tag{40}$$

where

$$\zeta_{01} = \frac{-m_1}{2} - \sqrt{\frac{m_1^2}{4} - \frac{\nu m_1}{1-\nu} + 1}, \quad \zeta_{02} = \frac{-m_1}{2} + \sqrt{\frac{m_1^2}{4} - \frac{\nu m_1}{1-\nu} + 1}$$

$$\zeta_{03} = 1, \quad \zeta_{04} = -(m_1+1)$$

$$A_{m0k1} = \frac{e_1}{e_5|_{n=0}}, \quad A_{m0k2} = \frac{e_2}{e_7|_{n=0}}, \quad p_o = p|_{n=o} \tag{41}$$

Using Eqs. (27), (28), (36) and (40), complete solutions for the displacement components are obtained. Using the strain-displacement and Hooke's relations, result into expression for stresses as

$$\sigma_{rr} = \frac{E_0}{(1+\nu)(1-2\nu)}\sum_{n=-\infty}^{\infty}\left\{\sum_{j=1, n\neq 0}^{4}[(1-\nu)\zeta_{nj} + \nu(inM_{nj}+1)]d_{nj}r^{\zeta_{nj}+m_1-1}\right.$$
$$+\sum_{m=1}^{\infty}<\sum_{k=0}^{\infty}[((1-\nu)(\beta_{pk1}+2)A_{mnk1} + \nu(inA_{mnk3}+A_{mnk1}))r^{\beta_{pk1}+m_1+1} + ((1-\nu)(\beta_{pk2}+2)A_{mnk2}$$
$$\left.+\nu(inA_{mnk4}+A_{mnk2}))r^{\beta_{pk2}+m_1+1}]G_{mn}(t) - (1+\nu)\alpha_0 G_{mn}(t)r^{m_2+m_1-\frac{m_3}{2}}C_p\left(\xi_{mn}\frac{r^f}{f}\right)>\right\}e^{in\theta}$$
$$+\frac{E_0}{(1+\nu)(1-2\nu)}\sum_{j=1}^{2}[(1-\nu)\zeta_{0j}+\nu]d_{0j}r^{\zeta_{0j}-1+m_1}$$

$$\sigma_{\theta\theta} = \frac{E_0}{(1+\nu)(1-2\nu)}\sum_{n=-\infty}^{\infty}\left\{\sum_{j=1, n\neq 0}^{4}[\nu\zeta_{nj} + (1-\nu)(inM_{nj}+1)]d_{nj}r^{\zeta_{nj}+m_1-1}\right.$$
$$+\sum_{m=1}^{\infty}<\sum_{k=0}^{\infty}[(\nu(\beta_{pk1}+2)A_{mnk1} + (1-\nu)(inA_{mnk3}+A_{mnk1}))r^{\beta_{pk1}+m_1+1} + (\nu(\beta_{pk2}+2)A_{mnk2}$$
$$\left.+(1-\nu)(inA_{mnk4}+A_{mnk2}))r^{\beta_{pk2}+m_1+1}]G_{mn}(t) - (1+\nu)\alpha_0 G_{mn}(t)r^{m_2+m_1-\frac{m_3}{2}}C_p\left(\xi_{mn}\frac{r^f}{f}\right)>\right\}e^{in\theta}$$
$$+\frac{E_0}{(1+\nu)(1-2\nu)}\sum_{j=1}^{2}[\nu\zeta_{0j}+(1-\nu)]d_{0j}r^{\zeta_{0j}-1+m_1}$$

$$\sigma_{zz} = \frac{E_0}{(1+\nu)(1-2\nu)}\sum_{n=-\infty}^{\infty}\left\{\sum_{j=1,n\neq 0}^{4}\nu[\zeta_{nj}+(inM_{nj}+1)]d_{nj}r^{\zeta_{nj}+m_1-1}\right.$$
$$+\sum_{m=1}^{\infty}<\sum_{k=0}^{\infty}\nu[((\beta_{pk1}+3)A_{mnk1}+inA_{mnk3})r^{\beta_{pk1}+m_1+1}+((\beta_{pk2}+3)A_{mnk2}$$
$$\left.+inA_{mnk4})r^{\beta_{pk2}+m_1+1}]G_{mn}(t)-(1+\nu)\alpha_0 G_{mn}(t)r^{m_2+m_1-\frac{m_3}{2}}C_p\left(\xi_{mn}\frac{r^f}{f}\right)>\right\}e^{in\theta}$$
$$+\frac{E_0\nu}{(1+\nu)(1-2\nu)}\sum_{j=1}^{2}[\zeta_{0j}+1]d_{0j}r^{\zeta_{0j}-1+m_1}$$

$$\sigma_{r\theta} = \frac{E_0}{2(1+\nu)}\sum_{n=-\infty,n\neq 0}^{\infty}\left\{\sum_{j=1}^{4}[in+(\zeta_{nj}-1)M_{nj}]d_{nj}r^{\zeta_{nj}+m_1+1}\right.$$
$$+\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}[inA_{mnk1}+(\beta_{pk1}+1)A_{mnk3})r^{\beta_{pk1}+m_1+1}$$
$$\left.+(inA_{mnk2}+(\beta_{pk2}+1)A_{mnk4})r^{\beta_{pk2}+m_1+1}]G_{mn}(t)\right\}e^{in\theta}$$
$$\frac{E_0}{2(1+\nu)}\sum_{j=3}^{4}d_{0j}(\zeta_{0j}-1)r^{\zeta_{0j}-1} \tag{42}$$

where d_{nj}, j = 1, 2, 3, 4, are four unknowns to be obtained from a suitable selection of the following mechanical boundary conditions

$$\begin{aligned} u(r_i,\theta) &= \varsigma_1(\theta), & v(r_i,\theta) &= \varsigma_3(\theta)\\ u(r_o,\theta) &= \varsigma_2(\theta), & v(r_o,\theta) &= \varsigma_4(\theta)\\ \sigma_{rr}(r_i,\theta) &= \varsigma_5(\theta), & \sigma_{r\theta}(r_i,\theta) &= \varsigma_7(\theta)\\ \sigma_{rr}(r_o,\theta) &= \varsigma_6(\theta), & \sigma_{r\theta}(r_o,\theta) &= \varsigma_8(\theta) \end{aligned}$$

The functions $\varsigma_1(\theta)$ to $\varsigma_8(\theta)$ are the mechanical boundary conditions known on the inner and outer radii. Expanding these boundary conditions into the complex Fourier series lead to a system of four linear equations to be solved for the constants d_{nj}, j = 1, 2, 3, 4.

4. Results and discussion

The proposed analytical solution is programmed into MATLAB (1994~2008) and an example of a cylinder with internal heat generation and initial temperature is solved. Consider a hollow functionally graded cylinder of inner radius r_i = 0.02 m and outer radius r_o = 0.024 m. The Poisson's ratio is assumed 0.3. The modulus of elasticity, the thermal coefficient of expansion, and the thermal conduction coefficient of the inner and outer radii are E_{in} = 66.2 Gpa, α_{in} = 10.3E-6/°C,

k_{in} = 18.1 W/mK, c_{in} = 808.3 J/kgK, ρ_{in} = 4410 kg/m^3, E_{out} = 117 Gpa, α_{out} = 7.11E-6/°C, k_{out} = 2.036 W/mK, c_{out} = 615.6 J/kgK, and ρ_{out} = 5600 kg/m^3, respectively (Reddy and Chin 1998). These numerical values correspond to $m_1 = 3.1236$, $m_2 = -2.0329$, $m_3 = -11.9839$, $m_4 = 1.3103$ and $m_5 = -1.4937$. The inside boundary is traction-free with zero temperature, and the outside boundary is fixed with zero temperature. The cylinder is assumed to be at the initial temperature of $T(r, \theta, 0) = 50\Gamma(100r)\cos(\theta)$°C, where Γ is the mathematical Gamma function. Therefore, the assumed boundary conditions result in $u(r_o, \theta) = 0$, $v(r_o, \theta) = 0$, $\sigma_{rr}(r_i, \theta) = 0$, and $\sigma_{r\theta}(r_i, \theta) = 0$. The cylinder is heated by the rate of energy generation per unit time and unit volume of $R = 6 \times 10^6 \times \frac{1}{r}\sin(5t)\cos(\theta)\frac{W}{m^3}$.

Fig. 1 illustrates the cylinder temperature at various angles θ over the course of 10 seconds. The temperature on the vertical axis is plotted against the time in seconds on the horizontal axis. All graph lines show that temperature decrease sharply in magnitude. The results are the sum of transient and steady state solutions that depend upon the initial condition for temperature and heat source, respectively. All graphs show that the transient solution damp after five seconds and the steady state solution remains. Fig. 2 shows the temperature along the radial direction. Temperature profiles are drawn for different times and angles θ. Because the magnitude of thermal conductivity of metal is higher than that of ceramic, cylinder is cooler in surfaces closer to the inside surface. Fig. 3 shows the hoop stress distribution versus time. Stress distribution is compressive at $\theta = 0$ and tensile at $\theta = 3\pi/4$. In case when $R = 0$, stresses vanish steadily. The curves associated with the non-zero heat source follow the sine-form pattern of the assumed heat source. Similar to the temperature distribution, the hoop stress distributions reach the steady state condition after five seconds. Fig. 4 highlights the radial stress distributions of the cylinder. The radial stress on the vertical axis is plotted against the radius on the horizontal axis. According to the given mechanical boundary conditions, stresses are zero at the inside surface. As may be seen, stresses decrease as

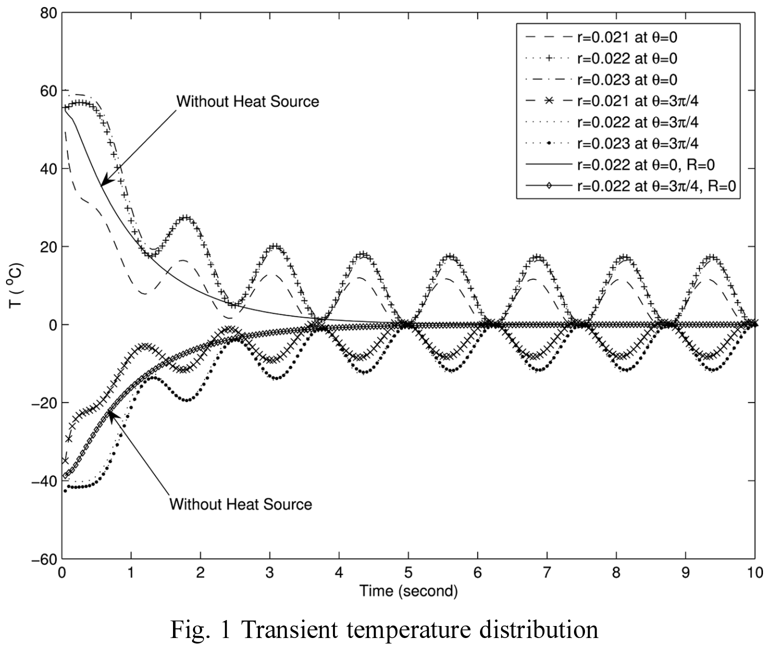

Fig. 1 Transient temperature distribution

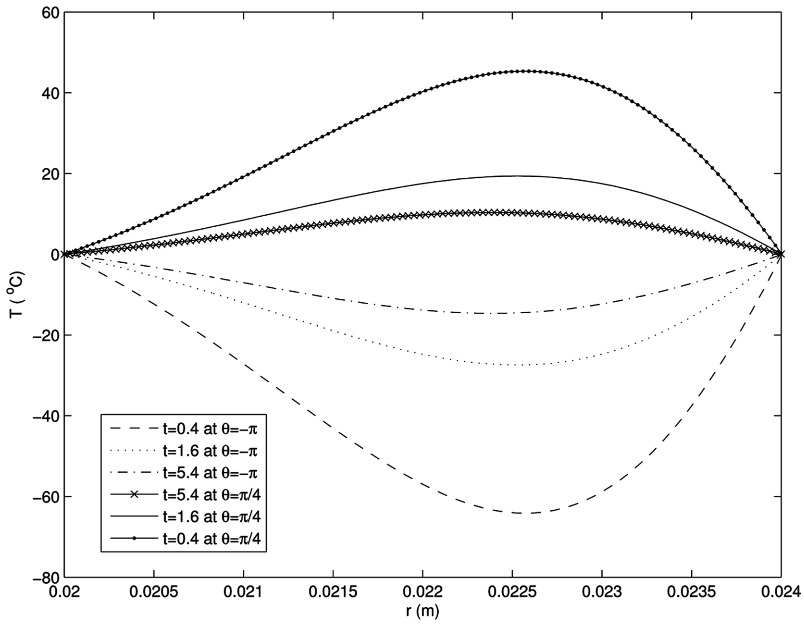

Fig. 2 Temperature distribution in the radial direction

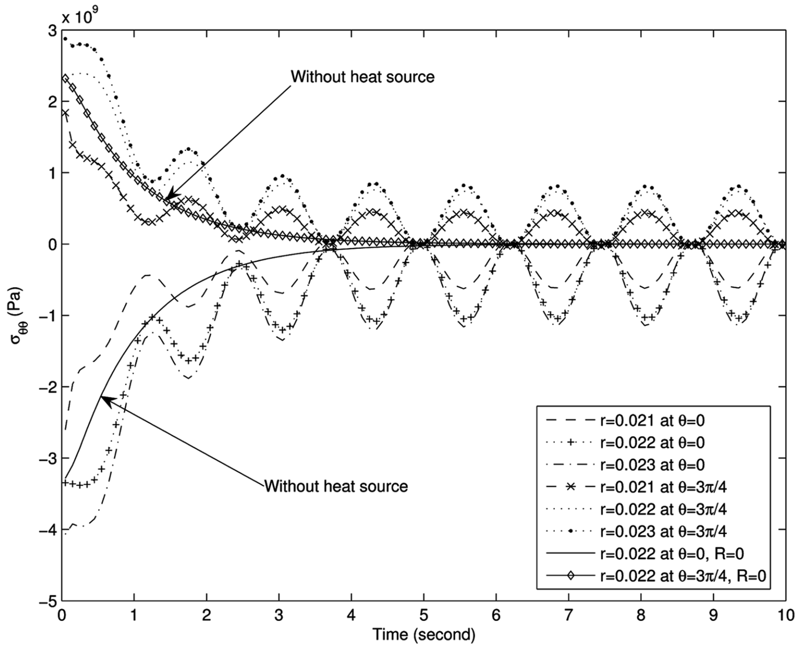

Fig. 3 Hoop stress

time increase. Fig. 5 shows the hoop stress along the radial direction. The shear stress distribution is plotted in Fig. 6. Shear stresses are zero at the inside surface. Comparing Figs. (4) and (5) indicates that the radial stress is larger than the hoop stress in magnitude. The reason is that the inside surface

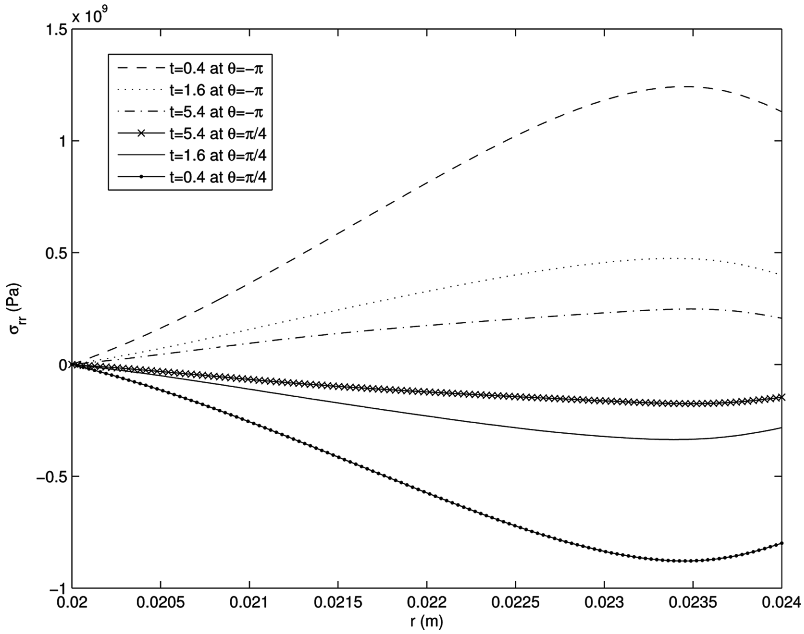

Fig. 4 Radial stress

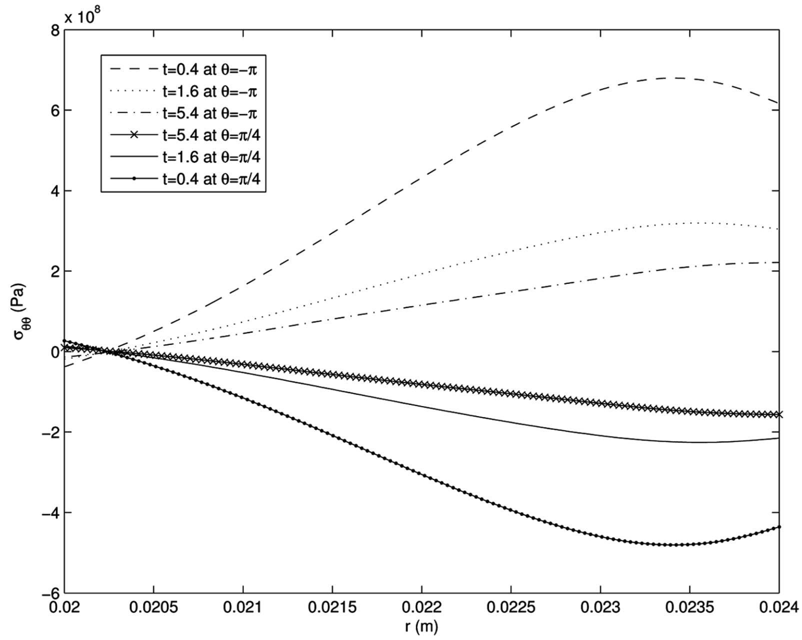

Fig. 5 Hoop stress in the radial direction

is traction free and the outside surface is fixed, with the given heat source which is inversely proportional to radius. The result, due to the assumed restraint on the outside surface, is larger radial thermal stress. With the assumed boundary conditions, the radial displacement is more restrained

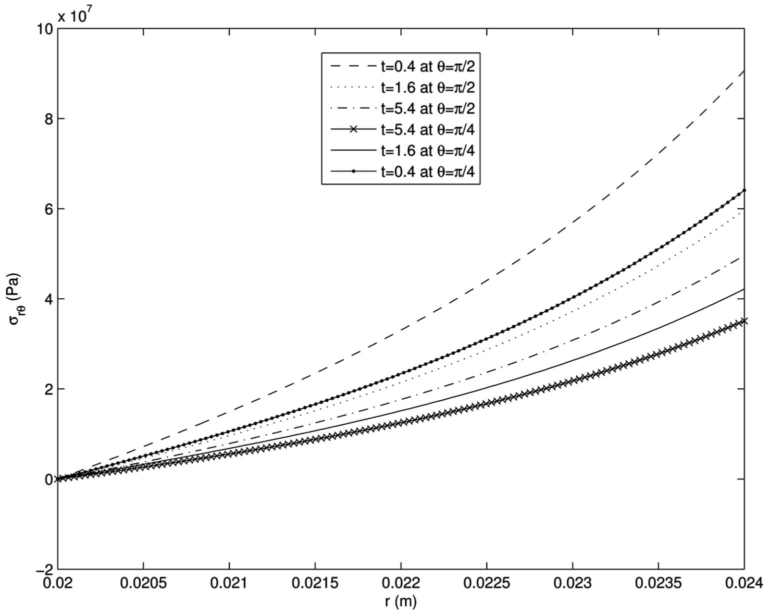

Fig. 6 Shear stress distribution

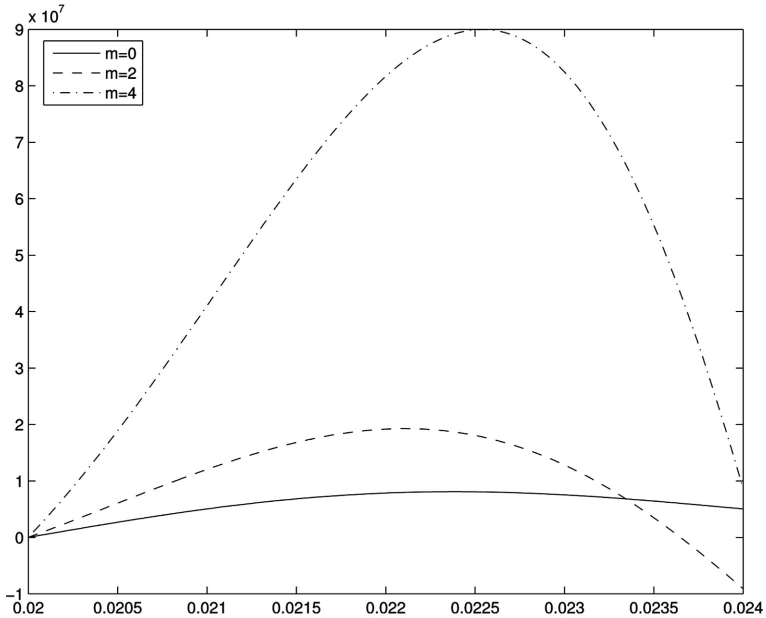

Fig. 7 Radial stress along the thickness with various power law indices at $t = 2$

than the hoop displacement, which produces larger radial stress compared to the hoop stress.

Now, consider the first example with identical power law indices of material properties, $m_1 = m_2 = m_3 = m_4 = m_5 = m$. The inside material properties are identical with the first example. Fig. 7 illustrates the effect of the power law index on the distribution of the radial thermal stress.

This figure is the plot of radial stress versus radial direction of the thick cylinder for different power law indices at $t = 2$ sec. The value of $m = 0$ corresponds to pure metal. According to Eq. (19) thermal stresses depend on modules of elasticity and thermal expansion coefficient. Since these parameters increase with the increase of power law parameter m, the radial thermal stress increases by the increase of m.

5. Conclusions

This paper presents a direct method of solution to obtain the transient mechanical and thermal stresses in a functionally graded hollow cylinder with heat source. The advantage of this method, compared to the conventional potential function method, is its mathematical strength to handle more general types of the mechanical and thermal boundary conditions. More complicated mechanical and thermal boundary conditions may be handled using the proposed method.

The distribution of radial and hoop stresses along the radial direction for some different types of boundary conditions are derived and plotted for the functionally graded cylinder. Study of Figs. 2, 4, and 5 indicates that a functionally graded thick cylinder may be tailored, with the selection of proper power law index, where the stress distribution along the radial direction become almost uniform. This is the very advantage of the use of functionally graded materials, where proper stress optimization may be obtained by the selection of a proper FGM profile.

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Appendix

$$e_1 = \frac{1+\nu}{1-\nu}\alpha_0\left(\frac{\xi_{mn}}{2f}\right)^{2k+p}\left[m_1 + m_2 - \frac{m_3}{2} + (2k+p)f\right]\frac{(-1)^k}{k!\Gamma(k+p+1)}$$

$$e_2 = \frac{1+\nu}{1-\nu}\alpha_0 c_{mn}\left(\frac{\xi_{mn}}{2f}\right)^{2k-p}\left[m_1 + m_2 - \frac{m_3}{2} + (2k-p)f\right]\frac{(-1)^k}{k!\Gamma(k-p+1)}$$

$$e_3 = \frac{2(1+\nu)}{1-2\nu}in\alpha_0\left(\frac{\xi_{mn}}{2f}\right)^{2k+p}\frac{(-1)^k}{k!\Gamma(k+p+1)}$$

$$e_4 = \frac{2(1+\nu)}{1-2\nu}in\alpha_0 c_{mn}\left(\frac{\xi_{mn}}{2f}\right)^{2k-p}\frac{(-1)^k}{k!\Gamma(k-p+1)}$$

$$e_5 = (\beta_{pk1}+2)(\beta_{pk1}+m_1+2) + \frac{\nu m_1}{1-\nu} - 1 - \frac{(1-2\nu)n^2}{2(1-\nu)}$$

$$e_6 = in\frac{(4+2m_1)\nu - 1 + \beta_{pk1}}{2(1-\nu)}$$

$$e_7 = (\beta_{pk2}+2)(\beta_{pk2}+m_1+2) + \frac{\nu m_1}{1-\nu} - 1 - \frac{(1-2\nu)n^2}{2(1-\nu)}$$

$$e_8 = in\frac{(4+2m_1)\nu - 1 + \beta_{pk2}}{2(1-\nu)}$$

$$e_9 = (\beta_{pk1}+2)(\beta_{pk1}+m_1+2) - m_1 - 1 - \frac{2(1-\nu)n^2}{1-2\nu}$$

$$e_{10} = \frac{in}{1-2\nu}(\beta_{pk1}+2) + in\left(\frac{3-4\nu}{1-2\nu} + m_1\right)$$

$$e_{11} = (\beta_{pk2}+2)(\beta_{pk2}+m_1+2) - m_1 - 1 - \frac{2(1-\nu)n^2}{1-2\nu}$$

$$e_{12} = \frac{in}{1-2\nu}(\beta_{pk2}+2) + in\left(\frac{3-4\nu}{1-2\nu} + m_1\right)$$