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Mesh distortion, locking and the use of metric trial functions for displacement type finite elements

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Abstract. The use of metric trial functions to represent the real stress field in what is called the unsymmetric finite element formulation is an effective way to improve predictions from distorted finite elements. This approach works surprisingly well because the use of parametric functions for the test functions satisfies the continuity conditions while the use of metric (Cartesian) shape functions for the trial functions attempts to ensure that the stress representation during finite element computation can retrieve in a best-fit manner, the actual variation of stress in the metric space. However, the issue of how to handle situations where there is locking along with mesh distortion has never been addressed. In this paper, we show that the use of a consistent definition of the constrained strain field in the metric space can ensure a lock-free solution even when there is mesh distortion. The three-noded Timoshenko beam element is used to illustrate the principles. Some significant conclusions are drawn regarding the optimal strategy for finite element modelling where distortion effects and field-consistency requirements have to be reconciled simultaneously.

Keywords: mesh distortion; locking; unsymmetric parametric-metric formulation; metric trial function; Timoshenko theory; three-node beam element.

1. Introduction

Conventional displacement type finite element formulations use identical trial and test functions (Galerkin elements) and perform well when used in regular meshes. When these meshes are distorted, their performance degrades rapidly and this has been well known (Stricklin *et al.* 1977, Backlund 1978, Gifford 1979, Arnold *et al.* 2002). Many efforts have been made over several decades to improve mesh distortion sensitivity.

Recently, Rajendran and co-workers (Rajendran and Liew 2003, Ooi *et al.* 2004, Rajendran and Subramanian 2004) introduced what they called the unsymmetric formulation. Here, two separate sets of shape functions, *viz*., the so-called *compatibility* (or *continuity*) enforcing isoparametric shape

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functions (in natural space) and the so-called *completeness* enforcing metric shape functions (in Cartesian space) are used tactically. The former satisfy exactly the minimum inter- as well as intra-element displacement continuity requirements, while the latter ensure all the (linear and where necessary, higher order) completeness requirements. Numerical results from test problems reveal that the unsymmetric elements (e.g., the new plane stress element, Rajendran and Liew 2003) can reproduce accurately displacement fields under various types of admissible mesh distortions only if the *continuity* enforcing shape functions are based on *isoparametric* functions and are actually used as the *test* functions while the *completeness* enforcing shape functions are based on *metric* forms and are used as the *trial* functions. The physical insight into why the test functions should be *continuity* enforcing and why this is ensured if *isoparametric* functions are adopted is easy to understand (Rajendran and Subramanian 2004).

Prathap *et al.* (2006) showed recently that the reason why the unsymmetric parametric-metric (PM) formulation has the greatest mesh distortion immunity is because the stress representation is managed in the metric (Cartesian) space. However, Prathap *et al.* (2007) showed later that the unsymmetric PM formulation, even though it is practically a very useful device to meet the continuity requirements and the best-fit stress recovery requirements simultaneously in a distorted element, is not strictly variationally correct. This is a very small price to pay for the great improvement in performance even under severe distortion.

So far, the issue of the performance of these elements in a regime where there is locking has not been specifically addressed. Prathap and Naganarayana (1992) showed that where there is locking, the distortion of the mesh causes this to be aggravated. Even if locking is removed by some enabling device (reduced integration, use of substitute functions, assumed strain field approached, etc.) for a uniform mesh, the performance will degrade dramatically when the mesh is distorted. The field-consistency requirement under mesh distortion is a tricky condition as their studies with the three-noded Timoshenko beam element showed.

In this paper, we implement an unsymmetric as well as a symmetric three-noded Timoshenko beam element with field-consistency features so that they perform well even under severe distortion. Both use metric shape functions for the trial functions. The shear strain is made field-consistent (Prathap 1993) in the metric space. Carefully designed numerical experiments show that in a one dimensional situation where continuity across element edges or surfaces is a non-issue, the symmetric consistent metric element performs more accurately than the unsymmetric element. The extension of this to general two-dimensional and three-dimensional will be an interesting challenge, as now the metric-metric (MM) approach will fail to ensure continuity, and only the parametric-metric (PM) will remain as a viable candidate. This will be attempted in future work.

2. The 3-noded Timoshenko beam element using parametric and metric functions

The three-noded isoparametric quadratic shear flexible beam element based on Timoshenko theory (henceforth, TB3 element) is well known and excellent descriptions of it are found in most textbooks. Therefore, the preliminary details of the formulation of the element, separately maintaining a bending stiffness matrix and a shear stiffness matrix, are omitted here. We shall instead focus attention on how the solution to the locking problem is complicated if the element's mid-node is displaced.

Fig. 1 shows the 3-noded beam element of length L with nodes at x_1, x_2 and x_3. We assume that

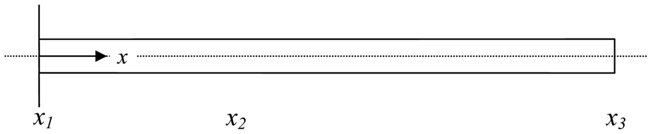

Fig. 1 The 3-noded beam element with nodes at x_1, x_2 and x_3

the node x_2 is not at the centre of the beam so that the distortion parameter $d = x_2 - L/2$. The parametric (better known as isoparametric) formulation requires the interpolation of the x coordinate using $x = N_i x_i$ and the displacement fields $w = N_i w_i$, $\theta = N_i \theta_i$, where N_i are the quadratic interpolation parametric functions, x_i are the nodal coordinates, w_i are the nodal vertical displacements and θ_i are the nodal rotation terms.

If r is the natural coordinate so that x_1 is at $r = -1$, x_2 is at $r = 0$ and x_3 is at $r = 1$, then x and the displacements and rotations are interpolated by

$$x = r(r-1)/2x_1 + (1-r^2)x_2 + r(r+1)/2x_3 \tag{1a}$$

$$w = r(r-1)/2w_1 + (1-r^2)w_2 + r(r+1)/2w_3 \tag{1b}$$

$$\theta = r(r-1)/2\theta_1 + (1-r^2)\theta_2 + r(r+1)/2\theta_3 \tag{1c}$$

Although a quadratic interpolation is assumed for the displacement field, we note that when node x_2 is not at the centre of the beam so that the distortion parameter d is non-zero, the bending strain $\kappa = d\theta/dx$ is no longer a linear function of x.

Indeed we have,

$$\kappa = \frac{d\theta}{dx} = \frac{(\theta_3 - \theta_1)/2 + r(\theta_1 + \theta_3 - 2\theta_2)}{(x_3 - x_1)/2 + r(x_1 + x_3 - 2x_2)} \tag{2}$$

where the denominator represents the Jacobian, J, governing the transformation from x to r spaces. If x_2 is not exactly mid-way between x_1 and x_3, J is no longer a constant and it is this that accounts partly for the inaccuracy of the distorted element (Prathap *et al.* 2006). Rajendran and co-workers (Rajendran and Liew 2003, Ooi *et al.* 2004, Rajendran and Subramanian 2004) proposed that distortion immunity is obtained if the interpolation for the real strain/stress is derived from trial functions in the metric, i.e., x space. The metric part of the formulation now uses $w = M_i w_i$, $\theta = M_i \theta_i$ where M_i are the quadratic metric functions. It is a simple exercise to derive these functions (Rajendran and Subramanian 2004).

A major part of the inaccuracy is, however, due to what is called the shear locking problem which is initiated by the shear strain term. The transverse shear strain can be written as

$$\gamma = \theta - \frac{dw}{dx} = \theta - \frac{dw/dr}{dx/dr} \tag{3}$$

The field-consistency requirements become very complicated (Prathap and Naganarayana 1992) due to the fact that θ is quadratic in x, and also in r, while the remaining term, which originates from the transverse displacement w, is linear in x but is transcendental in r due to the presence of the Jacobian (dx/dr) term in the denominator. It is this inconsistency that causes the poor accuracy (locking, slow convergence, stress oscillations) of the exactly integrated element. Prathap and Naganarayana (1992) experimented with several techniques to remove locking even under distortion. The optimal element was found to be one which had a linear variation of the strain fields in the natural (parametric) space. At that time, the question did not arise whether a better performance could have been achieved if the stress/strain had been defined to have a linear variation in the metric (Cartesian) space. The approach where the stress field is represented using metric (Cartesian) trial functions now allows this possibility and we shall implement it here. Thus a symmetric (MM) and unsymmetric (PM) formulations are potential candidates for further study.

3. Formulation of PP, MM and PM versions of the 3-noded beam element

The parametric shape functions for 3-noded beam element are defined as

$$N_1 = r(r-1)/2, \quad N_2 = (1-r^2) \text{ and } N_3 = r(r+1)/2 \tag{4}$$

From Eqs. (2), (3) and (4), the strain displacement matrix can be written as

$$\boldsymbol{B}_p = \begin{bmatrix} 0 & \frac{1}{J}\frac{\partial N_1}{\partial r} & 0 & \frac{1}{J}\frac{\partial N_2}{\partial r} & 0 & \frac{1}{J}\frac{\partial N_3}{\partial r} \\ -\frac{1}{J}\frac{\partial N_1}{\partial r} & N_1 & -\frac{1}{J}\frac{\partial N_2}{\partial r} & N_2 & -\frac{1}{J}\frac{\partial N_3}{\partial r} & N_3 \end{bmatrix} \tag{5}$$

Rajendran and co-workers have given a very elaborate account of the formulation of the unsymmetric problem and derivation of metric shape functions (Rajendran and Liew 2003, Ooi *et al.* 2004, Rajendran and Subramanian 2004). However, for the sake of completeness, this is summarized for 3-noded beam element as shown below.

The quadratic variation of transverse displacement w and rotation θ in metric space is of the form

$$w = a_0 + a_1 x + a_2 x^2 \tag{6a}$$

$$\theta = b_0 + b_1 x + b_2 x^2 \tag{6b}$$

w and θ interpolated with the metric shape functions are expressed as

$$w = M_i w_i \tag{7a}$$

$$\theta = M_i \theta_i \tag{7b}$$

The metric shape functions M_i at any point x for this element can be derived by solving the equations representing completeness conditions (Rajendran and Subramanian 2004) which must be satisfied in order to reproduce the displacement and rotation fields exactly as given by Eq. (6). These equations can be written in compact form as

$$\sum_{i=1}^{3} (M)_i x_i^p = x^p; \quad p = 0, 1, 2 \tag{8}$$

where the term i indicates node number and x^p correspond to those present in the displacement and rotation fields of Eq. (6).

Using Eqs. (2), (3) and (7), strain displacement matrix corresponding to metric shape functions are

$$\boldsymbol{B}_m = \begin{bmatrix} 0 & \frac{\partial M_1}{\partial x} & 0 & \frac{\partial M_2}{\partial x} & 0 & \frac{\partial M_3}{\partial x} \\ -\frac{\partial M_1}{\partial x} & M_1 & -\frac{\partial M_2}{\partial x} & M_2 & -\frac{\partial M_3}{\partial x} & M_3 \end{bmatrix} \tag{9}$$

The stiffness matrices for the PP, MM and PM elements are then

$$\mathbf{K}_{pp} = \int (\mathbf{B}_p)^T \mathbf{D}\mathbf{B}_p dx \tag{10}$$

$$\mathbf{K}_{mm} = \int (\mathbf{B}_m)^T \mathbf{D}\mathbf{B}_m dx \tag{11}$$

and

$$\mathbf{K}_{pm} = \int (\mathbf{B}_p)^T \mathbf{D}\mathbf{B}_m dx \tag{12}$$

The consistent load vector for the PP and PM cases is identical and is given by

$$\mathbf{f}_{pp} = \mathbf{f}_{pm} = \int (\mathbf{N})^T \mathbf{b} dx \tag{13}$$

where $\boldsymbol{b}$ is the body force. The consistent load vector for the MM element will then be

$$\mathbf{f}_{mm} = \int (\mathbf{M})^T \mathbf{b} dx \tag{14}$$

Numerical integration is used to derive the various matrices. The exactly integrated versions (taking a 4-pt Gaussian rule to be sufficiently accurate to achieve this even under severe distortion) will be called the PP4, MM4 and PM4 elements. One simple, and simplistic way, to eliminate locking is to apply the selective reduced integration using a 3-pt Gaussian integration rule for the bending energy and a 2-pt Gaussian rule for the shear energy. Where required to be introduced, we shall call these versions the PP2, MM2 and PM2 elements. Note that the PP and MM elements are symmetric elements while the PM element is based on an unsymmetric formulation. Prathap et al. (2007) showed that only the symmetric elements gave stresses which are a best fit of the exact solution while the unsymmetric element is slightly in error on this account.

4. Formulation of the field-consistent versions of the PM and MM beam elements

It has been well established (Prathap and Naganarayana 1992) that the PP element locks (poor performance, stress oscillations, etc.) in its original form based on isoparametric shape functions and exact numerical integration. The version based on selective numerical integration, PP2, is free of locking if the element is undistorted, but locking reappears when the mid-node is moved from the mid-point of the element. In the PP form, the element can be made free of locking where there is mesh distortion only by a non-trivial approach of ensuring consistency of the shear strains in the parametric space.

We shall now try to see how a field-consistent element, where consistency is taken to imply a linear variation of shear strain in the Cartesian (metric) space, can be implemented. From Eq. (3), we can see that if we can derive a substitute function $\overline{\theta}$, which is linear in x, to replace θ which is quadratic in x, $\overline{\theta}$ will be consistent with dw/dx in the Cartesian space. The substitute shear strain field $\overline{\gamma}$ will then be field-consistent even under severe distortion. Let the metric interpolation for the section rotation be written as

$$\theta = \frac{\theta_1(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + \frac{\theta_2(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + \frac{\theta_3(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} \tag{15}$$

Let this be re-written as

$$\theta = \sum_{i=1}^{3}(a_o + a_1 x + a_2 x^2)_i \theta_i \tag{16}$$

where typically, for node 1

$$(a_0)_1 = \frac{x_2 x_3}{(x_1-x_2)(x_1-x_3)}$$

$$(a_1)_1 = \frac{-(x_2+x_3)}{(x_1-x_2)(x_1-x_3)}$$

$$(a_2)_1 = \frac{1}{(x_1-x_2)(x_1-x_3)}$$

The coefficients for the other nodes follow a similar cyclic pattern.

Let the substitute shape function be written as

$$\overline{\theta} = \sum_{i=1}^{3}(b_o + b_1 x)_i \theta_i \tag{17}$$

It is known that the variationally correct manner to determine the coefficients $(b_j)_i$ from $(a_j)_i$ is to use the orthogonality condition (Prathap 1993)

$$\int_{x_1}^{x_3} \partial\overline{\theta}(\overline{\theta} - \theta)dx = 0 \tag{18}$$

The variational statement as given by Eq. (18) yields a set of two equations as follows:

$$\int_{x_1}^{x_3} (\overline{\theta} - \theta)dx = 0 \tag{19a}$$

$$\int_{x_1}^{x_3} x(\overline{\theta} - \theta)dx = 0 \tag{19b}$$

From Eq. (19), coefficients b_0 and b_1 can be easily derived.

Fig. 2 shows a typical representation of the parametric (N), metric (M) and smoothed metric (SM) shape functions at Node 1 for a case where the element spans from $x = 0$ to $x = 1$ and the mid node is at $x = 0.3$. These functions are then used to derive the field-consistent versions of PM and MM formulations, which we will call the PM4-C and MM4-C elements. It is to be emphasized here that for the shear terms only, the test functions will remain the original parametric and metric functions and the trial functions will now use the consistent smoothed metric functions. Only then will continuity be assured where several elements are assembled together.

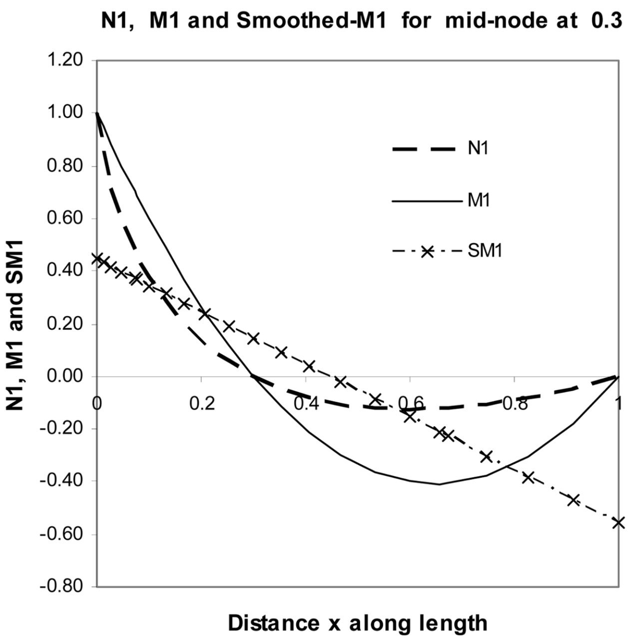

Fig. 2 A typical representation of the parametric (N), metric (M) and smoothed metric (SM) shape functions at Node 1

5. Numerical experiments with the 3-noded beam element

We take up a single element test where the beam is fixed at $x_1 = 0$ and is free at $x_3 = 10$. The length of the bar is therefore L = 10. The Young's modulus is taken as E = 1500, width of section w = 2 and depth t = 0.2 so that area of cross-section A = 0.4. Simpler loading cases like a concentrated moment M applied at the tip, or a concentrated load P applied at the tip which results in constant bending stress or linear bending stress and constant shear stress variations along the length of the beam do not allow us to discriminate clearly between the merits of the various symmetric and unsymmetric formulations. Therefore, we examine carefully the performance of the various elements for the case where a uniformly distributed load of intensity q = 0.16 is applied over the length of the beam. For the units assumed above, this will lead to a tip deflection w = 100, and shear stress $\tau = 0.4(L - x)$ and the bending stress at outermost fibre in the beam is $\sigma = 6(L - x)^2$.

As already outlined earlier, several elements are developed for the purpose of the present investigation. The PP elements are based on the standard symmetric formulation using the parametric interpolations for both trial and test functions. The PP4 element uses the 4-pt. Gaussian integration rule for the evaluation of the bending and shear stiffness matrices. The PP2 is the conventional way of using selective reduced integration to improve the performance of the three-noded Timoshenko beam element, using a 3-pt Gaussian integration rule for the bending energy and a 2-pt Gaussian rule for the shear energy. The PM element uses parametric interpolations for the test functions and metric interpolations for the trial functions and this is an unsymmetric formulation. The PM4 and PM3 and PM2 use the various exact and selectively reduced integration strategies respectively. The PM4-C version is based on using the substitute shape function for the section rotation in the manner described above. Similar interpretations apply for the MM variations.

Table 1 displays results from a single element test where the mid-node is placed at x = 5 and a uniformly distributed load of intensity q = 0.16 is applied vertically. This is a case where there is no distortion and the PP, PM and MM versions are identical. However, the consistent versions PM-C and MM-C are newly introduced. Table 1 shows that without reduced integration (PP4, PP3, etc.),

Table 1 Results from single element test where mid-node is placed at x = 5 and a uniformly distributed load of intensity q = 0.16 is applied vertically

x	Exact	PP4	PP3	PP2	PM4-C	PM3-C	MM4-C	MM3-C
Deflection in z-direction								
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5.00	35.42	16.75	16.74	33.34	33.36	33.36	33.35	33.33
10.00	100.00	66.82	66.77	99.99	100.03	100.06	100.03	99.94
Bending stress								
0.00	600.00	201.18	201.02	499.89	500.09	500.16	500.01	499.57
2.11	373.51	200.67	200.51	373.10	373.25	373.33	373.22	372.89
5.00	150.00	199.98	199.82	199.90	199.99	200.08	200.01	199.84
7.89	26.71	199.29	199.13	26.71	26.73	26.83	26.81	26.78
10.00	0.00	198.79	198.63	−100.08	−100.11	−100.00	−99.99	−99.90
Shear stress								
0.00	4.00	13.96	13.96	2503.87	4.00	4.00	4.00	4.00
2.11	3.16	3.15	3.15	3.16	3.16	3.15	3.15	3.15
5.00	2.00	−2.98	−2.98	−1247.94	2.00	2.00	2.00	2.00
7.89	0.84	0.85	0.84	0.84	0.84	0.84	0.84	0.84
10.00	0.00	9.96	9.96	2499.87	0.00	0.00	0.00	0.00

Table 2 Results from single element test where mid-node is placed at x = 4 and a uniformly distributed load of intensity q = 0.16 is applied vertically

x	Exact	PP4	PM4	MM4	PM4-C	MM4-C	Best-fit
Deflection in z-direction							
0.000	0.00	0.00	0.00	0.00	0.00	0.00	
4.000	24.32	9.66	17.44	10.95	25.10	24.02	
10.000	100.00	40.35	108.77	68.16	102.74	100.02	
Bending stress							
0	600.00	259.91	327.07	205.20	524.26	499.98	500.00
1.447	438.97	164.46	326.71	204.86	430.47	413.18	413.18
4.000	216.00	86.23	326.06	204.24	264.92	259.97	260.00
7.220	46.37	37.35	325.26	203.47	56.14	66.76	66.80
10.000	0.00	11.79	324.56	202.81	−124.10	−100.04	−100.00
Shear stress							
0	4.00	−212.20	11.96	13.89	4.00	4.00	4.00
1.447	3.42	75.36	4.34	5.93	3.42	3.42	3.42
4.000	2.40	−1.07	−2.71	−2.00	2.40	2.40	2.40
7.220	1.11	−43.84	0.09	−0.89	1.11	1.11	1.11
10.000	0.00	102.59	12.99	10.05	0.00	0.00	0.00

the elements are poor in performance both in prediction of deflections and stresses. The reduced integrated version (PP2), as is well known, shows very accurate deflections and bending stresses (consistent with a best fit of exact stresses) but the shear stresses, if computed directly from the strain-displacement relations, show wild oscillations about the Gauss points where they are correct. The PM-C and MM-C are remarkably accurate and for the undistorted element, there is no difference that the order of integration makes.

Table 2 displays results from a single element test where the mid-node is placed at x = 4 and a uniformly distributed load of intensity q = 0.16 is applied vertically. We see clearly that the best performance is obtained with the MM-C element. The bending stresses are now exactly the best-fit of the exact stresses in metric (Cartesian) space, as predicted. However, as we have seen for the bar element earlier (Prathap *et al.* 2007), the bending stresses from the PM4-C are clearly not a best-fit as the unsymmetric formulation no longer ensures this. There is a slight departure of the bending stress from the best fit trend and this is probably a function of the distortion parameter d. The elements where field-consistent trial functions have not been employed in representing the shear strain (PP4, PM4 and MM4) are clearly poor in performance.

Table 3 Results from single element test where mid-node is placed at $x = 5 + d$ and a uniformly distributed load of intensity $q = 0.16$ is applied vertically

d	Bending stress				Shear stress		
	x	Exact	PM4-C	MM4-C	Exact	PM4-C	MM4-C
−2.0	0.00	600.00	548.15	500.37	4.00	4.00	4.00
	0.78	510.06	493.87	453.54	3.69	3.69	3.69
	3.00	294.00	339.35	320.24	2.80	2.80	2.80
	6.55	71.27	92.03	106.87	1.38	1.38	1.38
	10.00	0.00	−147.85	−100.09	0.00	0.00	0.00
−1.5	0.00	600.00	535.93	500.55	4.00	4.00	4.00
	1.11	473.85	461.12	433.70	3.55	3.55	3.56
	3.50	253.50	300.74	290.37	2.60	2.60	2.60
	6.89	58.15	73.16	86.98	1.25	1.25	1.25
	10.00	0.00	−136.04	−99.98	0.00	0.00	0.00
−1.0	0.00	600.00	524.26	499.98	4.00	4.00	4.00
	1.45	438.97	430.47	413.18	3.42	3.42	3.42
	4.00	216.00	264.92	259.97	2.40	2.40	2.40
	7.22	46.37	56.14	66.76	1.11	1.11	1.11
	10.00	0.00	−124.10	−100.04	0.00	0.00	0.00
−0.5	0.00	600.00	511.74	499.95	4.00	4.00	4.00
	1.78	405.42	400.72	393.18	3.29	3.29	3.29
	4.50	181.50	231.06	230.00	2.20	2.20	2.20
	7.55	35.91	40.61	46.83	0.98	0.98	0.98
	10.00	0.00	−111.99	−99.94	0.00	0.00	0.00
0.0	0.00	600.00	500.09	500.01	4.00	4.00	4.00
	2.11	373.21	373.25	373.22	3.15	3.16	3.15
	5.00	150.00	199.99	200.01	2.00	2.00	2.00
	7.89	26.79	26.73	26.81	0.85	0.84	0.85
	10.00	0.00	−100.11	−99.99	0.00	0.00	0.00

Table 3 Continued

d	Bending stress				Shear stress		
	x	Exact	PM4-C	MM4-C	Exact	PM4-C	MM4-C
0.5	0.00	600.00	487.59	500.33	4.00	4.00	4.00
	2.45	342.32	346.80	353.44	3.02	3.02	3.02
	5.50	121.50	171.10	170.11	1.80	1.80	1.80
	8.22	19.01	14.59	6.80	0.71	0.71	0.71
	10.00	0.00	−87.83	−100.07	0.00	0.00	0.00
1.0	0.00	600.00	476.33	500.13	4.00	4.01	4.00
	2.78	312.78	322.75	333.32	2.89	2.89	2.89
	6.00	96.00	144.85	140.09	1.60	1.60	1.60
	8.55	12.56	3.79	−13.14	0.58	0.58	0.58
	10.00	0.00	−76.13	−99.95	0.00	0.00	0.00
1.5	0.00	600.00	464.33	500.75	4.00	4.00	4.00
	3.11	284.56	299.83	313.74	2.75	2.76	2.76
	6.50	73.50	120.88	110.30	1.40	1.40	1.40
	8.89	7.44	−5.23	−33.08	0.45	0.45	0.45
	10.00	0.00	−64.06	−99.95	0.00	0.00	0.00
2.0	0.00	600.00	452.61	501.31	4.00	4.01	4.00
	3.45	257.68	278.67	294.03	2.62	2.62	2.62
	7.00	54.00	99.34	80.33	1.20	1.20	1.20
	9.22	3.65	−12.70	−53.19	0.31	0.31	0.32
	10.00	0.00	−52.06	−100.09	0.00	0.00	0.00

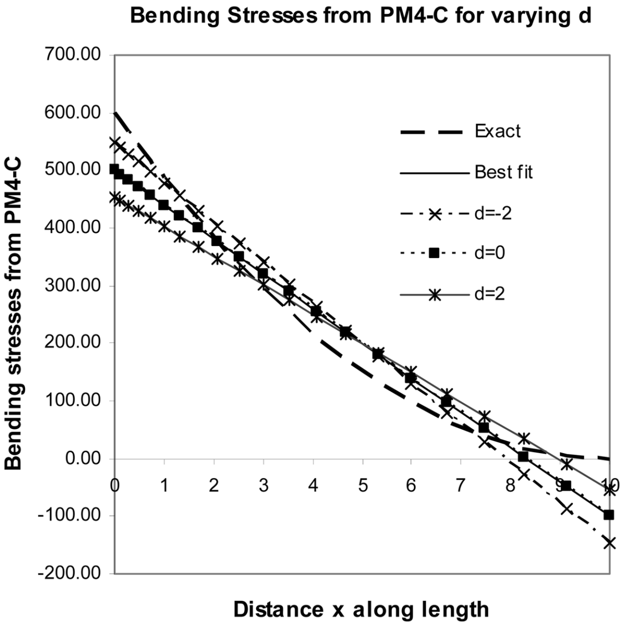

Fig. 3 Bending stresses from PM4-C for varying distortion parameter *d* show that the best-fit nature of finite element stresses is disturbed in the PM formulation

Table 3 shows that the departure of the bending stresses from the best fit is a function of the distortion parameter *d*. This implies that the loss of variational correctness (interpreted here as the closeness to the best fit) of the PM-C formulation is directly related to the degree of distortion. This is captured in Fig. 3 where the bending stresses from PM4-C for varying distortion parameter *d* show that the best-fit nature of finite element stresses is disturbed in the PM formulation. However, the MM-C is totally immune to mesh distortion.

5. Conclusions

In this paper, we have demonstrated that the use of a consistent definition of the constrained strain field along with the use of metric trial functions approach can ensure a lock-free solution even when there is mesh distortion. The three-noded Timoshenko beam element is used to illustrate the principles. Some key conclusions that emerge from this study regarding the optimal strategy for finite element modelling where distortion effects and field-consistency requirements have to be reconciled simultaneously can be tabulated as:

1. The PM or MM approach *per se* does not address the locking problem.
2. The stresses/strains that need to be consistently defined (no spurious constraints) must be represented in consistent form in the Cartesian (metric) space.
3. In a one-dimensional problem, continuity between elements is enforced at a point (and not across element edges or surfaces as in two and three dimensions) and therefore both PM and MM approaches are viable.
4. We have seen (Prathap *et al.* 2007) that the PM formulation is not variationally correct (i.e., it is not a best fit to the exact solution) while the MM formulation is. Thus where lack of continuity is not an issue, as in the one-dimensional problem here, the MM-C approach gives the best result.

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