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Two collinear Mode-I cracks in piezoelectric/piezomagnetic materials

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Abstract. In this paper, the behavior of two collinear Mode-I cracks in piezoelectric/piezomagnetic materials subjected to a uniform tension loading was investigated by the generalized Almansi's theorem. Through the Fourier transform, the problem can be solved with the help of two pairs of triple integral equations, in which the unknown variables were the jumps of displacements across the crack surfaces. To solve the triple integral equations, the jumps of displacements across the crack surfaces were directly expanded as a series of Jacobi polynomials to obtain the relations among the electric displacement intensity factors, the magnetic flux intensity factors and the stress intensity factors at the crack tips. The interaction of two collinear cracks was also discussed in the present paper.

Keywords: crack; piezoelectric/piezomagnetic materials; Fourier integral transform.

1. Introduction

The piezoelectric/piezomagnetic materials possesses piezoelectric, piezomagnetic and magneto-electric effects, thereby making the composite sensitive to elastic, electric and magnetic fields. Consequently, they are extensively used as electric packaging, sensors and actuators, e.g., magnetic field probes, acoustic/ultrasonic devices, hydrophones, and transducers with the responsibility of electro-magneto-mechanical energy conversion (Wu and Huang 2000). When subjected to mechanical, magnetic and electrical loads in service, magneto-electro-elastic composites can fail prematurely due to some defects, e.g., cracks, holes, etc. arising during their manufacturing processes. Therefore, it is of great importance to study the magneto-electro-elastic interaction and fracture behaviors of magneto-electro-elastic materials (Wu and Huang 2000, Sih and Song 2003, Song and Sih 2003, Wang and Mai 2003, Gao *et al.* 2003c,d, Spyropoulos *et al.* 2003).

For the fracture problem of piezoelectric/piezomagnetic materials, Liu *et al.* (2001) studied the generalized two-dimensional problem of an infinite magnetoelectroelastic plane with an elliptical hole using the Green's functions; Chung and Ting (1995) obtained the two-dimensional Green's functions for a magnetoelectroelastic anisotropic medium with an elliptical cavity or rigid inclusion; Pan (2002) derived the three-dimensional Green's functions in anisotropic

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magnetoelectroelastic bimaterials; Gao *et al*. (2003a,b) and Wang and Mai (2004) also studied the fracture problem of piezoelectric/piezomagnetic composites by the Stroh formalism; Chen *et al.* (2004) obtained the exact three-dimensional expressions for a full-space magneto-electro-thermo-elastic field with a penny-shaped crack subject to a uniform load on the crack surfaces using six harmonic functions; Wang and Shen (2002) obtained the general solution of three-dimensional problems in magnetoelectroelastic media using five potential functions. The development of piezoelectric/piezomagnetic composites has its roots in the early work of Van Suchtelen (1972)who proposed the combination of piezoelectric/piezomagnetic phases may exhibit a new material property—the magnetoelectric coupling effect. Since then, there have not been many researchers studying magnetoelectric coupling effect in $BaTiO_3$-$CoFe_2O_4$ composites, and most research results published were obtained in recent years (Wu and Huang 2000, Sih and Song 2003, Song and Sih 2003, Wang and Mai 2003, Gao *et al.* 2003a,b,c,d, Spyropoulos *et al.* 2003, Liu *et al.* 2001, Chung and Ting 1995, Pan 2002, Wang and Mai 2004, Chen *et al.* 2004, Wang and Shen 2002, Harshe *et al.* 1993, Avellaneda and Harshe 1994, Nan 1994, Benveniste 1995, Huang and Kuo 1997, Li 2000). Recently, the static fracture behavior of two parallel symmetry interface cracks and two collinear cracks in piezoelectric/piezomagnetic materials had been investigated in (Zhou and Wang 2004, Zhou *et al.* 2004, 2005b,c) by the Schmidt method (Morse and Feshbach 1958). However, they just concentrated on the anti-plane shear fracture problems in piezoelectric/ piezomagnetic materials.

In this paper, the similar problem that was treated by Gao *et al*. (2003b) was reworked using a somewhat different approach, named the Schmidt method (Morse and Feshbach 1958), i.e., the behavior of two collinear Mode-I cracks in piezoelectric/piezomagnetic materials subjected to a uniform tension loading was investigated by the generalized Almansi's theorem. The Fourier transform was used to reduce the mixed boundary value problem was reduced to two pairs of triple integral equations, in which the unknown variables are the jumps of displacements across the crack surfaces. To solve the triple integral equations, the jumps of displacements across the crack surface were directly expanded as a series of Jacobi polynomials to obtain the solution of the present paper. The solving process of the present paper was quite different from that adopted in the previous works (Wu and Huang 2000, Sih and Song 2003, Song and Sih 2003, Wang and Mai 2003, Gao *et al.* 2003a,b,c,d, Spyropoulos *et al.* 2003, Liu *et al.* 2001, Chung and Ting 1995, Pan 2002, Wang and Mai 2004, Chen *et al.* 2004, Wang and Shen 2002).

2. Basic equations of the piezoelectric/piezomagnetic materials

For the plane problem of linear elastic, homogeneous, transversely isotropic magnetoelectroelastic composite materials with vanishing body force, free charges and free magnetic fields, the basic equations are as follows (Song and Sih 2003, Wang and Mai 2003, Gao *et al.* 2003c,d, Spyropoulos *et al.* 2003, Liu *et al*. 2001)

$$\begin{cases} \dfrac{\partial \sigma_{xx}^{(j)}(x,z)}{\partial x} + \dfrac{\partial \sigma_{xz}^{(j)}(x,z)}{\partial z} = 0 \\ \dfrac{\partial \sigma_{xz}^{(j)}(x,z)}{\partial x} + \dfrac{\partial \sigma_{zz}^{(j)}(x,z)}{\partial z} = 0 \\ \dfrac{\partial D_x^{(j)}(x,z)}{\partial x} + \dfrac{\partial D_z^{(j)}(x,z)}{\partial z} = 0 \\ \dfrac{\partial B_x^{(j)}(x,z)}{\partial x} + \dfrac{\partial B_z^{(j)}(x,z)}{\partial z} = 0 \end{cases} \tag{1}$$

$$\begin{cases} \sigma_{xx}^{(j)} = c_{11}\dfrac{\partial u^{(j)}(x,z)}{\partial x} + c_{13}\dfrac{\partial w^{(j)}(x,z)}{\partial z} + e_{31}\dfrac{\partial \phi^{(j)}(x,z)}{\partial z} - f_{31}\dfrac{\partial \psi^{(j)}(x,z)}{\partial z} \\ \sigma_{zz}^{(j)} = c_{13}\dfrac{\partial u^{(j)}(x,z)}{\partial x} + c_{33}\dfrac{\partial w^{(j)}(x,z)}{\partial z} + e_{33}\dfrac{\partial \phi^{(j)}(x,z)}{\partial z} - f_{33}\dfrac{\partial \psi^{(j)}(x,z)}{\partial z} \\ \sigma_{xz}^{(j)} = c_{44}\Big(\dfrac{\partial u^{(j)}(x,z)}{\partial z} + \dfrac{\partial w^{(j)}(x,z)}{\partial x}\Big) + e_{15}\dfrac{\partial \phi^{(j)}(x,z)}{\partial x} - f_{15}\dfrac{\partial \psi^{(j)}(x,z)}{\partial x} \\ D_x^{(j)} = e_{15}\Big(\dfrac{\partial u^{(j)}(x,z)}{\partial z} + \dfrac{\partial w^{(j)}(x,z)}{\partial x}\Big) - \varepsilon_{11}\dfrac{\partial \phi^{(j)}(x,z)}{\partial x} - g_{11}\dfrac{\partial \psi^{(j)}(x,z)}{\partial x} \\ D_z^{(j)} = e_{31}\dfrac{\partial u^{(j)}(x,z)}{\partial x} + e_{33}\dfrac{\partial w^{(j)}(x,z)}{\partial z} - \varepsilon_{33}\dfrac{\partial \phi^{(j)}(x,z)}{\partial z} - g_{33}\dfrac{\partial \psi^{(j)}(x,z)}{\partial z} \\ B_x^{(j)} = f_{15}\Big(\dfrac{\partial u^{(j)}(x,z)}{\partial z} + \dfrac{\partial w^{(j)}(x,z)}{\partial x}\Big) + g_{11}\dfrac{\partial \phi^{(j)}(x,z)}{\partial x} - \mu_{11}\dfrac{\partial \psi^{(j)}(x,z)}{\partial x} \\ B_z^{(j)} = f_{31}\dfrac{\partial u^{(j)}(x,z)}{\partial x} + f_{33}\dfrac{\partial w^{(j)}(x,z)}{\partial z} + g_{33}\dfrac{\partial \phi^{(j)}(x,z)}{\partial z} - \mu_{33}\dfrac{\partial \psi^{(j)}(x,z)}{\partial z} \end{cases} \tag{2}$$

where $\sigma_{ik}^{(j)}(x,z), D_k^{(j)}(x,z)$ and $B_k^{(j)}(x,z)$ $(i = x,z, k = x,z, j = 1,2)$ are plane stresses, in-plane electric displacements and in-plane magnetic fluxes, respectively; $u^{(j)}(x,z)$ and $w^{(j)}(x,z)$ represent displacement components in the x- and z-directions, respectively; and $\phi^{(j)}(x,z)$ and $\psi^{(j)}(x,z)$ are electric potential and magnetic potential, respectively; c_{11}, c_{13}, c_{33} and c_{44} are elastic stiffness, respectively; ε_{11} and ε_{33} are dielectric constants, respectively; e_{15}, e_{31} and e_{33} are piezoelectric constants, respectively; f_{15}, f_{31} and f_{33} are piezomagnetic constants, respectively; g_{11} and g_{33} are electromagnetic constants, respectively; μ_{11} and μ_{33} are magnetic permeabilities, respectively. It should be noted that all the quantities with superscript j (j = 1, 2) correspond to the upper half plane 1 and the lower half plane 2 as shown in Fig. 1, respectively.

Substitution of Eq. (2) into Eq. (1) yields

$$\Big(c_{11}\frac{\partial^2}{\partial x^2} + c_{44}\frac{\partial^2}{\partial z^2}\Big)u^{(j)}(x,z) + (c_{13}+c_{44})\frac{\partial^2}{\partial x \partial z}w^{(j)}(x,z)$$
$$+(e_{31}+e_{15})\frac{\partial^2}{\partial x \partial z}\phi^{(j)}(x,z) + (-f_{31}-f_{15})\frac{\partial^2}{\partial x \partial z}\psi^{(j)}(x,z) = 0 \tag{3}$$

$$(c_{13}+c_{44})\frac{\partial^2}{\partial x\partial z}u^{(j)}(x,z)+\left(c_{44}\frac{\partial^2}{\partial x^2}+c_{33}\frac{\partial^2}{\partial z^2}\right)w^{(j)}(x,z)$$
$$+\left(e_{15}\frac{\partial^2}{\partial x^2}+e_{33}\frac{\partial^2}{\partial z^2}\right)\phi^{(j)}(x,z)+\left(-f_{15}\frac{\partial^2}{\partial x^2}-f_{33}\frac{\partial^2}{\partial z^2}\right)\psi^{(j)}(x,z)=0 \tag{4}$$

$$(e_{15}+e_{31})\frac{\partial^2}{\partial x\partial z}u^{(j)}(x,z)+\left(e_{15}\frac{\partial^2}{\partial x^2}+e_{33}\frac{\partial^2}{\partial z^2}\right)w^{(j)}(x,z)$$
$$+\left(-\varepsilon_{11}\frac{\partial^2}{\partial x^2}-\varepsilon_{33}\frac{\partial^2}{\partial z^2}\right)\phi^{(j)}(x,z)-\left(g_{11}\frac{\partial^2}{\partial x^2}+g_{33}\frac{\partial^2}{\partial z^2}\right)\psi^{(j)}(x,z)=0 \tag{5}$$

$$(f_{15}+f_{31})\frac{\partial^2}{\partial x\partial z}u^{(j)}(x,z)+\left(f_{15}\frac{\partial^2}{\partial x^2}+f_{33}\frac{\partial^2}{\partial z^2}\right)w^{(j)}(x,z)$$
$$+\left(g_{11}\frac{\partial^2}{\partial x^2}+g_{33}\frac{\partial^2}{\partial z^2}\right)\phi^{(j)}(x,z)-\left(\mu_{11}\frac{\partial^2}{\partial x^2}+\mu_{33}\frac{\partial^2}{\partial z^2}\right)\psi^{(j)}(x,z)=0 \tag{6}$$

3. The Mode-I crack

It is assumed that there are two collinear Mode-I Griffith cracks of length $1-l$ along the x-axis in piezoelectric/piezomagnetic materials as shown in Fig. 1. $2l$ is the distance between the two collinear cracks (The solution of two collinear cracks of length $r-l$ in piezoelectric/piezomagnetic materials can easily be obtained by a simple change in the numerical values of the present paper for crack length $1-l/r$. $r>l>0$). As discussed in Parton (1976), the crack is very thin. So, it is assumed that the electric potential, the magnetic potential, the normal electric displacement and the normal magnetic flux are continuous across the crack surfaces in the present paper, i.e. the permeable crack mode is adopted in the present paper. It is assumed that a distributed normal stress loading $\sigma_{zz}(x,0)=-\tau_0(x)$ was directly applied on the upper and lower crack surfaces, which is equivalent to investigating the perturbation fields for a remotely loaded cracked-body through the standard superposition technique in fracture mechanics. So the boundary conditions along the crack surfaces can be written as follows

$$\sigma_{xz}^{(1)}(x,0^+)=\sigma_{xz}^{(2)}(x,0^-)=0,\quad \sigma_{zz}^{(1)}(x,0^+)=\sigma_{zz}^{(2)}(x,0^-)=-\tau_0,\quad l\le|x|\le 1 \tag{7}$$

$$\begin{cases} u^{(1)}(x,0^+)=u^{(2)}(x,0^-), & w^{(1)}(x,0^+)=w^{(2)}(x,0^-) \\ \sigma_{zz}^{(1)}(x,0^+)=\sigma_{zz}^{(2)}(x,0^-), & \sigma_{xz}^{(1)}(x,0^+)=\sigma_{xz}^{(2)}(x,0^-) \end{cases},\quad |x|<l,|x|>1 \tag{8}$$

$$\begin{cases} \phi^{(1)}(x,0^+)=\phi^{(2)}(x,0^-) \\ \psi^{(1)}(x,0^+)=\psi^{(2)}(x,0^-) \\ D_z^{(1)}(x,0^+)=D_z^{(2)}(x,0^-) \\ B_z^{(1)}(x,0^+)=B_z^{(2)}(x,0^-) \end{cases},\quad |x|\ge 0 \tag{9}$$

where τ_0 is a magnitude of the uniform stress loading.

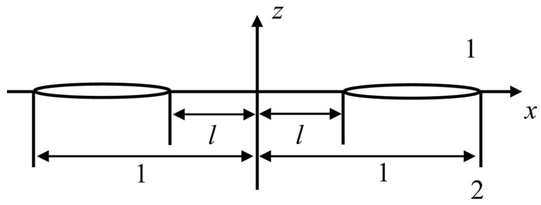

Fig. 1 The coordinate system for two collinear crack in piezoelectric/piezomagnetic materials

4. Solution procedures

Eqs. (3)-(6) can be solved by use of the method given by Yang (2001). As expression in Yang's work (2001), Eqs. (3)-(6) can be rewritten as follows

$$[MD]\begin{Bmatrix} u^{(j)}(x,z) \\ w^{(j)}(x,z) \\ \phi^{(j)}(x,z) \\ \psi^{(j)}(x,z) \end{Bmatrix} = 0 \tag{10}$$

where the operator is

$$[MD] = \begin{bmatrix} c_{11}\frac{\partial^2}{\partial x^2}+c_{44}\frac{\partial^2}{\partial z^2} & (c_{13}+c_{44})\frac{\partial^2}{\partial x\partial z} & (e_{31}+e_{15})\frac{\partial^2}{\partial x\partial z} & (-f_{31}-f_{15})\frac{\partial^2}{\partial x\partial z} \\ (c_{13}+c_{44})\frac{\partial^2}{\partial x\partial z} & c_{44}\frac{\partial^2}{\partial x^2}+c_{33}\frac{\partial^2}{\partial z^2} & e_{15}\frac{\partial^2}{\partial x^2}+e_{33}\frac{\partial^2}{\partial z^2} & -f_{15}\frac{\partial^2}{\partial x^2}-f_{33}\frac{\partial^2}{\partial z^2} \\ (e_{15}+e_{31})\frac{\partial^2}{\partial x\partial z} & e_{15}\frac{\partial^2}{\partial x^2}+e_{33}\frac{\partial^2}{\partial z^2} & -\varepsilon_{11}\frac{\partial^2}{\partial x^2}-\varepsilon_{33}\frac{\partial^2}{\partial z^2} & -g_{11}\frac{\partial^2}{\partial x^2}-g_{33}\frac{\partial^2}{\partial z^2} \\ (f_{15}+f_{31})\frac{\partial^2}{\partial x\partial z} & f_{15}\frac{\partial^2}{\partial x^2}+f_{33}\frac{\partial^2}{\partial z^2} & g_{11}\frac{\partial^2}{\partial x^2}+g_{33}\frac{\partial^2}{\partial z^2} & -\mu_{11}\frac{\partial^2}{\partial x^2}-\mu_{33}\frac{\partial^2}{\partial z^2} \end{bmatrix}$$

The determinant of matrix $[MD]$ is

$$\det[MD] = a\frac{\partial^8}{\partial z^8}+b\frac{\partial^8}{\partial x^2\partial z^6}+c\frac{\partial^8}{\partial x^4\partial z^4}+d\frac{\partial^8}{\partial x^6\partial z^2}+e\frac{\partial^8}{\partial x^8}$$

where a, b, c, d and e are given in the Appendix. They are constants which only depend on the properties of materials.

Based on the cofactors Δ_{ik} of $\det[MD]$ (i, k = 1, 2, 3, 4), and the method developed in Chen *et al.* (2004), the general solution of Eq. (10) can be expressed as follows

$$[u^{(j)}(x,z), w^{(j)}(x,z), \phi^{(j)}(x,z), \psi^{(j)}(x,z)]^T = (\Delta_{i1},\Delta_{i2},\Delta_{i3},\Delta_{i4})^T F^{(j)}(x,z),\quad (i=1,2,3,4) \tag{11}$$

with $F^{(j)}(x,z)$ satisfying the following equation

$$\det[MD]F^{(j)}(x,z) = 0 \tag{12}$$

In the following analysis, we use only $(\Delta_{21},\Delta_{22},\Delta_{23},\Delta_{24})$ for the present problem, which can be expressed as follows

$$\Delta_{21} = \alpha_{11}\frac{\partial^6}{\partial x^5 \partial z} + \alpha_{12}\frac{\partial^6}{\partial x^3 \partial z^3} + \alpha_{13}\frac{\partial^6}{\partial x \partial z^5} \tag{13}$$

$$\Delta_{22} = \alpha_{21}\frac{\partial^6}{\partial x^6} + \alpha_{22}\frac{\partial^6}{\partial x^4 \partial z^2} + \alpha_{23}\frac{\partial^6}{\partial x^2 \partial z^4} + \alpha_{24}\frac{\partial^6}{\partial z^6} \tag{14}$$

$$\Delta_{23} = \alpha_{31}\frac{\partial^6}{\partial x^6} + \alpha_{32}\frac{\partial^6}{\partial x^4 \partial z^2} + \alpha_{33}\frac{\partial^6}{\partial x^2 \partial z^4} + \alpha_{34}\frac{\partial^6}{\partial z^6} \tag{15}$$

$$\Delta_{24} = \alpha_{41}\frac{\partial^6}{\partial x^6} + \alpha_{42}\frac{\partial^6}{\partial x^4 \partial z^2} + \alpha_{43}\frac{\partial^6}{\partial x^2 \partial z^4} + \alpha_{44}\frac{\partial^6}{\partial z^6} \tag{16}$$

where α_{ik} $(i = 1,2,3,4;\ k = 1,2,3,4)$ can be obtained as shown in the Appendix. They are constants which only depend on the properties of materials.

Using the symmetry on x-axis and the Fourier transform on x, $F^{(j)}(x,z)$ can be expressed as follow

$$F^{(j)}(x,z) = \frac{2}{\pi}\int_0^\infty f^{(j)}(s,z)\cos(sx)ds \tag{17}$$

Substitution of Eq. (17) into Eq. (12) yields

$$a\frac{\partial^8 f^{(j)}(x,z)}{\partial z^8} - bs^2\frac{\partial^6 f^{(j)}(x,z)}{\partial z^6} + cs^4\frac{\partial^4 f^{(j)}(x,z)}{\partial z^4} - ds^6\frac{\partial^2 f^{(j)}(x,z)}{\partial z^2} + es^8 f^{(j)}(x,z) = 0 \tag{18}$$

which is a homogeneous equation and the solution of $f^{(j)}(s,z)$ is a function of $\exp(-\lambda sz)$ in which λ is the root of the following algebraic equation

$$a\lambda^8 - b\lambda^6 + c\lambda^4 - d\lambda^2 + e = 0 \tag{19}$$

which is determined by

$$\begin{cases} \lambda_1^2 = \dfrac{b}{4a} - \dfrac{1}{2}\sqrt{R_5+R_6} - \dfrac{1}{2}\sqrt{2R_5 - R_6 - \dfrac{\dfrac{b^3}{a^3} - \dfrac{4bc}{a^2} + \dfrac{8d}{a}}{4\sqrt{R_5+R_6}}} \\ \lambda_2^2 = \dfrac{b}{4a} - \dfrac{1}{2}\sqrt{R_5+R_6} + \dfrac{1}{2}\sqrt{2R_5 - R_6 - \dfrac{\dfrac{b^3}{a^3} - \dfrac{4bc}{a^2} + \dfrac{8d}{a}}{4\sqrt{R_5+R_6}}} \\ \lambda_3^2 = \dfrac{b}{4a} + \dfrac{1}{2}\sqrt{R_5+R_6} - \dfrac{1}{2}\sqrt{2R_5 - R_6 + \dfrac{\dfrac{b^3}{a^3} - \dfrac{4bc}{a^2} + \dfrac{8d}{a}}{4\sqrt{R_5+R_6}}} \\ \lambda_4^2 = \dfrac{b}{4a} + \dfrac{1}{2}\sqrt{R_5+R_6} + \dfrac{1}{2}\sqrt{2R_5 - R_6 + \dfrac{\dfrac{b^3}{a^3} - \dfrac{4bc}{a^2} + \dfrac{8d}{a}}{4\sqrt{R_5+R_6}}} \end{cases} \tag{20}$$

where $R_1 = 2c^3-9bcd+27ad^2+27b^2e-72ace$, $R_2 = c^2-3bd+12ae$, $R_3 = \sqrt{-4(R_2)^3+(R_1)^2}$

$R_4 = \dfrac{(R_1+R_3)^{1/3}}{2^{1/3}}$, $R_5 = \dfrac{b^2}{4a^2}-\dfrac{2c}{3a}$, $R_6 = \dfrac{R_2}{3aR_4}+\dfrac{R_4}{3a}$

Depending on the properties of λ^2, the function $f^{(j)}(s,z)$ has five different general solutions (for $z\geq 0, j = 1$) (Other cases can be obtained using a similar method, but they are omitted in the present paper for brevity.)

(a) If $\lambda_1^2\neq\lambda_2^2\neq\lambda_3^2\neq\lambda_4^2>0$, then

$$f^{(1)}(s,z) = A_1(s)e^{-\lambda_1 sz}+A_2(s)e^{-\lambda_2 sz}+A_3(s)e^{-\lambda_3 sz}+A_4(s)e^{-\lambda_4 sz} \tag{21}$$

(b) If $\lambda_1^2\neq\lambda_2^2\neq\lambda_3^2=\lambda_4^2>0$, then

$$f^{(1)}(s,z) = A_1(s)e^{-\lambda_1 sz}+A_2(s)e^{-\lambda_2 sz}+A_3(s)e^{-\lambda_3 sz}+A_4(s)sze^{-\lambda_3 sz} \tag{22}$$

(c) If $\lambda_1^2\neq\lambda_2^2=\lambda_3^2=\lambda_4^2>0$, then

$$f^{(1)}(s,z) = A_1(s)e^{-\lambda_1 sz}+A_2(s)e^{-\lambda_2 sz}+A_3(s)sze^{-\lambda_2 sz}+A_4(s)s^2z^2e^{-\lambda_2 sz} \tag{23}$$

(d) If $\lambda_1^2=\lambda_2^2=\lambda_3^2=\lambda_4^2>0$, then

$$f^{(1)}(s,z) = A_1(s)e^{-\lambda_2 sz}+A_2(s)sze^{-\lambda_2 sz}+A_3(s)s^2z^2e^{-\lambda_2 sz}+A_4(s)s^3y^3e^{-\lambda_2 sz} \tag{24}$$

(e) If $\lambda_1^2>0, \lambda_2^2>0, \lambda_1^2\neq\lambda_2^2$ and $\lambda_3^2, \lambda_4^2<0$ or λ_3^2 and λ_4^2 being a pair of conjugate complex roots, and therefore λ_3 and λ_4 are a pair of conjugate complexes $-\delta\pm i\omega$, the solution of the function $f^{(1)}(s,z)$ is

$$f^{(1)}(s,z) = A_1(s)e^{-\lambda_1 sz}+A_2(s)e^{-\lambda_2 sz}+A_3(s)e^{-\delta sy}\cos(s\omega z)+A_4(s)e^{-\delta sz}\sin(s\omega z) \tag{25}$$

where δ and $\omega > 0$ and $A_i(s)$ ($i = 1, 2, 3, 4$) is a function of s to be determined by the boundary conditions.

Based on the solution of auxiliary function $f^{(j)}(s,z)$, the displacement, stress, electric displacement and electric potential fields are calculated by using Mathematica and using Eqs. (21)-(25) and Eq. (11). Because of the symmetry, it suffices to consider the problem for $x\geq 0, |z|<\infty$. For the case of $\lambda_1^2\neq\lambda_2^2\neq\lambda_3^2\neq\lambda_4^2>0$, the displacements, stresses, electric displacements, electric potentials, magnetic fluxes and magnetic potentials can be expressed, respectively, as follows (The other cases can be obtained using a similar method. Here, they are omitted in the present paper for brevity.)

$$\begin{cases} u^{(1)}(x,z) = \dfrac{2}{\pi}\displaystyle\sum_{i=1}^{4}\beta_i^{(1)}\int_0^\infty A_i(s)s^6\sin(sx)e^{-\lambda_i sz}ds \\ w^{(1)}(x,z) = \dfrac{2}{\pi}\displaystyle\sum_{i=1}^{4}\beta_i^{(2)}\int_0^\infty A_i(s)s^6\cos(sx)e^{-\lambda_i sz}ds \\ \phi^{(1)}(x,z) = \dfrac{2}{\pi}\displaystyle\sum_{i=1}^{4}\beta_i^{(3)}\int_0^\infty A_i(s)s^6\cos(sx)e^{-\lambda_i sz}ds \\ \psi^{(1)}(x,z) = \dfrac{2}{\pi}\displaystyle\sum_{i=1}^{4}\beta_i^{(4)}\int_0^\infty A_i(s)s^6\cos(sx)e^{-\lambda_i sz}ds \end{cases} \tag{26}$$

$$\begin{cases} u^{(2)}(x,z) = -\dfrac{2}{\pi}\sum_{i=1}^{4}\beta_i^{(1)}\int_0^\infty B_i(s)s^6\sin(sx)e^{\lambda_i sz}ds \\ w^{(2)}(x,z) = \dfrac{2}{\pi}\sum_{i=1}^{4}\beta_i^{(2)}\int_0^\infty B_i(s)s^6\cos(sx)e^{\lambda_i sz}ds \\ \phi^{(2)}(x,z) = \dfrac{2}{\pi}\sum_{i=1}^{4}\beta_i^{(3)}\int_0^\infty B_i(s)s^6\cos(sx)e^{\lambda_i sz}ds \\ \psi^{(2)}(x,z) = \dfrac{2}{\pi}\sum_{i=1}^{4}\beta_i^{(4)}\int_0^\infty B_i(s)s^6\cos(sx)e^{\lambda_i sz}ds \end{cases} \tag{27}$$

where $\begin{cases} \beta_i^{(1)} = \lambda_i(\alpha_{11} - \alpha_{12}\lambda_i^2 + \alpha_{13}\lambda_i^4), \quad \beta_i^{(2)} = -\alpha_{21} + \alpha_{22}\lambda_i^2 - \alpha_{23}\lambda_i^4 + \alpha_{24}\lambda_i^6 \\ \beta_i^{(3)} = -\alpha_{31} + \alpha_{32}\lambda_i^2 - \alpha_{33}\lambda_i^4 + \alpha_{34}\lambda_i^6, \quad \beta_i^{(4)} = -\alpha_{41} + \alpha_{42}\lambda_i^2 - \alpha_{43}\lambda_i^4 + \alpha_{44}\lambda_i^6 \end{cases}$

$$\begin{cases} \sigma_{zz}^{(1)}(x,z) = \dfrac{2}{\pi}\sum_{i=1}^{4}\chi_i^{(1)}\int_0^\infty A_i(s)s^7e^{-\lambda_i sz}\cos(sx)ds \\ \sigma_{xz}^{(1)}(x,z) = \dfrac{2}{\pi}\sum_{i=1}^{4}\chi_i^{(2)}\int_0^\infty A_i(s)s^7e^{-\lambda_i sz}\sin(sx)ds \\ D_z^{(1)}(x,z) = \dfrac{2}{\pi}\sum_{i=1}^{4}\chi_i^{(3)}\int_0^\infty A_i(s)s^7e^{-\lambda_i sz}\cos(sx)ds \\ B_z^{(1)}(x,z) = \dfrac{2}{\pi}\sum_{i=1}^{4}\chi_i^{(4)}\int_0^\infty A_i(s)s^7e^{-\lambda_i sz}\cos(sx)ds \end{cases} \tag{28}$$

$$\begin{cases} \sigma_{zz}^{(2)}(x,z) = -\dfrac{2}{\pi}\sum_{i=1}^{4}\chi_i^{(1)}\int_0^\infty B_i(s)s^7e^{\lambda_i sz}\cos(sx)ds \\ \sigma_{xz}^{(2)}(x,z) = \dfrac{2}{\pi}\sum_{i=1}^{4}\chi_i^{(2)}\int_0^\infty B_i(s)s^7e^{\lambda_i sz}\sin(sx)ds \\ D_z^{(2)}(x,z) = -\dfrac{2}{\pi}\sum_{i=1}^{4}\chi_i^{(3)}\int_0^\infty B_i(s)s^7e^{\lambda_i sz}\cos(sx)ds \\ B_z^{(2)}(x,z) = -\dfrac{2}{\pi}\sum_{i=1}^{4}\chi_i^{(4)}\int_0^\infty B_i(s)s^7e^{\lambda_i sz}\cos(sx)ds \end{cases} \tag{29}$$

where $\chi_i^{(1)} = c_{13}\beta_i^{(1)} - c_{33}\lambda_i\beta_i^{(2)} - e_{33}\lambda_i\beta_i^{(3)} + f_{33}\lambda_i\beta_i^{(4)}$, $\chi_i^{(2)} = -c_{44}\lambda_i\beta_i^{(1)} - c_{44}\beta_i^{(2)} - e_{15}\beta_i^{(3)} + f_{15}\beta_i^{(4)}$, $\chi_i^{(3)} = e_{31}\beta_i^{(1)} - e_{33}\lambda_i\beta_i^{(2)} + \varepsilon_{33}\lambda_i\beta_i^{(3)} + g_{33}\lambda_i\beta_i^{(4)}$, $\chi_i^{(4)} = c_{13}\beta_i^{(1)} - f_{33}\lambda_i\beta_i^{(2)} - g_{33}\lambda_i\beta_i^{(3)} + \mu_{33}\lambda_i\beta_i^{(4)}$

To solve the problem, the jumps of displacements across the crack surfaces are defined as follows

$$f_1(x) = u^{(1)}(x,0) - u^{(2)}(x,0) \tag{30}$$

$$f_2(x) = w^{(1)}(x,0) - w^{(2)}(x,0) \tag{31}$$

We can prove that $f_1(x)$ is an odd function and $f_2(x)$ is an even function.

Substituting Eqs. (26)-(27) into Eqs. (30)-(31), applying Eqs. (28)-(29), the Fourier transform and the boundary conditions (7)-(9), we have

$$[X_1]\begin{bmatrix} A_1(s) \\ A_2(s) \\ A_3(s) \\ A_4(s) \end{bmatrix} + [X_1]\begin{bmatrix} B_1(s) \\ B_2(s) \\ B_3(s) \\ B_4(s) \end{bmatrix} = \begin{bmatrix} \dfrac{\overline{f}_1(s)}{s^6} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{32}$$

$$[X_2]\begin{bmatrix} A_1(s) \\ A_2(s) \\ A_3(s) \\ A_4(s) \end{bmatrix} - [X_2]\begin{bmatrix} B_1(s) \\ B_2(s) \\ B_3(s) \\ B_4(s) \end{bmatrix} = \begin{bmatrix} \dfrac{\overline{f}_2}{s^6} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{33}$$

Here a superposed bar indicates the Fourier transform.

Solving eight Equations of Eqs. (32)-(33) with eight unknown functions, substituting the solution into Eq. (28) and applying the boundary conditions (7)-(9), we have (The solving processes can be obtained as shown in the Appendix)

$$\sigma_{zz}^{(1)}(x,0) = \frac{\beta_1}{\pi}\int_0^\infty s\overline{f}_2(s)\cos(sx)ds = -\tau_0, \quad l \le x \le 1 \tag{34}$$

$$\sigma_{xz}^{(1)}(x,0) = \frac{\beta_2}{\pi}\int_0^\infty s\overline{f}_1(s)\sin(sx)ds = 0, \quad l \le x \le 1 \tag{35}$$

$$\begin{cases} \int_0^\infty \overline{f}_1(s)\sin(sx)ds = 0 \\ \int_0^\infty \overline{f}_2(s)\cos(sx)ds = 0 \end{cases}, 0 \le x < l,\ x > l \tag{36}$$

where $\beta_j (j = 1,2)$ are non-zero constants which are dependent on the material properties, which can be obtained as shown in the Appendix. Here, we just give these constants for the case of $\lambda_1^2 \ne \lambda_2^2 \ne \lambda_3^2 \ne \lambda_4^2 > 0$. The other cases can be obtained using the same method. The two pairs of triple integral Eqs. (34)-(36) must be solved to determine the unknown functions $\overline{f}_1(s)$ and $\overline{f}_2(s)$.

5. Solution of the triple integral equations

The Schmidts method (Morse and Feshbach (1958) is used to solve the triple integral Eqs. (34)-(36). The jumps of displacements across the crack surfaces were expanded by the following series

$$f_1(x) = \sum_{n=0}^{\infty} a_n P_n^{(1/2,1/2)}\left(\frac{x - \frac{1+l}{2}}{\frac{1-l}{2}}\right)\left(1 - \frac{\left(x - \frac{1+l}{2}\right)^2}{\left(\frac{1-l}{2}\right)^2}\right)^{1/2}, \quad \text{for } l \le x \le 1 \tag{37}$$

$$f_1(x) = 0, \text{ for } 0 \le x < l,\ 1 < x \tag{38}$$

$$f_2(x) = \sum_{n=0}^{\infty} b_n P_n^{(1/2,\,1/2)}\left(\frac{x - \frac{1+l}{2}}{\frac{1-l}{2}}\right)\left(1 - \frac{\left(x - \frac{1+l}{2}\right)^2}{\left(\frac{1-l}{2}\right)^2}\right)^{1/2}, \quad \text{for } l \le x \le 1 \tag{39}$$

$$f_2(x) = 0, \text{ for } 0 \le x < l,\ 1 < x \tag{40}$$

where a_n and b_n are unknown coefficients, $P_n^{(1/2,\,1/2)}(x)$ is a Jacobi polynomial (Gradshteyn and Ryzhik 1980). The Fourier Transform of Eqs. (37)-(40) is (Erdelyi 1954)

$$\bar{f}_1(s) = \sum_{n=0}^{\infty} a_n F_n G_n^{(1)}(s)\frac{1}{s}J_{n+1}\left(s\frac{1-l}{2}\right) \tag{41}$$

$$\bar{f}_2(s) = \sum_{n=0}^{\infty} b_n F_n G_n^{(2)}(s)\frac{1}{2}J_{n+1}\left(s\frac{1-l}{2}\right) \tag{42}$$

where $F_n = 2\sqrt{\pi}\dfrac{\Gamma\left(n+1+\frac{1}{2}\right)}{n!}$, $G_n^{(1)}(s) = \begin{cases} (-1)^{n/2}\sin\left(s\frac{1+l}{2}\right), & n = 0, 2, 4, 6, \ldots \\ (-1)^{n-1/2}\cos\left(s\frac{1+l}{2}\right), & n = 1, 3, 5, 7, \ldots \end{cases}$

$G_n^{(2)}(s) = \begin{cases} (-1)^{n/2}\cos\left(s\frac{1+l}{2}\right), & n = 0, 2, 4, 6, \ldots \\ (-1)^{n+1/2}\sin\left(s\frac{1+l}{2}\right), & n = 1, 3, 5, 7, \ldots \end{cases}$, $\Gamma(x)$ and $J_n(x)$ are the Gamma and Bessel functions, respectively.

Substituting Eqs. (41)-(42) into Eqs. (34)-(36), it can be shown that Eq. (36) are automatically satisfied. After integration with respect to x in $[l, x]$, Eqs. (34) and (35) are reduced to the following forms

$$\frac{\beta_1}{\pi}\sum_{n=0}^{\infty} b_n F_n \int_0^{\infty} \frac{1}{s} G_n^{(2)}(s) J_{n+1}\left(s\frac{1-l}{2}\right)[\sin(sx) - \sin(sl)]ds = -\tau_0(x-l), \ l \le x \le 1 \tag{43}$$

$$\frac{\beta_2}{\pi}\sum_{n=0}^{\infty} a_n F_n \int_0^{\infty} \frac{1}{s} G_n^{(1)}(s) J_{n+1}\left(s\frac{1-l}{2}\right)[\cos(sx) - \cos(sl)]ds = 0, \ l \le x \le 1 \tag{44}$$

From Eq. (44), it can be derived that $a_n = 0$ ($n = 0, 1, 2, 3, \ldots$). So $f_1(x) = 0$.

From the relationships (Gradshteyn and Ryzhik 1980)

$$\int_0^\infty \frac{1}{s} J_n(as)\sin(bs)ds = \begin{cases} \dfrac{\sin[n\sin^{-1}(b/a)]}{n}, & a>b \\ \dfrac{a^n \sin(n\pi/2)}{n[b+\sqrt{b^2-a^2}]^n}, & b>a \end{cases}$$

$$\int_0^\infty \frac{1}{s} J_n(as)\cos(bs)ds = \begin{cases} \dfrac{\cos[n\sin^{-1}(b/a)]}{n}, & a>b \\ \dfrac{a^n \cos(n\pi/2)}{n[b+\sqrt{b^2-a^2}]^n}, & b>a \end{cases}$$

the semi-infinite integral in Eq. (43) can be modified as follows

$$\int_0^\infty \frac{1}{s} J_{n+1}\left(s\frac{1-l}{2}\right)\cos\left(s\frac{1+l}{2}\right)\sin(sx)ds$$

$$= \frac{1}{2(n+1)}\left\{ \frac{\left(\frac{1-l}{2}\right)^{n+1}\sin\left(\frac{(n+1)\pi}{2}\right)}{\left\{x+\frac{1+l}{2}+\sqrt{\left(x+\frac{1+l}{2}\right)^2-\left(\frac{1-l}{2}\right)^2}\right\}^{n+1}} - \sin\left[(n+1)\sin^{-1}\left(\frac{1+l-2x}{1-l}\right)\right]\right\} \tag{45}$$

$$\int_0^\infty \frac{1}{s} J_{n+1}\left(s\frac{1-l}{2}\right)\sin\left(s\frac{1+l}{2}\right)\sin(sx)ds$$

$$= \frac{1}{2(n+1)}\left\{ \cos\left[(n+1)\sin^{-1}\left(\frac{1+l-2x}{1-l}\right)\right] - \frac{\left(\frac{1-l}{2}\right)^{n+1}\cos\left(\frac{(n+1)\pi}{2}\right)}{\left\{x+\frac{1+l}{2}+\sqrt{\left(x+\frac{1+l}{2}\right)^2-\left(\frac{1-l}{2}\right)^2}\right\}^{n+1}}\right\} \tag{46}$$

Thus the semi-infinite integral in Eq. (43) can be evaluated directly. Eq. (43) can now be solved for coefficients b_n by the Schmidt method (Morse and Feshbach 1958). For brevity, the Eq. (43) can be rewritten as follow

$$\sum_{n=0}^{\infty} b_n E_n(x) = U(x), \quad l<x<1 \tag{47}$$

where $E_n(x)$ and $U(x)$ are known functions and coefficients b_n are unknown and will be determined. A set of functions $P_n(x)$ which satisfy the following orthogonality conditions

$$\int_l^1 P_m(x)P_n(x)dx = N_n\delta_{mn}, \quad N_n = \int_l^1 P_n^2(x)dx \tag{48}$$

can be constructed from the function, $E_n(x)$, such that

$$P_n(x) = \sum_{i=0}^{n} \frac{M_{in}}{M_{nn}} E_i(x) \tag{49}$$

where M_{ij} is the cofactor of the element d_{ij} of matrix D_n, which is defined as follows

$$D_n = \begin{bmatrix} d_{00}, & d_{01}, & d_{02}, & \ldots, & d_{0n} \\ d_{10}, & d_{11}, & d_{12}, & \ldots, & d_{1n} \\ d_{20}, & d_{21}, & d_{22}, & \ldots, & d_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ d_{n0}, & d_{n1}, & d_{n2}, & \ldots, & d_{nn} \end{bmatrix}, \quad d_{ij} = \int_l^1 E_i(x) E_j(x) dx \tag{50}$$

Using Eqs. (48)-(50), we obtain

$$b_n = \sum_{j=n}^{\infty} q_j \frac{M_{nj}}{M_{jj}} \quad \text{with} \quad q_j = \frac{1}{N_j} \int_l^1 U(x) P_j(x) dx \tag{51}$$

6. Intensity factors

Once we have coefficients a_n and b_n, we can obtain the entire stress fields, the electric displacement fields and the magnetic flux fields. However, in fracture mechanics, it is important to determine the stresses $\sigma_{zz}^{(1)}, \sigma_{xz}^{(1)}$, the electric displacements $D_x^{(1)}, D_z^{(1)}$ and the magnetic fluxes $B_x^{(1)}, B_z^{(1)}$ in the vicinity of the crack tips, respectively. In the present study, $\sigma_{zz}^{(1)}, \sigma_{xz}^{(1)}$, $D_x^{(1)}, D_z^{(1)}$ $B_x^{(1)}$ and $B_z^{(1)}$ along the crack line can be expressed, respectively, as follows

$$\sigma_{zz}^{(1)}(x,0) = \frac{\beta_1}{\pi} \sum_{n=0}^{\infty} b_n F_n \int_0^{\infty} G_n^{(2)}(s) J_{n+1}\left(s \frac{1-l}{2}\right) \cos(xs) ds \tag{52}$$

$$\sigma_{xz}^{(1)}(x,0) = 0 \tag{53}$$

$$D_z^{(1)}(x,0) = \frac{\beta_3}{\pi} \sum_{n=0}^{\infty} b_n F_n \int_0^{\infty} G_n^{(2)}(s) J_{n+1}\left(s \frac{1-l}{2}\right) \cos(xs) ds \tag{54}$$

$$D_x^{(1)}(x,0) = \frac{\beta_4}{\pi} \sum_{n=0}^{\infty} b_n F_n \int_0^{\infty} G_n^{(2)}(s) J_{n+1}\left(s \frac{1-l}{2}\right) \sin(xs) ds \tag{55}$$

$$B_z^{(1)}(x,0) = \frac{\beta_5}{\pi} \sum_{n=0}^{\infty} b_n F_n \int_0^{\infty} G_n^{(2)}(s) J_{n+1}\left(s \frac{1-l}{2}\right) \cos(xs) ds \tag{56}$$

$$B_x^{(1)}(x,0) = \frac{\beta_6}{\pi} \sum_{n=0}^{\infty} b_n F_n \int_0^{\infty} G_n^{(2)}(s) J_{n+1}\left(s \frac{1-l}{2}\right) \sin(xs) ds \tag{57}$$

where $\beta_i (i = 3, 4, 5, 6)$ are non-zero constants which depend on the properties of materials. These constants can be obtained as shown in the Appendix.

Observing the expressions in Eqs. (52)-(57) and using the following relationships (Gradshteyn and Ryzhik 1980)

$$\int_0^\infty J_n(sa)\cos(bs)ds = \begin{cases} \dfrac{\cos[n\sin^{-1}(b/a)]}{\sqrt{a^2-b^2}}, & a>b \\ -\dfrac{a^n\sin(n\pi/2)}{\sqrt{b^2-a^2}[b+\sqrt{b^2-a^2}]^n}, & b>a \end{cases}$$

$$\int_0^\infty J_n(sa)\sin(bs)ds = \begin{cases} \dfrac{\sin[n\sin^{-1}(b/a)]}{\sqrt{a^2-b^2}}, & a>b \\ -\dfrac{a^n\cos(n\pi/2)}{\sqrt{b^2-a^2}[b+\sqrt{b^2-a^2}]^n}, & b>a \end{cases}$$

the singular parts of the stress field, the electric displacement and the magnetic flux can be expressed, respectively, as follows ($x > 1$ or $x < l$)

$$\sigma_{zz0}^{(1)} = \frac{\beta_1}{2\pi}\sum_{n=0}^{\infty} b_n F_n H_n(l,x), \quad \sigma_{xz0}^{(1)} = 0 \tag{58}$$

$$D_{z0}^{(1)} = \frac{\beta_3}{2\pi}\sum_{n=0}^{\infty} b_n F_n H_n(l,x), \quad D_{x0}^{(1)} = 0 \tag{59}$$

$$B_{z0}^{(1)} = \frac{\beta_5}{2\pi}\sum_{n=0}^{\infty} b_n F_n H_n(l,x), \quad B_{x0}^{(1)} = 0 \tag{60}$$

where $H_n(l,x) = \begin{cases} (-1)^{n+1}R(l,x,n), & 0<x<l \\ -R(l,x,n), & x>1 \end{cases}$

$$R(l,x,n) = \frac{2(1-l)^{n+1}}{\sqrt{|1+l-2x|^2-(1-l)^2}[|1+l-2x|+\sqrt{|1+l-2x|^2-(1-l)^2}]^{n+1}}$$

At the left tip of the right crack, we obtain the normal stress intensity factor K_{IL} as follow

$$K_{IL} = \lim_{x\to l^{-1}} \sqrt{2\pi(1-x)}\cdot\sigma_{zz0}^{(1)} = \beta_1\sqrt{\frac{1}{2\pi(1-x)}}\sum_{n=0}^{\infty}(-1)^{n+1}b_nF_n \tag{61}$$

At the right tip of the right crack, we obtain the normal stress intensity factor K_{IR} as follow

$$K_{IR} = \lim_{x\to 1^{+}} \sqrt{2\pi(x-1)}\cdot\sigma_{zz0}^{(1)} = -\beta_1\sqrt{\frac{1}{2\pi(1-l)}}\sum_{n=0}^{\infty}b_nF_n \tag{62}$$

However, at the right and the left tips of the right crack, we obtain the shear stress intensity factors $K_{IIR} = \lim_{x\to 1^{+}} \sqrt{2\pi(x-1)}\cdot\sigma_{xz0}^{(1)}$ and $K_{IIL} = \lim_{x\to l^{-1}} \sqrt{2\pi(l-x)}\cdot\sigma_{xz0}^{(1)}$ are all equal to zero.

At the left tip of the right crack, we obtain the electric displacement intensity factor K_{IL}^D in zdirection as follow

$$K_{IL}^{D} = \lim_{x \to l^{-1}} \sqrt{2\pi(1-x)} \cdot D_{z0}^{(1)} = \beta_3 \sqrt{\frac{1}{2\pi(1-l)}} \sum_{n=0}^{\infty} (-1)^{n+1} b_n F_n = \frac{\beta_3}{\beta_1} K_{IL} \tag{63}$$

At the right tip of the right crack, we obtain the electric displacement intensity factor K_{IR}^{D} in z-direction as follow

$$K_{IR}^{D} = \lim_{x \to 1^{+}} \sqrt{2\pi(x-1)} \cdot D_{z0}^{(1)} = -\beta_3 \sqrt{\frac{1}{2\pi(1-l)}} \sum_{n=0}^{\infty} b_n F_n = \frac{\beta_3}{\beta_1} K_{IR} \tag{64}$$

However, at the right and the left tips of the right crack, we obtain the electric displacement intensity factors $K_{IIR}^{D} = \lim_{x \to 1^{+}} \sqrt{2\pi(x-1)} \cdot D_{x0}^{(1)}$ and $K_{IIL}^{D} = \lim_{x \to l^{-1}} \sqrt{2\pi(l-x)} \cdot D_{x0}^{(1)}$ in x-direction are all equal to zero.

At the left tip of the right crack, we obtain the magnetic flux intensity factor K_{IL}^{B} in z-direction as follo

$$K_{IL}^{B} = \lim_{x \to l^{-1}} \sqrt{2\pi(l-x)} \cdot B_{z0}^{(1)} = \beta_5 \sqrt{\frac{1}{2\pi(1-l)}} \sum_{n=0}^{\infty} (-1)^{n+1} b_n F_n = \frac{\beta_5}{\beta_1} K_{IL} \tag{65}$$

At the right tip of the right crack, we obtain the magnetic flux intensity factor K_{IR}^{B} in z-direction as follow

$$K_{IR}^{B} = \lim_{x \to 1^{+}} \sqrt{2\pi(x-1)} \cdot D_{z0}^{(1)} = -\beta_5 \sqrt{\frac{1}{2\pi(1-l)}} \sum_{n=0}^{\infty} b_n F_n = \frac{\beta_5}{\beta_1} K_{R} \tag{66}$$

However, at the right and the left tips of the right crack, we obtain the magnetic flux intensity factors $K_{IIR}^{B} = \lim_{x \to 1^{+}} \sqrt{2\pi(x-1)} \cdot B_{x0}^{(1)}$ and $K_{IIL}^{B} = \lim_{x \to l^{-1}} \sqrt{2\pi(l-x)} \cdot B_{x0}^{(1)}$ in x-direction are all equal to zero.

7. Numerical results and discussion

As discussed in the works (Zhou and Wang 2004, Zhou *et al*. 2005a, Zhou and Wang 2006), it can be seen that the Schmidt method performs satisfactorily if the first ten terms of the infinite series in Eq. (47) are retained. The numerical results are plotted as shown in Fig. 2. From the results of the solution, the following observations are very significant:

(i) In the present paper, the similar problem that was treated by Gao *et al.* (2003b) was reworked using a somewhat different approach, named the Schmidt method (Morse and Feshbach 1958), i.e. the behavior of two collinear Mode-I cracks in piezoelectric/piezomagnetic materials subjected to a uniform tension loading was investigated by the generalized Almansi's theorem. This generalized Almansi's theorem in the present paper is feasible for general cases, as discussed in Eqs. (21)-(25), and thus the obtained solution is valid to general cases. However, the Eshelby-Stroh's method which adopted in Gao *et al.* (2003b) is valid only for the cases of non-degenerate materials. The unknown variables of triple integral equations are the jumps of displacements across the crack surfaces, not the analytic functions or the dislocation density functions. This is the major difference between the current work and the available work in the literature (Wu and Huang 2000, Sih and Song 2003, Song and Sih 2003, Wang and Mai 2003, Gao *et al.* 2003a,b,c,d, Spyropoulos *et al.* 2003, Liu *et al.* 2001, Chung and Ting 1995, Pan 2002, Wang and Mai 2004, Chen *et al.* 2004, Wang and

Shen 2002). The problem in the present paper is a special case in the Gao *et al.* (2003b). The multiple collinear cracks in Gao *et al.* (2003b) can be also solved by using the Schmidt method, the generalized Almansi's theorem and the representative crack unit method. We will consider this problem in the future. Certainly, the problem of multiple cracks is a more general case in the practice.

(ii) In the present paper, it was also assumed that two collinear cracks only subject to a uniform tension stress loading, do not subject to an electric field or a magnetic flux loading at the same time. Certainly, the loading and the geometry of cracks are symmetry. However, the uniform mechanical-electric-magnetic loads are considered at the same time in Gao *et al.* (2003b). In the Gao *et al.* (2003b), it is assumed that the medium is only subject to the remote uniform loading $\sum_1^\infty = [\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty, D_2^\infty, B_2^\infty]$ and $\sum_2^\infty = [\sigma_{11}^\infty, \sigma_{12}^\infty, \sigma_{13}^\infty, E_1^\infty, H_1^\infty]$. It was also assumed that the electric potential, the magnetic potential, the normal electric displacement and the normal magnetic flux are continuous across the crack surfaces. Certainely, the solution in the Gao *et al.* (2003) can be returned to one of the present paper.

(iii) The solution of the stress intensity factors in the present paper was the same as one in Gao *et al.* (2003b). However, the electric displacement and the magnetic flux intensity factors were different from ones in Gao *et al.* (2003b) because the boundary conditions were not the same as each other.

(iv) From the solution, it can be obtained that the singular stress, the singular electric displacement and the singular magnetic flux in piezoelectric/piezomagnetic materials carry the same forms as those in elastic materials.

(v) From Eqs. (43) and (61)-(62), it can be obtained that the stress field does not depend on the material properties except the crack length. So in all computation, the material constants were not considered. However, the electric displacement and magnetic flux intensity factors depend on the stress intensity factors and the properties of materials as shown in Eqs. (63)-(66). The electro-magneto-elastic coupling effects can be obtained as shown in Eqs. (63)-(66). This means that an applied mechanical load alone can produce the electric displacement and the magnetic flux singularities. The results of the electric displacement intensity factors and the magnetic flux

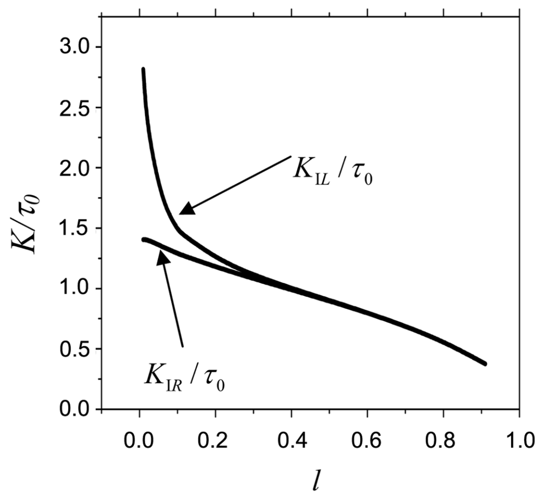

Fig. 2 The stress intensity factor versus l

intensity factors can be directly obtained form the results of the stress intensity factors through Eqs. (63)-(66). In the present paper, they are omitted for brevity.

(vi) The stress intensity factors decrease with the increase in the distance between two cracks as shown in Fig. 2. The stress, the electric displacement and the magnetic flux fields near the inner crack tips are larger than ones near the outer crack tips. It can be also obtained that the interaction of two collinear cracks decreases with the increase in the distance between two collinear cracks. The electric displacement and the magnetic flux intensity factors have the same changing tendency as the stress intensity factors. However, the magnitudes of the electric displacement intensity factors or the magnetic flux intensity factors are different from the stress intensity factors.

8. Conclusions

In the present paper, the similar problem that was treated by Gao *et al.* (2003b) was reworked using a somewhat different approach, named the Schmidt method. From the solution, it can be obtained that the singular stress, the singular electric displacement and the singular magnetic flux in piezoelectric/piezomagnetic materials carry the same forms as those in elastic materials. It can be also founded that the stress intensity factors do not depend on the material properties for the electrically and magnetically permeable mode-I crack in piezoelectric/piezomagnetic materials as shown in isotropic materials. However, the electric displacement intensity factors and the magnetic flux intensity factors depend on the stress intensity factors and the properties of piezoelectric/ piezomagnetic materials. The electro-magneto-elastic coupling effects can be also obtained as shown in Eqs. (63)-(66).

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Appendix: Coefficients

$$a = c_{44}[-2e_{33}f_{33}g_{33} + e_{33}^2\mu_{33} - f_{33}^2\varepsilon_{33} + c_{33}(g_{33}^2 + \mu_{33}\varepsilon_{33})]$$

$$\begin{aligned} b = -\{ & e_{31}^2 f_{33}^2 + 2c_{33}e_{31}f_{15}g_{33} + 2c_{33}e_{31}f_{31}g_{33} - 2c_{13}e_{31}f_{33}g_{33} - 2c_{44}e_{31}f_{33}g_{33} - 2c_{33}c_{44}g_{11}g_{33} \\ & + c_{13}^2 g_{33}^2 - c_{11}c_{33}g_{33}^2 + 2c_{13}c_{44}g_{33}^2 - c_{33}e_{31}^2\mu_{33} + e_{33}^2(f_{15}^2 + 2f_{15}f_{31} + f_{31}^2 - c_{44}\mu_{11} - c_{11}\mu_{33}) \\ & + e_{15}^2(f_{33}^2 - c_{33}\mu_{33}) + 2e_{15}\{[c_{33}(f_{15} + f_{31}) - c_{13}f_{33}]g_{33} + e_{31}(f_{33}^2 - c_{33}\mu_{33})\} \\ & -2e_{33}\{-c_{44}f_{33}g_{11} + c_{13}f_{15}g_{33} + c_{13}f_{31}g_{33} + c_{44}f_{31}g_{33} - c_{11}f_{33}g_{33} \\ & + e_{15}(f_{15}f_{33} + f_{31}f_{33} - c_{13}\mu_{33}) + e_{31}[f_{15}f_{33} + f_{31}f_{33} - \mu_{33}(c_{13} + c_{44})]\} \\ & + c_{44}f_{33}^2\varepsilon_{11} - c_{33}c_{44}\mu_{33}\varepsilon_{11} + c_{33}f_{15}^2\varepsilon_{33} + 2c_{33}f_{15}f_{31}\varepsilon_{33} + c_{33}f_{31}^2\varepsilon_{33} - 2c_{13}f_{15}f_{33}\varepsilon_{33} - 2c_{13}f_{31}f_{33}\varepsilon_{33} \\ & - 2c_{44}f_{31}f_{33}\varepsilon_{33} + c_{11}f_{33}^2\varepsilon_{33} - c_{33}c_{44}\mu_{11}\varepsilon_{33} + c_{13}^2\mu_{33}\varepsilon_{33} - c_{11}c_{33}\mu_{33}\varepsilon_{33} + 2c_{13}c_{44}\mu_{33}\varepsilon_{33}\} \end{aligned}$$

$$\begin{aligned} c = {} & 2c_{13}e_{33}f_{15}g_{11} + 2c_{13}e_{33}f_{31}g_{11} + 2c_{44}e_{33}f_{31}g_{11} - 2c_{11}e_{33}f_{33}g_{11} + c_{33}c_{44}g_{11}^2 - 2c_{11}e_{33}f_{15}g_{33} \\ & - 2c_{13}^2 g_{11}g_{33} + 2c_{11}c_{33}g_{11}g_{33} - 4c_{13}c_{44}g_{11}g_{33} + c_{11}c_{44}g_{33}^2 + c_{11}e_{33}^2\mu_{11} \\ & + e_{15}^2(2f_{31}f_{33} + c_{33}\mu_{11} - 2c_{13}\mu_{33}) + e_{31}^2(-2f_{15}f_{33} + c_{33}\mu_{11} + c_{44}\mu_{33}) \\ & -2e_{15}[c_{33}(f_{15} + f_{31})g_{11} - c_{13}f_{33}g_{11} - 2c_{13}f_{15}g_{33} - c_{13}f_{31}g_{33} + c_{11}f_{33}g_{33} \\ & + e_{33}(f_{15}f_{31} + f_{31}^2 + c_{13}\mu_{11} - c_{11}\mu_{33})] \\ & + 2e_{31}\{-c_{33}f_{15}g_{11} - c_{33}f_{31}g_{11} + c_{13}f_{33}g_{11} + c_{44}f_{33}g_{11} + c_{13}f_{15}g_{33} - c_{44}f_{31}g_{33} \\ & + e_{33}[f_{15}^2 + f_{15}f_{31} - (c_{13} + c_{44})\mu_{11}] + e_{15}(-f_{15}f_{33} + f_{31}f_{33} + c_{33}\mu_{11} - c_{13}\mu_{33})\} \\ & - c_{33}f_{15}^2\varepsilon_{11} - 2c_{33}f_{15}f_{31}\varepsilon_{11} - c_{33}f_{31}^2\varepsilon_{11} + 2c_{13}f_{15}f_{33}\varepsilon_{11} + 2c_{13}f_{31}f_{33}\varepsilon_{11} + 2c_{44}f_{31}f_{33}\varepsilon_{11} \\ & - c_{11}f_{33}^2\varepsilon_{11} + c_{33}c_{44}\mu_{11}\varepsilon_{11} - c_{13}^2\mu_{33}\varepsilon_{11} + c_{11}c_{33}\mu_{33}\varepsilon_{11} - 2c_{13}c_{44}\mu_{33}\varepsilon_{11} + 2c_{13}f_{15}^2\varepsilon_{33} \\ & + 2c_{13}f_{15}f_{31}\varepsilon_{33} - c_{44}f_{31}^2\varepsilon_{33} - 2c_{11}f_{15}f_{33}\varepsilon_{33} - c_{13}^2\mu_{11}\varepsilon_{33} + c_{11}c_{33}\mu_{11}\varepsilon_{33} - 2c_{13}c_{44}\mu_{11}\varepsilon_{33} + c_{11}c_{44}\mu_{33}\varepsilon_{33} \end{aligned}$$

$$\begin{aligned} d = {} & -2c_{11}e_{33}f_{15}g_{11} - c_{13}^2 g_{11}^2 + c_{11}c_{33}g_{11}^2 - 2c_{13}c_{44}g_{11}^2 + 2c_{11}c_{44}g_{11}g_{33} + e_{31}^2(-f_{15}^2 + c_{44}\mu_{11}) \\ & + 2e_{31}(e_{15}f_{15}f_{31} + c_{13}f_{15}g_{11} - c_{44}f_{31}g_{11} - c_{13}e_{15}\mu_{11}) \\ & + 2e_{15}[c_{13}(2f_{15} + f_{31})g_{11} - c_{11}(f_{33}g_{11} + f_{15}g_{33} - e_{33}\mu_{11})]] \\ & -e_{15}^2(f_{31}^2 + 2c_{13}\mu_{11} - c_{11}\mu_{33}) + 2c_{13}f_{15}^2\varepsilon_{11} + 2c_{13}f_{15}f_{31}\varepsilon_{11} - c_{44}f_{31}^2\varepsilon_{11} - 2c_{11}f_{15}f_{33}\varepsilon_{11} \\ & - c_{13}^2\mu_{11}\varepsilon_{11} + c_{11}c_{33}\mu_{11}\varepsilon_{11} - 2c_{13}c_{44}\mu_{11}\varepsilon_{11} + c_{11}c_{44}\mu_{33}\varepsilon_{11} - c_{11}f_{15}^2\varepsilon_{33} + c_{11}c_{44}\mu_{11}\varepsilon_{33} \end{aligned}$$

$$e = -2c_{11}e_{15}f_{15}g_{11} + c_{11}c_{44}g_{11}^2 + c_{11}e_{15}^2\mu_{11} - c_{11}f_{15}^2\varepsilon_{11} + c_{11}c_{44}\mu_{11}\varepsilon_{11}$$

$$\begin{aligned} \alpha_{11} = -[& -e_{31}f_{15}g_{11} + c_{13}g_{11}^2 + c_{44}g_{11}^2 + e_{15}^2\mu_{11} + e_{15}(-2f_{15}g_{11} - f_{31}g_{11} + e_{31}\mu_{11}) \\ & - f_{15}^2\varepsilon_{11} - f_{15}f_{31}\varepsilon_{11} + c_{13}\mu_{11}\varepsilon_{11} + c_{44}\mu_{11}\varepsilon_{11}] \end{aligned}$$

$$\begin{aligned} \alpha_{12} = {} & e_{31}f_{33}g_{11} + e_{31}f_{15}g_{33} - 2c_{13}g_{11}g_{33} - 2c_{44}g_{11}g_{33} + e_{33}[f_{15}g_{11} + f_{31}g_{11} - (e_{15} + e_{31})\mu_{11}] \\ & - e_{15}^2\mu_{33} + e_{15}(f_{33}g_{11} + 2f_{15}g_{33} + f_{31}g_{33} - e_{31}\mu_{33}) + f_{15}f_{33}\varepsilon_{11} + f_{31}f_{33}\varepsilon_{11} - c_{13}\mu_{33}\varepsilon_{11} - c_{44}\mu_{33}\varepsilon_{11} \\ & + f_{15}^2\varepsilon_{33} + f_{15}f_{31}\varepsilon_{33} - c_{13}\mu_{11}\varepsilon_{33} - c_{44}\mu_{11}\varepsilon_{33} \end{aligned}$$

$$\alpha_{13} = -\{-e_{15}f_{33}g_{33} - e_{31}f_{33}g_{33} + c_{13}g_{33}^2 + c_{44}g_{33}^2 + e_{33}[-f_{15}g_{33} - f_{31}g_{33} + (e_{15} + e_{31})\mu_{33}] - f_{15}f_{33}\varepsilon_{33} - f_{31}f_{33}\varepsilon_{33} + c_{13}\mu_{33}\varepsilon_{33} + c_{44}\mu_{33}\varepsilon_{33}\}$$

$$\alpha_{21} = c_{11}(g_{11}^2 + \mu_{11}\varepsilon_{11})$$

$$\alpha_{22} = -2e_{31}(f_{15} + f_{31})g_{11} + c_{44}g_{11}^2 + 2c_{11}g_{11}g_{33} + e_{15}^2\mu_{11} + e_{31}^2\mu_{11} - 2e_{15}(f_{15}g_{11} + f_{31}g_{11} - e_{31}\mu_{11}) - f_{15}^2\varepsilon_{11} - 2f_{15}f_{31}\varepsilon_{11} - f_{31}^2\varepsilon_{11} + c_{44}\mu_{11}\varepsilon_{11} + c_{11}\mu_{33}\varepsilon_{11} + c_{11}\mu_{11}\varepsilon_{33}$$

$$\alpha_{23} = -2e_{31}(f_{15} + f_{31})g_{33} + 2c_{44}g_{11}g_{33} + c_{11}g_{33}^2 + e_{15}^2\mu_{33} + e_{31}^2\mu_{33} - 2e_{15}(f_{15}g_{33} + f_{31}g_{33} - e_{31}\mu_{33}) + c_{44}\mu_{33}\varepsilon_{11} - f_{15}^2\varepsilon_{33} - 2f_{15}f_{31}\varepsilon_{33} - f_{31}^2\varepsilon_{33} + c_{44}\mu_{11}\varepsilon_{33} + c_{11}\mu_{33}\varepsilon_{33}$$

$$\alpha_{24} = c_{44}(g_{33}^2 + \mu_{33}\varepsilon_{33})$$

$$\alpha_{31} = -c_{11}(f_{15}g_{11} - e_{15}\mu_{11})$$

$$\alpha_{32} = -\{-c_{13}f_{15}g_{11} - c_{13}f_{31}g_{11} - c_{44}f_{31}g_{11} + c_{11}f_{33}g_{11} + c_{11}f_{15}g_{33} - c_{11}e_{33}\mu_{11} + e_{31}[-f_{15}^2 - f_{15}f_{31} + (c_{13} + c_{44})\mu_{11}] + e_{15}(f_{15}f_{31} + f_{31}^2 + c_{13}\mu_{11} - c_{11}\mu_{33})\}$$

$$\alpha_{33} = -[-e_{15}f_{15}f_{33} - e_{31}f_{15}f_{33} - e_{15}f_{31}f_{33} - e_{31}f_{31}f_{33} + c_{44}f_{33}g_{11} - c_{13}f_{15}g_{33} - c_{13}f_{31}g_{33} - c_{44}f_{31}g_{33} + c_{11}f_{33}g_{33} + c_{13}e_{15}\mu_{33} + c_{13}e_{31}\mu_{33} + c_{44}e_{31}\mu_{33}$$

$$\alpha_{34} = -c_{44}(f_{33}g_{33} - e_{33}\mu_{33})$$

$$\alpha_{41} = c_{11}(e_{15}g_{11} + f_{15}\varepsilon_{11})$$

$$\alpha_{42} = -\{-e_{31}^2f_{15} + e_{15}^2f_{31} + (c_{13} + c_{44})e_{31}g_{11} - c_{11}e_{33}g_{11} + e_{15}[e_{31}(-f_{15} + f_{31}) + c_{13}g_{11} - c_{11}g_{33}] + c_{13}f_{15}\varepsilon_{11} + c_{13}f_{31}\varepsilon_{11} + c_{44}f_{31}\varepsilon_{11} - c_{11}f_{33}\varepsilon_{11} - c_{11}f_{15}\varepsilon_{33}\}$$

$$\alpha_{43} = e_{15}^2f_{33} + e_{31}^2f_{33} + c_{44}e_{33}g_{11} + c_{11}e_{33}g_{33} - e_{15}[e_{33}(f_{15} + f_{31}) - 2e_{31}f_{33} + c_{13}g_{33}] - e_{31}[e_{33}(f_{15} + f_{31}) + (c_{13} + c_{44})g_{33}] + c_{44}f_{33}\varepsilon_{11} - c_{13}f_{15}\varepsilon_{33} - c_{13}f_{31}\varepsilon_{33} - c_{44}f_{31}\varepsilon_{33} + c_{11}f_{33}\varepsilon_{33}$$

$$\alpha_{44} = c_{44}(e_{33}g_{33} + f_{33}\varepsilon_{33})$$

The solving processes of the constants β_1, β_2, β_3, β_4, β_5 and β_6

For $\lambda_1^2 \neq \lambda_2^2 \neq \lambda_3^2 \neq \lambda_4^2 > 0$ case, the matrices $[X_i]$ $(i = 1, 2)$ can be expressed as follows

$$[X_1] = \begin{bmatrix} \beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} & \beta_4^{(1)} \\ \chi_1^{(1)} & \chi_2^{(1)} & \chi_3^{(1)} & \chi_4^{(1)} \\ \chi_1^{(3)} & \chi_2^{(3)} & \chi_3^{(3)} & \chi_4^{(3)} \\ \chi_1^{(4)} & \chi_2^{(4)} & \chi_3^{(4)} & \chi_4^{(4)} \end{bmatrix}, \quad [X_2] = \begin{bmatrix} \beta_1^{(2)} & \beta_2^{(2)} & \beta_3^{(2)} & \beta_4^{(2)} \\ \beta_1^{(3)} & \beta_2^{(3)} & \beta_3^{(3)} & \beta_4^{(3)} \\ \beta_1^{(4)} & \beta_2^{(4)} & \beta_3^{(4)} & \beta_3^{(4)} \\ \chi_1^{(2)} & \chi_2^{(2)} & \chi_3^{(2)} & \chi_4^{(2)} \end{bmatrix} \tag{A-1}$$

From Eqs. (32)-(33), it can be obtained

$$a = \frac{1}{2}[X_1]^{-1}\begin{bmatrix} \frac{\bar{f}_1}{s^6} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2}[X_2]^{-1}\begin{bmatrix} \frac{\bar{f}_2}{s^6} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad b = \frac{1}{2}[X_1]^{-1}\begin{bmatrix} \frac{\bar{f}_1}{s^6} \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2}[X_2]^{-1}\begin{bmatrix} \frac{\bar{f}_2}{s^6} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{A-2}$$

where $a = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}, \quad b = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}$

So the unknown functions A_i and B_i can be expressed as follows

$$A_i = \frac{1}{2s^6}[m_{i1}\bar{f}_1 + n_{i1}\bar{f}_2], \quad B_i = \frac{1}{2s^6}[m_{i1}\bar{f}_1 - n_{i1}\bar{f}_2] \tag{A-3}$$

where $[m_{ij}]_{4\times 4} = [X_1]^{-1}$, $[n_{ij}]_{4\times 4} = [X_2]^{-1}$.

Substituting Eqs. (A-3) into Eqs. (32)-(33), it can be obtained

$$\frac{\bar{f}_1}{2}\begin{bmatrix} \beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} & \beta_4^{(1)} \\ \chi_1^{(1)} & \chi_2^{(1)} & \chi_3^{(1)} & \chi_4^{(1)} \\ \chi_1^{(3)} & \chi_2^{(3)} & \chi_3^{(3)} & \chi_4^{(3)} \\ \chi_1^{(4)} & \chi_2^{(4)} & \chi_3^{(4)} & \chi_4^{(4)} \end{bmatrix}\begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \\ m_{41} \end{bmatrix} + \frac{\bar{f}_1}{2}\begin{bmatrix} \beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} & \beta_4^{(1)} \\ \chi_1^{(1)} & \chi_2^{(1)} & \chi_3^{(1)} & \chi_4^{(1)} \\ \chi_1^{(3)} & \chi_2^{(3)} & \chi_3^{(3)} & \chi_4^{(3)} \\ \chi_1^{(4)} & \chi_2^{(4)} & \chi_3^{(4)} & \chi_4^{(4)} \end{bmatrix}\begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \\ m_{41} \end{bmatrix} = \begin{bmatrix} \bar{f}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{A-4}$$

$$\frac{\bar{f}_2}{2}\begin{bmatrix} \beta_1^{(2)} & \beta_2^{(2)} & \beta_3^{(2)} & \beta_4^{(2)} \\ \beta_1^{(3)} & \beta_2^{(3)} & \beta_3^{(3)} & \beta_4^{(3)} \\ \beta_1^{(4)} & \beta_2^{(4)} & \beta_3^{(4)} & \beta_3^{(4)} \\ \chi_1^{(2)} & \chi_2^{(2)} & \chi_3^{(2)} & \chi_4^{(2)} \end{bmatrix}\begin{bmatrix} n_{11} \\ n_{21} \\ n_{31} \\ n_{41} \end{bmatrix} + \frac{\bar{f}_2}{2}\begin{bmatrix} \beta_1^{(2)} & \beta_2^{(2)} & \beta_3^{(2)} & \beta_4^{(2)} \\ \beta_1^{(3)} & \beta_2^{(3)} & \beta_3^{(3)} & \beta_4^{(3)} \\ \beta_1^{(4)} & \beta_2^{(4)} & \beta_3^{(4)} & \beta_3^{(4)} \\ \chi_1^{(2)} & \chi_2^{(2)} & \chi_3^{(2)} & \chi_4^{(2)} \end{bmatrix}\begin{bmatrix} n_{11} \\ n_{21} \\ n_{31} \\ n_{41} \end{bmatrix} = \begin{bmatrix} \frac{\bar{f}_2}{s^6} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{A-5}$$

So it can be obtained

$$\begin{bmatrix} \beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} & \beta_4^{(1)} \\ \chi_1^{(1)} & \chi_2^{(1)} & \chi_3^{(1)} & \chi_4^{(1)} \\ \chi_1^{(3)} & \chi_2^{(3)} & \chi_3^{(3)} & \chi_4^{(3)} \\ \chi_1^{(4)} & \chi_2^{(4)} & \chi_3^{(4)} & \chi_4^{(4)} \end{bmatrix}\begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \\ m_{41} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \sum_{i=1}^{4} \chi_i^{(1)} m_{i1} = 0 \\ \sum_{i=1}^{4} \chi_i^{(3)} m_{i1} = 0 \\ \sum_{i=1}^{4} \chi_i^{(4)} m_{i1} = 0 \end{cases} \tag{A-6}$$

$$\begin{bmatrix} \beta_1^{(2)} & \beta_2^{(2)} & \beta_3^{(2)} & \beta_4^{(2)} \\ \beta_1^{(3)} & \beta_2^{(3)} & \beta_3^{(3)} & \beta_4^{(3)} \\ \beta_1^{(4)} & \beta_2^{(4)} & \beta_3^{(4)} & \beta_3^{(4)} \\ \chi_1^{(2)} & \chi_2^{(2)} & \chi_3^{(2)} & \chi_4^{(2)} \end{bmatrix}\begin{bmatrix} n_{11} \\ n_{21} \\ n_{31} \\ n_{41} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \sum_{i=1}^{4} \chi_i^{(2)} n_{i1} = 0 \tag{A-7}$$

Substituting Eqs. (A-3) into Eq. (28) and applying Eqs. (A-6)-(A-7), we have

$$\sigma_{zz}^{(1)}(x,0) = \frac{1}{\pi}\sum_{i=1}^{4}\chi_i^{(1)}m_{i1}\int_0^{\infty}s\bar{f}_1(s)\cos(sx)ds + \frac{1}{\pi}\sum_{i=1}^{4}\chi_i^{(1)}n_{i1}\int_0^{\infty}s\bar{f}_2(s)\cos(sx)ds$$

$$= \frac{1}{\pi}\sum_{i=1}^{4}\chi_i^{(1)}n_{i1}\int_0^{\infty}s\bar{f}_2(s)\cos(sx)ds = \frac{1}{\pi}\beta_1\int_0^{\infty}s\bar{f}_2(s)\cos(sx)ds \tag{A-8}$$

$$\sigma_{xz}^{(1)}(x,0) = \frac{1}{\pi}\sum_{i=1}^{4}\chi_i^{(2)}m_{i1}\int_0^{\infty}s\bar{f}_1(s)\sin(sx)ds + \frac{1}{\pi}\sum_{i=1}^{4}\chi_i^{(2)}n_{i1}\int_0^{\infty}s\bar{f}_2(s)\sin(sx)ds$$

$$= \frac{1}{\pi}\sum_{i=1}^{4}\chi_i^{(2)}m_{i1}\int_0^{\infty}s\bar{f}_1(s)\sin(sx)ds = \frac{1}{\pi}\beta_2\int_0^{\infty}s\bar{f}_1(s)\sin(sx)ds \tag{A-9}$$

$$D_z^{(1)}(x,0) = \frac{1}{\pi}\sum_{i=1}^{4}\chi_i^{(3)}m_{i1}\int_0^{\infty}s\bar{f}_1(s)\cos(sx)ds + \frac{1}{\pi}\sum_{i=1}^{4}\chi_i^{(3)}n_{i1}\int_0^{\infty}s\bar{f}_2(s)\cos(sx)ds$$

$$= \frac{1}{\pi}\sum_{i=1}^{4}\chi_i^{(3)}n_{i1}\int_0^{\infty}s\bar{f}_2(s)\cos(sx)ds = \frac{1}{\pi}\beta_3\int_0^{\infty}s\bar{f}_2(s)\cos(sx)ds \tag{A-10}$$

$$D_x^{(1)}(x,0) = \frac{1}{\pi}\sum_{i=1}^{4}[-\lambda_i e_{15}\beta_i^{(1)} - e_{15}\beta_i^{(2)} + \varepsilon_{11}\beta_i^{(3)} + g_{11}\beta_i^{(4)}]$$

$$(m_{i1}+n_{i1})[\int_0^{\infty}s\bar{f}_1(s)\sin(sx)ds + \int_0^{\infty}s\bar{f}_2(s)\sin(sx)ds]$$

$$= \frac{1}{\pi}\beta_4\int_0^{\infty}s\bar{f}_2(s)\sin(sx)ds(\bar{f}_1(s)=0) \tag{A-11}$$

$$B_z^{(1)}(x,0) = \frac{1}{\pi}\sum_{i=1}^{4}\chi_i^{(4)}m_{i1}\int_0^{\infty}s\bar{f}_1(s)\cos(sx)ds + \frac{1}{\pi}\sum_{i=1}^{4}\chi_i^{(4)}n_{i1}\int_0^{\infty}s\bar{f}_2(s)\cos(sx)ds$$

$$= \frac{1}{\pi}\sum_{i=1}^{4}\chi_i^{(4)}n_{i1}\int_0^{\infty}s\bar{f}_2(s)\cos(sx)ds = \frac{1}{\pi}\beta_5\int_0^{\infty}s\bar{f}_2(s)\cos(sx)ds \tag{A-12}$$

$$B_x^{(1)}(x,0) = \frac{1}{\pi}\sum_{i=1}^{4}[-\lambda_i f_{15}\beta_i^{(1)} - f_{15}\beta_i^{(2)} - g_{11}\beta_i^{(3)} + \mu_{11}\beta_i^{(4)}]$$

$$(m_{i1}+n_{i1})[\int_0^{\infty}s\bar{f}_1(s)\sin(sx)ds + \int_0^{\infty}s\bar{f}_2(s)\sin(sx)ds]$$

$$= \frac{1}{\pi}\beta_6\int_0^{\infty}s\bar{f}_2(s)\sin(sx)ds \tag{A-13}$$