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Discussion

The unsymmetric finite element formulation and variational incorrectness*

Discussion by S. Rajendran†

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The above paper by Prathap *et al.* presents an interesting investigation into the recently introduced *unsymmetric formulation* (Rajendran and Liew 2003, Rajendran and Subramanian 2004, Ooi *et al.* 2004, Prathap and Mukherjee 2004, Liew *et al.* 2006, Prathap *et al.* 2006). The paper concludes that although the elements based on this formulation work surprisingly well under mesh distortions, the formulation is "variationally incorrect". This discussion is aimed at revisiting this conclusion. This author is of opinion that the unsymmetric formulation strictly conforms to the principle of virtual work and hence must be variationally correct. On the other hand, the numerical examples reported by Prathap et al. do show clearly that the unsymmetric formulation does not give the best-fit approximation to the exact solution, and hence it seems correct to conclude that the unsymmetric formulation must be variationally incorrect. Why is this contraction? Is the formulation then variationally correct or incorrect?

Prathap *et al.* point out three inter-related aspects of a finite element formulation:

(i) *Variational correctness* (i.e., existence of a variational basis for the formulation).

(ii) *Orthogonality condition* (satisfaction of an orthogonality equation that arises from the projection theorem), and

(iii) *Best approximation property* (i.e., the ability to give best-fit solution in the trial space).

In the symmetric (Galerkin) formulations, the three aspects are closely inter-related and refer to the same idea. They mutually imply one another (see Fig. 1(a)). As a result, the variational correctness of a given symmetric formulation can be assessed simply by checking if the given formulation satisfies the best-approximation property. However, this approach to assess variational correctness does not seem appropriate for unsymmetric formulation. Prathap *et al.*, however, have used the same approach for the assessment of unsymmetric formulation.

One way to resolve the apparent contradiction is to disassociate the best-fit property from variational correctness, and view variational correctness and best-fit property as two independent aspects of an unsymmetric formulation. We may still associate the property of variational correctness with orthogonality condition (see Fig. 1(b)) because the orthogonality equation itself arises from the virtual work principle. Therefore, for a given unsymmetric formulation, the

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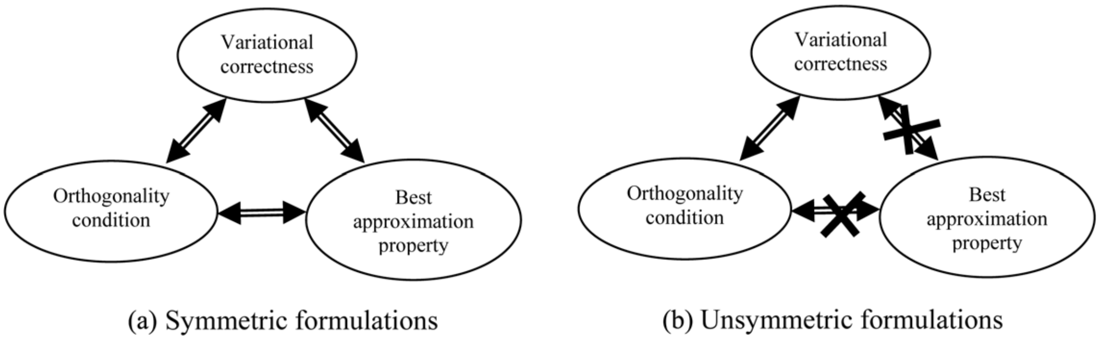

Fig. 1 Equivalence of properties

variational correctness may have to be assessed by checking if the formulation satisfies orthogonality condition. Thus, we have to conclude that the present unsymmetric formulation (Rajendran and Liew 2003, Rajendran and Subramanian 2004, Ooi *et al*. 2004, Prathap and Mukherjee 2004, Liew *et al.* 2006, Prathap *et al.* 2006) is variationally correct because we can show that it satisfies the orthogonality condition. And at the same time, we have to acknowledge that it does not have best-fit approximation property.

1. Orthogonality conditions for different formulations

The specific form of orthogonality condition applicable to the PP, MM and PM formulations (Rajendran and Subramanian 2004) depends on the choice of test and trial functions in the formulation concerned:

$$a(u-\bar{u}_{PP}, \bar{u}_w) = 0 \quad \text{(PP element)} \tag{2}$$

$$a(u-\hat{u}_{MM}, \hat{u}_w) = 0 \quad \text{(MM element)} \tag{3}$$

$$a(u-\hat{u}_{PM}, \bar{u}_w) = 0 \quad \text{(PM element)} \tag{4}$$

where $\bar{u}_{PP} \in \mathcal{P}$, $\hat{u}_{MM} \in \mathcal{M}$ and $\hat{u}_{PM} \in \mathcal{M}$ are the solutions given PP, MM and PM formulations, and $\bar{u}_w \in \mathcal{P}$ and $\hat{u}_w \in \mathcal{M}$ are the weight (test) functions. $\mathcal{P}$ is the *parametric space* defined using the parametric shape functions $\{N_1, N_2, N_3\}$ as the basis functions and $\mathcal{M}$ is the *metric space* defined using the metric shape functions $\{M_1, M_2, M_3\}$ as the basis functions.

For the one-element bar problem of Prathap *et al*., the conditions (2)-(4) can be easily verified to be satisfied by the PP, MM and PM formulations, respectively, using a computer algebra system (like Mathematica$^{@}$). The details are not shown here due to page limitation. Thus, it turns out that all the three formulations (viz., PP, MM and PM) do satisfy their respective orthogonal conditions which emerge directly from the virtual work principle. Thus, all the three formulations should be viewed variationally correct.

2. Graphical representation of orthogonality conditions (2)-(4)

The orthogonal conditions can be represented graphically as in Fig. 2. The graphical representation presented here is rather different from that of by Prathap et al. Before attempting to interpret this figure, it is important to realise that the vectors $(u-\bar{u}_{PP})$ and $(u-\hat{u}_{PM})$, which are represented by lines CA and BA respectively in Fig. 2, must be collinear. This is because both of them have to pass through point A and at the same time have to be orthogonal to $\mathcal{P}$-space (by virtue of Eqs. (2) and (4), respectively). Several interesting interpretations emerge from Fig. 2:

1. $\hat{u}_{MM}$ is an orthogonal projection of u in the $\mathcal{M}$-space and hence is the best-fit approximation of u in the $\mathcal{M}$-space. However, $\hat{u}_{PM}$ is not an orthogonal projection of u in the $\mathcal{M}$-space and hence is not the best-fit approximation of u in the $\mathcal{M}$-space (These observations are the same as that of Prathap *et al.*).
2. $\hat{u}_{PM}$ lies at the intersection between $\mathcal{M}$-space and a subspace spanned by u and $\bar{u}_{PP}$. Let us call the latter as the *$\mathcal{P}$-constraint space*. It is so named because this constraint space comes about because of our very choice of $\mathcal{P}$-space as the weighting space for PM formulation. Thus, the PM formulation gives a solution that lies at the intersection of $\mathcal{M}$-space and *$\mathcal{P}$-constraint space*.

Alternatively, we may interpret PM solution, $\hat{u}_{PM}$, as a projection of exact solution u in the $\mathcal{M}$-space subject to the constraint that $\hat{u}_{PM}$ must also lie in the *$\mathcal{P}$-constraint space*. Thus $\hat{u}_{PM}$ is a constrained optimal solution rather than the best-fit solution. On similar lines, we can re-interpret the MM and PP formulations: MM solution, $\hat{u}_{MM}$, is a projection of exact solution in the $\mathcal{M}$-space subject to the constraint that $\hat{u}_{MM}$ must also lie in the *$\mathcal{M}$-constraint space* (a subspace spanned by u and $\hat{u}_{MM}$). PP solution, $\bar{u}_{PP}$, is a projection of exact solution in the $\mathcal{P}$-space subject to the constraint that $\bar{u}_{PP}$ must also lie in the *$\mathcal{P}$-constraint space*.

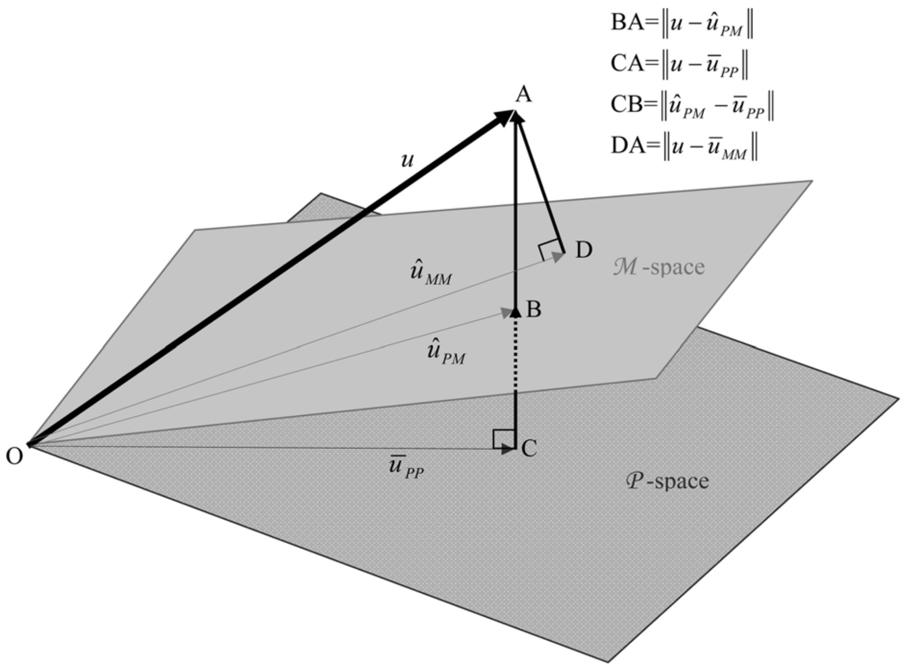

Fig. 2 Graphical representation of the orthogonality conditions related to PP, MM and PM formulations.

Thus, every formulation, whether it is PM, MM or PP, is variational in nature in the sense that we are dealing with a constrained minimization problem, i.e., we search for an optimum solution vector in the respective trial space subject to the condition that the solution vector also lies in the respective constraint space. In this sense, the PM formulation seems to have nothing contravariational about it. The only difference is that PP and MM formulations give the best-fit solutions whereas the PM formulation does not give a best-fit but gives only an 'optimal-fit' solution.

The conclusions of this discussion are as follows:

It seems reasonable to argue that the PM formulation is variationally correct although it does not give the best possible approximation of u in the $\mathcal{M}$-space. The idea mooted here is that for an unsymmetric formulation to be variationally correct, the formulation need not give the best-fit solution.

Theoretically, MM formulation gives the best approximation of u in the $\mathcal{M}$-space. Unfortunately, this best-fit property of MM formulation cannot always be taken for granted. Although the MM formulation does give good results for the 1-D element tried out by Prathap *et al.*, the MM formulation for 2-D and 3-D elements (Rajendran and Liew 2003, Rajendran and Subramanian 2004, Ooi *et al.* 2004, Liew *et al.* 2006) does not in general give a better approximation to exact solution than the PM formulation. The reason for this is that, for 2-D and 3-D elements, the metric shape functions loose their inter-element compatibility property particularly when mesh distortions are present. The incompatible metric shape functions then become unsuitable for use as test functions. (Note that virtual work principle demands the test function to be compatible. In this respect, we should rather say that the MM formulation tends to become variationally incorrect for 2D and 3D applications whenever mesh distortions are present.) Note that inter-element compatibility is never an issue in 1-D problems, but is of serious concern in 2-D problems and even more so in 3-D problems.

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Rajendran must be congratulated for offering a vastly improved graphical representation of the orthogonality conditions related to the PP, MM and PM formulations. It is clear from this that the strain/stress obtained from the PM formulation is not the best possible approximation of strain/stress in the *M*-space. However, it is the best possible solution where the continuity conditions (inter-element compatibility) are scrupulously maintained when mesh distortion is present.

Engineers are often concerned about whether the stress/strain computed is the best-fit of the actual state of stress. Prathap *et al.* interpret variational correctness in this utilitarian sense. Rajendran prefers the definition where variational correctness is linked to the orthogonality condition and we see this as a matter of preference rather than a contradiction.

It might be useful to point out that the quadrilateral area coordinate formulations which have been gaining attention recently as the answer to the problem of sensitivity to mesh distortion attempt solutions in the *M*-space but are forced to relax the continuity requirements through generalized conforming conditions. It appears therefore that the unsymmetric PM approach is the only one that maintains completeness and continuity rigorously, albeit at the cost of compromising on the best-fit aspect which the symmetric formulation permits.