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Free vibration analysis of a uniform beam carrying multiple spring-mass systems with masses of the springs considered

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Abstract. The reports regarding the free vibration analysis of uniform beams carrying single or multiple spring-mass systems are plenty, however, among which, those with inertia effect of the helical spring(s) considered are limited. In this paper, by taking the mass of the helical spring into consideration, the stiffness and mass matrices of a spring-mass system and an equivalent mass that may be used to replace the effect of a spring-mass system are derived. By means of the last element stiffness and mass matrices, the natural frequencies and mode shapes for a uniform cantilever beam carrying any number of spring-mass systems (or loaded beam) are determined using the conventional finite element method (FEM). Similarly, by means of the last equivalent mass, the natural frequencies and mode shapes of the same loaded beam are also determined using the presented equivalent mass method (EMM), where the cantilever beam elastically mounted by a number of lumped masses is replaced by the same beam rigidly attached by the same number of equivalent masses. Good agreement between the numerical results of FEM and those of EMM and/or those of the existing literature confirms the reliability of the presented approaches.

Keywords: mass of spring; cantilever beam; spring-mass system; equivalent mass; natural frequency; mode shape.

1. Introduction

In the field of aeronautics, naval architecture and civil engineering, the vibration characteristics of structures carrying various equipments, such as radar, oscillator, engine, absorbers, etc., are important information for structural engineers. Hence, the vibration problems, such as beams carrying elastically mounted concentrated masses, have been studied by many researchers. For example, Ercoli and Laura (1987), Larrondo *et al.* (1992), Rossit and Laura (2001), Gürgöze (1996, 1998, 1999), Wu and Chou (1998), Wu and Chen (2001) and Wang *et al.* (2007) have investigated the dynamic characteristics of beams carrying single or multiple one-degree-of-freedom (one-dof) spring-mass systems. Manikanahally and Crocker (1991), Dowell (1979), Nicholson and Bergman (1986) have researched the effects of single and multiple one-dof spring-mass absorbers. Wu *et al.* (1999) and Wu and Chen (2000) have studied the vibration characteristics of a uniform cantilever

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beam carrying multiple one-dof spring-damper-mass systems using the analytical-and-numerical-combined method. Chang and Chang (1998) have studied the free and forced vibrations of beams carrying a two-dof spring-mass system by means of Laplace transform with respect to the spatial variable. Wu and Whittaker (1999) and Wu (2002, 2003) have performed the free vibration analyses of beams carrying multiple two-dof spring-mass systems. Wu (2005) has studied the free vibration characteristics of a rectangular plate carrying multiple three-degree-of-freedom spring-mass systems. Chen and Liu (2006) have investigated the free and forced vibrations of a tapered beam carrying multiple point mass. Recently, Gürgöze (2005) has studied the fundamental natural frequency of a cantilever beam carrying a tip spring-mass system with mass of the spring considered. He took the mass of the helical spring into account by modeling the helical spring as an axially vibrating rod.

From the review of the preceding literature, it can be found that the inertia effect of the helical spring(s) is neglected in most of the existing vibration analyses except Gürgöze (2005). Because the total number of spring-mass systems attached to the beam is single in Gürgöze (2005), this paper aims at investigating the influence of mass of each helical spring on the dynamic characteristics of a cantilever beam carrying multiple spring-mass systems.

For convenience, a beam carries nothing is called the *bare beam*, while that carries any number of spring-mass systems is called the *loaded beam* in this paper. First, the equation of motion of the spring-mass system with mass of the helical spring considered is derived by means of Lagrange's equations. From the last equation of motion, the stiffness and mass matrices of the spring-mass system are obtained and the conventional finite element method (FEM) is used to determine the natural frequencies and mode shapes of the loaded beam. Next, from the force equilibrium equation, an equivalent mass with its dynamic effect to be the same as that of the spring-mass system is introduced so that the free vibration characteristics of a beam carrying any number of spring-mass systems can be obtained from those of the same beam carrying the same number of *rigidly attached* equivalent masses. The key point of the presented equivalent mass method (EMM) is to derive the characteristic equation of the loaded beam analytically using the above-mentioned *rigidly attached* equivalent masses together with the natural frequencies and mode shapes of the bare beam and then to solve the last characteristic equation numerically for the natural frequencies and mode shapes of the loaded beam. Since the equivalent mass of the spring-mass system is a function of the natural frequencies of the loaded beam, the cut and trial procedure is used in EMM. For validation, all numerical results obtained from EMM are compared with those obtained from FEM and good agreement was achieved. Because the order of the overall property matrices for the equations of motion of the loaded beam derived from EMM is much less than that derived from FEM, the computer storage memory required by EMM is much less than that required by FEM. This advantage of EMM will be more predominant if the total number of spring-mass systems attached to the beam is large. Besides, the presented EMM also provides an effective technique for evaluating the overall inertia effect of a spring-mass system attached to a beam.

2. Equation of motion of a spring-mass system

Fig. 1(a) shows an arbitrary spring-mass system attached to point i of the uniform beam. In which, $k^{(v)}, \overline{m}^{(v)}$ and $\ell^{(v)}$ are, respectively, the spring constant, mass per unit length and total length of the helical spring, while F_i and $F^{(v)}$ are, respectively, the interaction force at attaching point i and the external force on lumped mass $m^{(v)}$. Besides, u_i, $\dot{u}_i$ and $\ddot{u}_i$ are respectively the

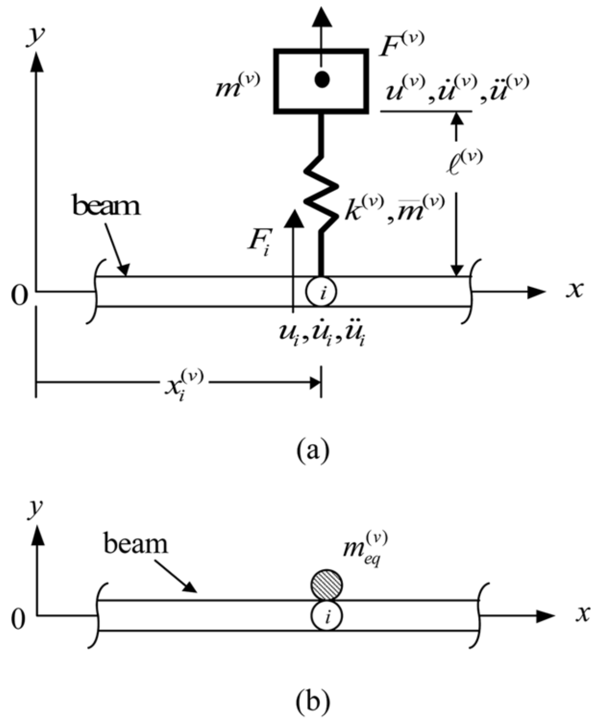

Fig. 1 (a) A uniform beam carrying an arbitrary spring-mass system can be replaced by (b) the same beam carrying a rigidly attached equivalent lumped mass $m_{eq}^{(v)}$

displacement, velocity and acceleration of attaching point i, while $u^{(v)}, \dot{u}^{(v)}$ and $\ddot{u}^{(v)}$ are those of lumped mass $m^{(v)}$. In the above symbols, the superscript v represents the numbering of the spring-mass system attached to the beam.

If the displacement for any infinitesimal element (dy) of the helical spring varies linearly from u_i to $u^{(v)}$, then the kinetic energy (T) and strain energy (V) of the spring-mass system are, respectively, given by

$$T = \frac{1}{2}m^{(v)}\dot{u}^{(v)^2} + \frac{1}{2}\int_0^{\ell^{(v)}} \bar{m}^{(v)}\left[\dot{u}_i + \frac{y}{\ell^{(v)}}(\dot{u}^{(v)} - \dot{u}_i)\right]^2 dy \tag{1}$$

$$V = \frac{1}{2}k^{(v)}[u^{(v)} - u_i]^2 \tag{2}$$

where the over dot $(\cdot)$ represents the differentiation with respect to time t and $\ell^{(v)}$ denotes the total length of the helical spring in static equilibrium position of the loaded beam.

Introducing Eqs. (1) and (2) into the following Lagrange's equations (Clough and Penzien 1975)

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{u}_i}\right) - \frac{\partial T}{\partial u_i} + \frac{\partial V}{\partial u_i} = F_i \tag{3a}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{u}^{(v)}}\right) - \frac{\partial T}{\partial u^{(v)}} + \frac{\partial V}{\partial u^{(v)}} = F^{(v)} \tag{3b}$$

one obtains

$$\frac{1}{3}m^{(v)}\ell^{(v)}\ddot{u}_i+\frac{1}{6}\overline{m}^{(v)}\ell^{(v)}\ddot{u}^{(v)}+k^{(v)}u_i-k^{(v)}u^{(v)} = F_i \tag{4a}$$

$$m^{(v)}\ddot{u}^{(v)}+\frac{1}{6}\overline{m}^{(v)}\ell^{(v)}\ddot{u}_i+\frac{1}{6}\overline{m}^{(v)}\ell^{(v)}\ddot{u}^{(v)}-k^{(v)}u_i+k^{(v)}u^{(v)} = F^{(v)} \tag{4b}$$

Writing Eqs. (4a) and (4b) in matrix form yields

$$\{F\}^{(v)} = [m]^{(v)}\{\ddot{u}\}^{(v)}+[k]^{(v)}\{u\}^{(v)} \tag{5}$$

where

$$[m]^{(v)} = m^{(v)}\begin{bmatrix} \frac{1}{3}\alpha^{(v)} & \frac{1}{6}\alpha^{(v)} \\ \frac{1}{6}\alpha^{(v)} & 1+\frac{1}{3}\alpha^{(v)} \end{bmatrix} \tag{6a}$$

$$k^{(v)} = \begin{bmatrix} k^{(v)} & -k^{(v)} \\ -k^{(v)} & k^{(v)} \end{bmatrix} \tag{6b}$$

$$\{\ddot{u}\}^{(v)} = [\ddot{u}_i \;\; \ddot{u}^{(v)}]^T \tag{6c}$$

$$\{u\}^{(v)} = [u_i \;\; u^{(v)}]^T \tag{6d}$$

$$\{F\}^{(v)} = [F_i \;\; F^{(v)}]^T \tag{6e}$$

with

$$\alpha^{(v)} = \overline{m}^{(v)}\ell^{(v)}/m^{(v)} \tag{6f}$$

In the preceding expressions, $[m]^{(v)}$ and $[k]^{(v)}$ are respectively the mass and stiffness matrices of the v-th spring-mass system, $\{\ddot{u}\}^{(v)}$ and $\{u\}^{(v)}$ are respectively the acceleration and displacement vectors, while $\{F\}^{(v)}$ is the external loading vector.

If the displacement and acceleration of attaching point i is zero, i.e., $u_i = 0$ and $\ddot{u}_i = 0$, then from Eqs. (5) and (6) one obtains

$$F^{(v)} = (m^{(v)})\cdot\left(1+\frac{1}{3}\alpha^{(v)}\right)\cdot\ddot{u}^{(v)}+k^{(v)}u^{(v)} = 0 \tag{7}$$

which is the equation of motion for free vibration of the v-th spring-mass system with respect to the *static* beam. From Eq. (7) one obtains the natural frequency of the v-th spring-mass system to be

$$\omega^{(v)} = \sqrt{\frac{k^{(v)}}{m^{(v)}\cdot\left(1+\frac{1}{3}\alpha^{(v)}\right)}} = \frac{\omega_0^{(v)}}{\sqrt{1+\frac{1}{3}\alpha^{(v)}}} \tag{8a}$$

where

$$\omega_0^{(v)} = \sqrt{k^{(v)}/m^{(v)}} \tag{8b}$$

In Eqs. (8a) and (8b), $\omega^{(v)}$ denotes the *local natural frequency* of the v-th spring-mass system with mass of helical spring considered (i.e., $\alpha^{(v)} \neq 0$), while $\omega_0^{(v)}$ denotes that with mass of helical spring neglected (i.e., $\alpha^{(v)} = 0$). From Eq. (8a) one sees that the influence of mass ratio ($\alpha^{(v)}$) on the *local natural frequency* $\omega^{(v)}$ is dependent on the value of $\omega_0^{(v)}$. If the numerical value of $\omega_0^{(v)}$ is large, then a small change of $\alpha^{(v)}$ will lead to larger variation of $\omega^{(v)}$, otherwise, the effect of $\alpha^{(v)}$ will be negligible.

3. Equivalent mass of the spring-mass system

The equation of motion for the lumped mass $m^{(v)}$ of the v-th spring-mass system shown in Fig. 1(a) is given by Eq. (4b) and the interaction force F_i at attaching point i is given by Eq. (4a). For a free vibrating loaded beam, one has

$$F^{(v)} = 0 \tag{9}$$

and

$$u_i = \bar{u}_i e^{j\bar{\omega}t} \tag{10a}$$

$$u^{(v)} = \bar{u}^{(v)} e^{j\bar{\omega}t} \tag{10b}$$

In the last equations, $\bar{u}_i$ and $\bar{u}^{(v)}$, respectively, represent the amplitude of u_i and $u^{(v)}$, $\bar{\omega}$ represents the natural frequency of the loaded beam, t is time and $j = \sqrt{-1}$.

From Eqs. (10a) and (10b), one obtains

$$u_i = -\frac{\ddot{u}_i}{\bar{\omega}^2} \tag{11a}$$

$$u^{(v)} = -\frac{\ddot{u}^{(v)}}{\bar{\omega}^2} \tag{11b}$$

Substituting Eqs. (9), (11a) and (11b) into Eq. (4b) yields

$$\ddot{u}^{(v)} = \left[\frac{6k^{(v)} + \bar{m}^{(v)}\ell^{(v)}\bar{\omega}^2}{6k^{(v)} - 2\bar{m}^{(v)}\ell^{(v)}\bar{\omega}^2 - 6m^{(v)}\bar{\omega}^2}\right]\ddot{u}_i \tag{12}$$

Introducing Eqs. (11a), (11b) and (12) into Eq. (4a) leads to

$$F_i = m_{eq}^{(v)}\ddot{u}_i \tag{13}$$

where

$$m_{eq}^{(v)} = \frac{\overline{m}^{(v)}\ell^{(v)}}{3} + \frac{1}{6}\left[\frac{\overline{m}^{(v)^2}\ell^{(v)^2}\overline{\omega}^2}{6k^{(v)} - 2\overline{m}^{(v)}\ell^{(v)}\overline{\omega}^2 - 6m^{(v)}\overline{\omega}^2}\right] + \frac{2\overline{m}^{(v)}\ell^{(v)}k^{(v)}}{6k^{(v)} - 2\overline{m}^{(v)}\ell^{(v)}\overline{\omega}^2 - 6m^{(v)}\overline{\omega}^2} - \frac{k^{(v)}}{\overline{\omega}^2}$$
$$+ \frac{6k^{(v)2}}{\overline{\omega}^2(6k^{(v)} - 2\overline{m}^{(v)}\ell^{(v)}\overline{\omega}^2 - 6m^{(v)}\overline{\omega}^2)} \tag{14a}$$

or

$$m_{eq}^{(v)} = \frac{m^{(v)}\alpha^{(v)}}{3} + \frac{1}{6}\left[\frac{m^{(v)^2}\alpha^{(v)^2}\overline{\omega}^2}{6k^{(v)} - 2m^{(v)}\alpha^{(v)}\overline{\omega}^2 - 6m^{(v)}\overline{\omega}^2}\right] + \frac{2m^{(v)}\alpha^{(v)}k^{(v)}}{6k^{(v)} - 2m^{(v)}\alpha^{(v)}\overline{\omega}^2 - 6m^{(v)}\overline{\omega}^2} - \frac{k^{(v)}}{\overline{\omega}^2}$$
$$+ \frac{6k^{(v)2}}{\overline{\omega}^2(6k^{(v)} - 2m^{(v)}\alpha^{(v)}\overline{\omega}^2 - 6m^{(v)}\overline{\omega}^2)} \tag{14b}$$

Eq. (13) reveals that the dynamic effects of the spring-mass system attached to the beam can be replaced by a *rigidly attached* equivalent mass (see Fig. 1(b)) with magnitude $m_{eq}^{(v)}$ given by Eqs. (14a,b). From Eq. (14a), one sees that the equivalent mass $m_{eq}^{(v)}$ is dependent on the following parameters of the spring-mass system: magnitude of lumped mass $m^{(v)}$, spring constant $k^{(v)}$, mass per unit spring length $\overline{m}^{(v)}$ and total length $\ell^{(v)}$ of the helical spring. Among the above-mentioned parameters of a spring-mass system, the parameter $\overline{m}^{(v)}$ denotes the inertial effect of the mass of the helical spring neglected by most of the existing literature.

4. Natural frequencies and mode shapes of the loaded beam

By neglecting the effects of shear deformation and rotatory inertia of the beam, the equation of motion for a uniform beam carrying p one-dof spring-mass systems takes the form (Wu and Chou 1998)

$$E_b I_b \frac{\partial^4 y(x,t)}{\partial x^4} + \overline{m}_b \frac{\partial^2 y(x,t)}{\partial t^2} + \sum_{v=1}^{p} F_i \delta(x - x_i^{(v)}) = 0 \tag{15}$$

where E_b and I_b are respectively the Young's modulus and area moment of inertia of the beam, $\overline{m}_b$ is the mass per unit length of the beam, $y(x,t)$ is the transverse deflection of the beam at position x and time t, F_i is the interaction force at the attaching point of the v^{th} spring-mass system, $x = x_i^{(v)}$, and $\delta(\cdot)$ is the Dirac delta function.

Substituting Eq. (13) into Eq. (15), one obtains

$$E_b I_b \frac{\partial^4 y(x,t)}{\partial x^4} + \overline{m}_b \frac{\partial^2 y(x,t)}{\partial t^2} = -\sum_{v=1}^{p} m_{eq}^{(v)} \frac{\partial^2 y(x,t)}{\partial t^2} \delta(x - x_i^{(v)}) \tag{16a}$$

Note that

$$u_i = y(x_i^{(v)}, t) \tag{16b}$$

$$\ddot{u}_i = \frac{\partial^2 y(x_i^{(v)}, t)}{\partial t^2} \tag{16c}$$

Based on the expansion theorem (Meirovitch 1967) or the mode superposition methodology (Clough and Penzien 1975), the transverse deflection of the beam is given by

$$y(x,t) = \sum_{s=1}^{n'} \overline{Y}_s(x) q_s(t) \tag{17}$$

where $\overline{Y}_s(x)$ represents the s^{th} mode shape of the bare beam, $q_s(t)$ is a generalized co-ordinate, and n' is the number of total modes considered.

Introducing Eq. (17) into Eq. (16a), multiplying the resulting expression by $\overline{Y}_r(x)$ and integrating the entire equation over the total beam length ℓ_b, one has

$$\int_0^{\ell_b} \sum_{s=1}^{n'} \overline{Y}_r(x) E_b I_b \overline{Y}_s''''(x) q_s(t) dx + \int_0^{\ell_b} \sum_{s=1}^{n'} \overline{Y}_r(x) \overline{m}_b \overline{Y}_s(x) \ddot{q}_s(t) dx$$
$$= -\int_0^{\ell_b} \sum_{v=1}^{p} \overline{Y}_r(x) m_{eq}^{(v)} \sum_{s=1}^{n'} \overline{Y}_s(x) \ddot{q}_s(t) \delta(x - x_i^{(v)}) dx \tag{18}$$

If the mode shapes $\overline{Y}_s(x)$ (s = 1 to n') are normalized with respect to $\overline{m}_b$, then application of orthogonality of normal mode shapes to Eq. (18) leads to

$$\ddot{q}_r(t) + \Omega_r^2 q_r(t) = -\sum_{v=1}^{p} \sum_{s=1}^{n'} m_{eq}^{(v)} \overline{Y}_r(x_i^{(v)}) \overline{Y}_s(x_i^{(v)}) \ddot{q}_s(t), \quad r = 1 \text{ to } n' \tag{19}$$

where Ω_r represents the r^{th} natural frequency of the bare beam.

If the loaded beam performs free harmonic vibration, the generalized co-ordinate $q_s(t)$ takes the form

$$q_s(t) = \overline{q}_s e^{j\overline{\omega} t} \tag{20}$$

where $\overline{q}_s$ is the amplitude of $q_s(t)$ and $\overline{\omega}$ is the natural frequency of the loaded beam, as stated previously.

Substituting Eq. (20) into Eq. (19) yields

$$\Omega_r^2 \overline{q}_r - \sum_{v=1}^{p} \sum_{s=1}^{n'} m_{eq}^{(v)} \overline{\omega}^2 \overline{Y}_r(x_i^{(v)}) \overline{Y}_s(x_i^{(v)}) \overline{q}_s = \overline{\omega}^2 \overline{q}_r, \quad r = 1 \text{ to } n' \tag{21}$$

Writing the last equation in matrix form, one obtains

$$([A] - \overline{\omega}^2 [B])\{\overline{q}\} = \{0\} \tag{22}$$

where

$$[A]_{n' \times n'} = [\Omega^2]_{n' \times n'} \tag{23a}$$

$$[B]_{n' \times n'} = [I]_{n' \times n'} + [B']_{n' \times n'} \tag{23b}$$

$$[I]_{n' \times n'} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & \vdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}_{n' \times n'} \tag{23c}$$

$$[B']_{n' \times n'} = \sum_{v=1}^{p} m_{eq}^{(v)} [\overline{Y}(x_i^{(v)})]_{n' \times n'} \tag{23d}$$

$$[\overline{Y}(x)]_{n' \times n'} = \begin{bmatrix} \overline{Y}_1(x)\overline{Y}_1(x) & \overline{Y}_1(x)\overline{Y}_2(x) & \dots & \overline{Y}_1(x)\overline{Y}_{n'}(x) \\ \overline{Y}_2(x)\overline{Y}_1(x) & \overline{Y}_2(x)\overline{Y}_2(x) & \dots & \overline{Y}_2(x)\overline{Y}_{n'}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \overline{Y}_{n'}(x)\overline{Y}_1(x) & \overline{Y}_{n'}(x)\overline{Y}_2(x) & \dots & \overline{Y}_{n'}(x)\overline{Y}_{n'}(x) \end{bmatrix} \tag{23e}$$

$$\{q\}_{n' \times 1} = \{\overline{q}_1 \ \ \overline{q}_2 \ \ \dots \ \ \overline{q}_{n'}\}_{n' \times 1} \tag{23f}$$

$$[\Omega^2]_{n' \times n'} = \begin{bmatrix} \Omega_1^2 & 0 & \dots & 0 & 0 \\ 0 & \Omega_2^2 & \dots & \vdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \Omega_{n'-1}^2 & 0 \\ 0 & 0 & \dots & 0 & \Omega_{n'}^2 \end{bmatrix}_{n' \times n'} \tag{23g}$$

Eq. (21) or (22) is the characteristic equation of the loaded beam. In the last expressions, the symbols, { } and [ ] represent the column matrix and square matrix, respectively.

From Eq. (17) one may infer that the mode shape of the loaded beam corresponding to the natural frequency $\overline{\omega}$ obtained from Eq. (22) is given by

$$Y(x) = \sum_{s=1}^{n'} \overline{Y}_s(x)\overline{q}_s \tag{24}$$

5. Solution of the problem

In this paper, the natural frequencies and the corresponding mode shapes of the beam carrying any number of spring-mass systems are firstly determined by means of the presented equivalent mass method (EMM), where each spring-mass system is replaced by a rigidly attached equivalent lumped mass $m_{eq}^{(v)}$ defined by Eqs. (14a,b). Then, the last results are compared with those obtained by using the conventional finite element method (FEM), where each spring-mass system is considered as a finite element with its mass and stiffness matrices defined by Eqs. (6a) and (6b). The solution procedures of EMM and FEM are briefly described below.

5.1 By using the equivalent mass method (EMM)

Eq. (22) is the characteristic equation of the loaded beam. Non-trivial solution of Eq. (22) requires that

$$\left|[A] - \overline{\omega}^2[B]\right| = 0 \tag{25}$$

Since the equivalent mass $m_{eq}^{(v)}$ for each spring-mass system is a function of natural frequency $\overline{\omega}$ of the loaded beam, as one may see from Eqs. (14a,b), so are the matrices $[B']$ and $[B]$ defined by Eqs. (23b,c, d). The half-interval method (Carnahan *et al.* 1997) is used to solve the eigenvalue $\overline{\omega}$ from Eq. (25). Then, the corresponding eigenvector $\{\overline{q}\}$ is obtained by substituting the values of $\overline{\omega}$ into Eq. (22). Finally, the corresponding mode shape of the loaded beam is determined by Eq. (24). It is to be noted that, each time, one may obtain only one natural frequency and one corresponding mode shape of the loaded beam from Eqs. (22)-(25). Thus, the same task must be repeated n times for the determination of n natural frequencies and n corresponding mode shapes of the loaded beam.

5.2 By using the finite element method (FEM)

For free vibration of an undamped uniform beam carrying any number of spring-mass systems, its equations of motion take the form

$$[M]\{\ddot{u}\} + [K]\{u\} = \{0\} \tag{26}$$

where $\{u\}$ and $\{\ddot{u}\}$ are the overall displacement and acceleration vectors, respectively; while $[M]$ and $[K]$ are the overall mass and stiffness matrices of the loaded beam, respectively. The matrices, $[M]$ and $[K]$, can be obtained by adding the element property matrices of each spring-mass system, given by Eqs. (6a) and (6b), to the overall ones of the beam itself by using the standard finite element assembly technique (Bathe 1982) and imposing the prescribed boundary conditions.

For free vibration, one has $\{u\} = \{\hat{u}\}e^{j\overline{\omega}t}$. The substitution of the last relation into Eq. (26) leads to

$$([K] - \overline{\omega}^2[M])\{\hat{u}\} = \{0\} \tag{27}$$

Eq. (27) is a typical eigenvalue problem, thus many techniques may be used to determine the eigenvalues ($\overline{\omega}_r$) and the corresponding eigenvectors $\{\hat{u}\}_r$, r = 1, 2, etc. In this paper, Eq. (27) was solved with the Lanczos algorithm (Golub 1972).

From the preceding descriptions, one sees that, in the conventional FEM, the order of the overall property matrices, $[M]$ and $[K]$, is equal to the total degree of freedom of the entire vibrating system and increases one when one more spring-mass system is attached to the bare beam. However, in the presented EMM, the order of the effective matrices, $[A]$ and $[B]$, is equal to the number of total modes (n') considered and does not increase with increasing the total number of the spring-mass systems attached to the beam. Since the order of the overall effective matrices, $[A]$ and $[B]$, derived from EMM is much lower than that of the overall property matrices, $[M]$ and $[K]$, derived from the conventional FEM, the computer storage memory required by EMM is much less than that required by FEM. This advantage will be more predominant if the total number of the spring-mass systems attached to the beam is large.

6. Numerical results and discussions

In this section, the natural frequencies and mode shapes of an undamped uniform cantilever beam carrying a spring-mass system and those carrying multiple ones are investigated. The beam length is ℓ_b = 1.0 m and its cross-sectional area is $A_b = \pi d^2/4$ = 1.9635 × 10^{-3} m^2, it is made of steel with mass density ρ = 7.8367 × 10^3 kg/m^3 and Young's modulus E_b = 2.069 × 10^{11} N/m^2. The beam is composed of 20 identical beam elements and 21 nodes (for FEM), and the total number of modes used for the mode superposition method is n' = 8 (in EMM).

6.1 Validation

Fig. 2 shows a cantilever beam carrying a spring-mass system at free end. The spring constant and lumped mass of the spring-mass system are $k^{(v)} = k^{(1)}$ = 6.34761 × 10^6 N/m and $m^{(v)} = m^{(1)}$ = 7.69375 kg. According to Eq. (6f), the mass ratio for the current spring-mass system is given by $\alpha^{(v)} = \alpha^{(1)} = \overline{m}^{(1)}\ell^{(1)}/m^{(1)}$.

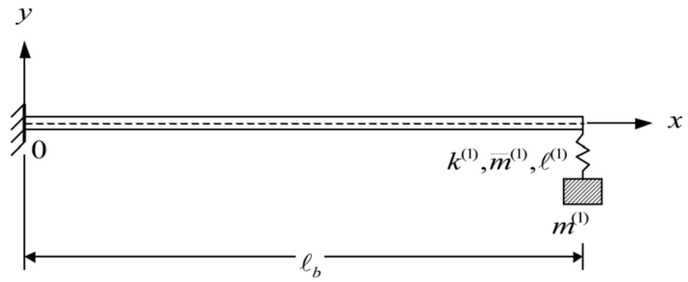

Fig. 2 A cantilever beam carrying a spring-mass system at free end

Table 1 Influence of mass ratio ($\alpha^{(1)} = \overline{m}^{(1)}\ell^{(1)}/m^{(1)}$) on the first five natural frequencies, $\overline{\omega}_i$ (i = 1 to 5), of a cantilever beam carrying a spring-mass system at free end, with spring constant $k^{(1)}$ = 6.34761 × 10^6 N/m and lumped mass $m^{(1)}$ = 7.69375 kg (see Fig. 2)

Mass ratios $\alpha^{(1)}$	Methods	Natural frequencies of the loaded beam (rad/s)					Local frequency $\omega^{(1)}$
		$\overline{\omega}_1$	$\overline{\omega}_2$	$\overline{\omega}_3$	$\overline{\omega}_4$	$\overline{\omega}_5$	
0.0	EMM	**128.6216**	**972.2608**	**2132.9760**	**4210.7803**	**7879.6440**	908.3142
	FEM	127.7321	971.8482	2131.3710	4210.0970	7879.7680	
0.025	EMM	**127.5566**	**971.5363**	**2117.7605**	**4167.9802**	**7809.5162**	904.5530
	FEM	126.5945	971.1980	2116.3230	4167.6300	7809.9880	
0.05	EMM	**126.5174**	**970.8274**	**2102.7833**	**4127.9276**	**7746.2056**	900.8382
	FEM	124.9216	970.4556	2101.4120	4127.9500	7747.0760	
0.075	EMM	**125.5031**	**970.1132**	**2088.0486**	**4090.4379**	**7688.9390**	897.1688
	FEM	123.5210	969.4702	2087.0270	4090.9410	7690.1840	
0.1	EMM	**124.5128**	**969.4532**	**2073.5592**	**4055.3322**	**7637.0086**	893.5438
	FEM	122.7895	969.0273	2072.8820	4056.4710	7638.6250	
0.0	(Wu 2002)	128.6211	971.9848	2131.5152	4210.2821	7879.7358	908.3142

To confirm the reliability of the presented theory and the developed computer programs, the values of mass ratios $\alpha^{(1)}$ are taken to be 0.0, 0.025, 0.05, 0.075 and 0.1. Clearly, $\alpha^{(1)}$ = 0.0 indicates that the mass of the spring is neglected in the analysis. In other words, the natural frequencies of the loaded beam with mass of the helical spring considered will approach those with mass of the helical spring neglected if one sets the value of mass ratio $\alpha^{(1)}$ to approach zero (i.e., $\alpha^{(1)} \to 0$).

Table 1 shows the first five natural frequencies, $\overline{\omega}_i$ (i = 1 to 5), of the cantilever beam carrying a spring-mass system at free end, as shown in Fig. 2. From the table, it can be seen that the natural frequencies obtained from the presented EMM are very close to the corresponding ones obtained from the conventional FEM. Besides, the natural frequencies of the loaded beam approach those with $\alpha^{(1)} = 0.0$ and the latter are very close to the corresponding ones obtained from (Wu 2002), listed in the final row of Table 1. The last result is to be expected because the mass of the helical spring is neglected in (Wu 2002).

Because all the first five mode shapes of the loaded beam with mass ratio $\alpha^{(1)}$ = 0.0, 0.025, 0.05, 0.075 or 0.1 look similar, only those with $\alpha^{(1)}$ = 0.1 are shown in Fig. 3. In which, the solid curves with circles, crosses, triangles, rectangles and stars (—○—, —+—, —Δ—, —□—, —☆—) respectively represent the 1st, 2nd, 3rd, 4th and 5th mode shapes obtained from EMM, while the dashed ones (--○--, --+--, --Δ--, --□--, --☆--) respectively represent those obtained from conventional FEM. From the Fig., one finds that the mode shapes obtained from the presented EMM are in good agreement with those obtained from FEM. Based on all the preceding facts, it is believed that the presented EMM is viable for the title problem.

In Table 1, the influence of the mass ratios $\alpha^{(1)}$ on the *local natural frequency* of the spring-

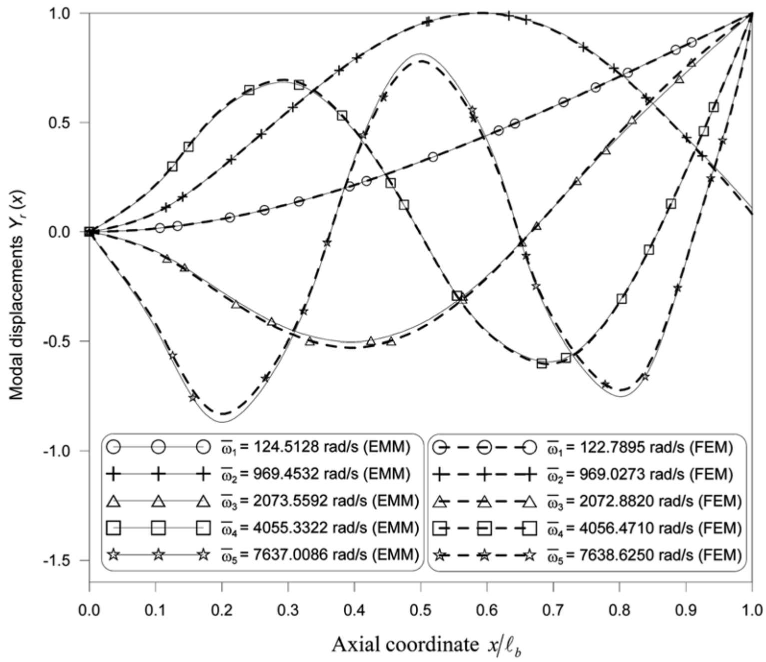

Fig. 3 The first five mode shapes, $Y_r(x)$ (r = 1 to 5), of a cantilever beam carrying a spring-mass system with mass ratio $\alpha^{(1)}$ = 0.1, obtained from EMM (——) and FEM (-----)

mass system with respect to the static cantilever, $\omega^{(1)} = \sqrt{\alpha^{(1)}/[m^{(1)}(1+1/3\,\alpha^{(1)})]}$, is also listed in the final column. It is evident that the local natural frequency $\omega^{(1)}$ decreases with increasing the mass ratio $\alpha^{(1)}$. From Table 1, one also sees that the influence of mass ratio $\alpha^{(1)}$ on the natural frequencies of the loaded beam is dependent on its influence on the local natural frequency of the spring-mass system, $\omega^{(1)}$. If consideration of the spring mass will lead to significant change of

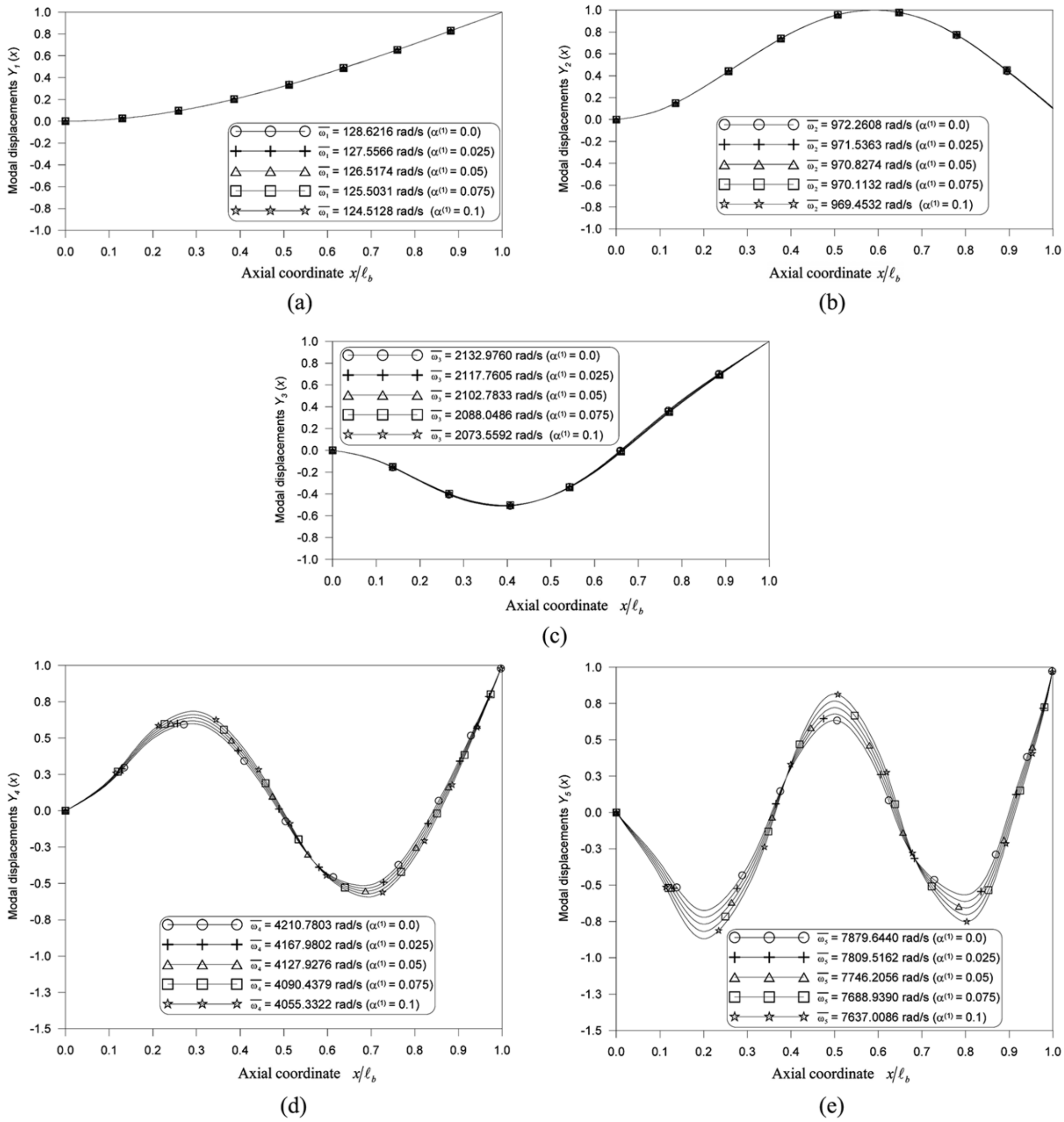

Fig. 4 Influence of mass ratios, $\alpha^{(1)}$, on the (a) 1st, (b) 2nd, (c) 3rd, (d) 4th and (e) 5th mode shapes of the cantilever beam carrying a spring-mass system at free end with spring constant $k^{(1)} = 6.34761 \times 10^6$ N/m and lumped mass $m^{(1)} = 7.69375$ kg

the value of $\omega^{(1)}$, then the effect of spring mass may be significant, otherwise, the effect will be negligible. The last phenomenon well agrees with the numerical results presented in (Gürgöze 2005).

Figs. 4(a)-4(e) show the first five mode shapes of the loaded beam, respectively. In these figures, the solid curves with circles, crosses, triangles, rectangles and stars (—○—, —+—, —Δ—, —□—, —★—) are, respectively, for the cases with mass ratios $\alpha^{(1)}$ = 0.0, 0.025, 0.05, 0.075 and 0.1. From the figures, one finds that the influence of mass ratio $\alpha^{(1)}$ on the 1st, 2nd and 3rd mode shapes of the loaded beam is negligible. However, this is not true for higher modes of the loaded beam. For the 4th and 5th mode shapes of the loaded beam, the larger the mass ratio $\alpha^{(1)}$, the larger the vibration amplitude of the loaded beam, as one may see from Figs. 4(d) and 4(e).

6.2 Influence of local natural frequency of the spring-mass system, $\omega_0^{(v)}$

All the physical parameters of the current cantilever beam are exactly the same as those of the last subsection, and the mass and mass ratio of the spring-mass system are taken to be $m^{(v)} = m^{(1)}$ = 7.69375 kg and $\alpha^{(v)} = \alpha^{(1)}$ = 0.1. If the spring constants are $k^{(v)} = k^{(1)}$ = 6.34761 × 10^5, 6.34761 × 10^6 and 6.34761 × 10^7 N/m, then the corresponding local natural frequencies of the spring-mass system are given by $\omega_0^{(v)} = \omega_0^{(1)} = \sqrt{k^{(1)}/m^{(1)}}$ = 287.234, 908.314 and 2872.342 rad/s, respectively.

Table 2 shows the first five natural frequencies of the loaded beam, $\overline{\omega}_i$ (i = 1 to 5). In which, the ones shown in the 3rd, 5th and 7th rows are obtained from EMM, while those in the 4th, 6th and 8th rows are obtained from FEM. From the table, it is seen that the natural frequencies obtained from the former are also in good agreement with those obtained from the latter. In addition, the first five natural frequencies of the loaded beam increase with increasing the magnitude of $\omega_0^{(1)}$.

Figs. 5(a)-(5e) show the first five mode shapes of the loaded beam, respectively. Where the solid curves with circles, crosses and triangles (—○—, —+—, —Δ—) are, respectively, for the cases with $\omega_0^{(1)}$ = 287.234, 908.314 and 2872.342 r/s. From the figures, one sees that the influence of local natural frequency $\omega_0^{(1)}$ of the spring-mass system on the first five mode shapes of the loaded beam is significant except the first mode.

Table 2 Influence of local natural frequency ($\overline{\omega}_0^{(1)} = \sqrt{k^{(1)}/m^{(1)}}$) on the first five natural frequencies, $\overline{\omega}_i$ (i = 1 to 5), of a cantilever beam carrying a spring-mass system at free end, with lumped mass $m^{(1)}$ = 7.69375 kg, mass ratio $\alpha^{(1)}$ = 0.1 and spring constants $k^{(1)}$ = 6.34761 × 10^5, 6.34761 × 10^6 and 6.34761 × 10^7 N/m. (see Fig. 2)

Local frequency $\overline{\omega}_0^{(1)} = \sqrt{k^{(1)}/m^{(1)}}$ (rad/s)	Methods	Natural frequencies of the loaded beam (rad/s)				
		$\overline{\omega}_1$	$\overline{\omega}_2$	$\overline{\omega}_3$	$\overline{\omega}_4$	$\overline{\omega}_5$
287.234	EMM	**117.3535**	**502.3222**	**1429.8305**	**3866.0634**	**7561.1117**
	FEM	115.9275	501.4681	1430.0860	3866.7790	7561.8920
908.314	EMM	**124.5128**	**969.4532**	**2073.5592**	**4055.3322**	**7637.0086**
	FEM	122.7895	969.0273	2072.8820	4056.4710	7638.6250
2872.342	EMM	**125.2592**	**1068.8252**	**3179.8941**	**6012.4805**	**8949.7939**
	FEM	124.5058	1068.2870	3175.8000	5992.6290	8930.0630

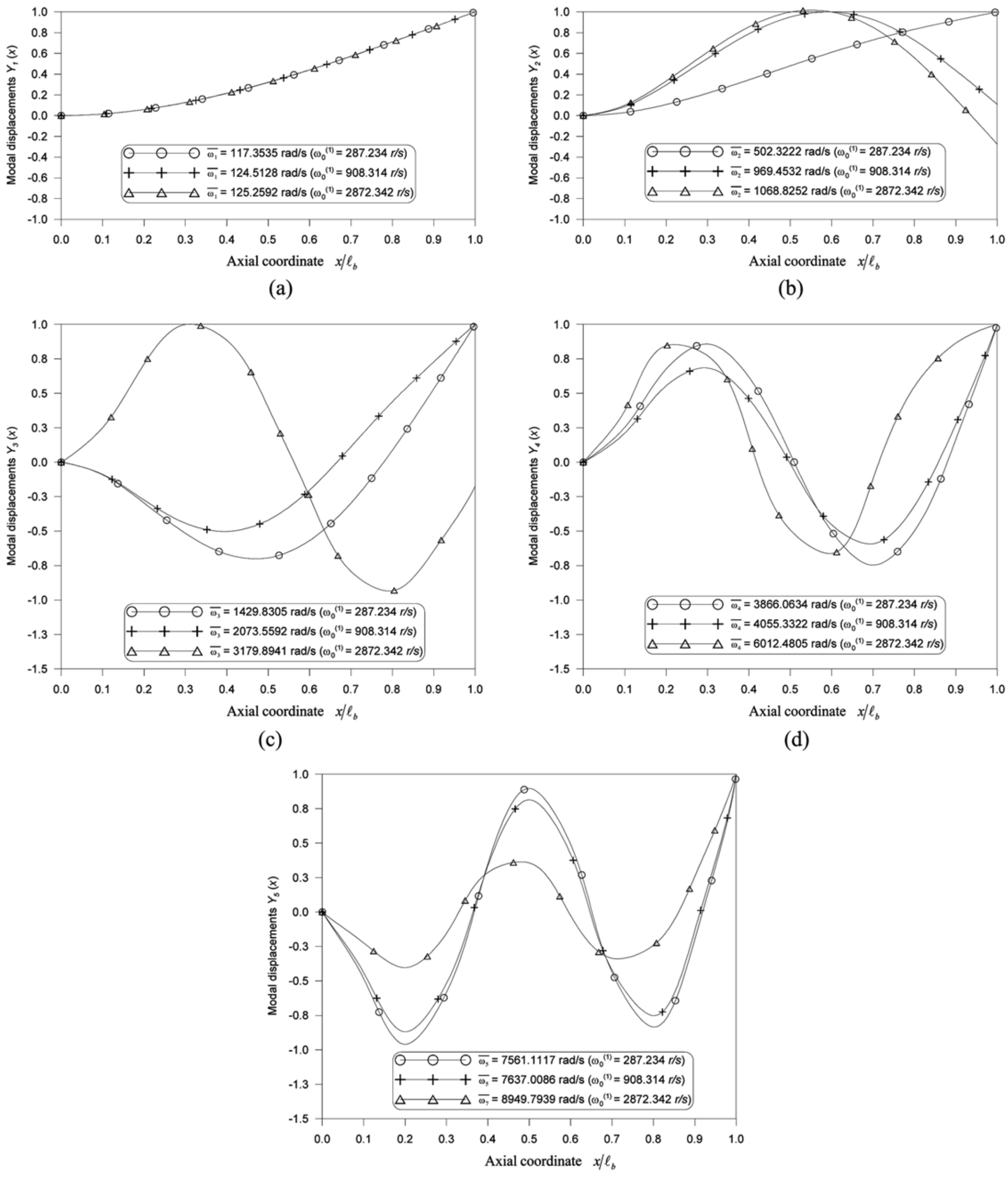

Fig. 5 Influence of local natural frequency $\overline{\omega}_0^{(1)} = \sqrt{k^{(1)}/m^{(1)}}$ of the spring-mass system on the (a) 1st, (b) 2nd, (c) 3rd, (d) 4th and (e) 5th mode shapes of the cantilever beam carrying a spring-mass system at free end with lumped mass $m^{(1)} = 7.69375$ kg and mass ratio $\alpha^{(1)} = \overline{m}^{(1)}\ell^{(1)}/m^{(1)} = 0.1$

6.3 A cantilever beam carrying three identical spring-mass systems

To show the applicability of the presented EMM, a cantilever beam carrying three identical

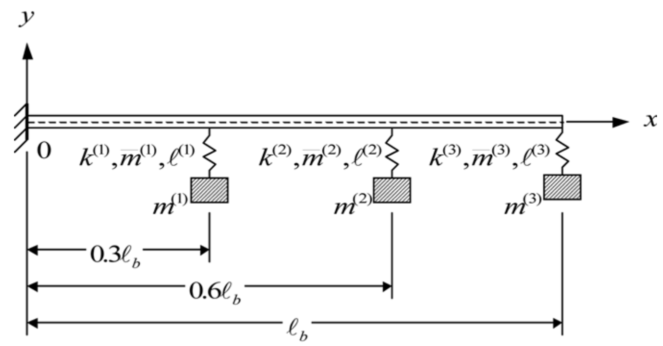

Fig. 6 A cantilever beam carrying three spring-mass systems

Table 3 Influence of mass ratio ($\alpha^{(1)} = \alpha^{(2)} = \alpha^{(3)}$) on the first five natural frequencies, $\overline{\omega}_i$ (i = 1 to 5), of a cantilever beam carrying three spring-mass systems, with spring constants $k^{(1)} = k^{(2)} = k^{(3)} = 2.11587 \times 10^6$ N/m and lumped masses $m^{(1)} = m^{(2)} = m^{(3)} = 2.56458$ kg (see Fig. 6)

Mass ratios $\alpha^{(1)} = \alpha^{(2)} = \alpha^{(3)}$	Methods	Natural frequencies of the loaded beam (rad/s)				
		$\overline{\omega}_1$	$\overline{\omega}_2$	$\overline{\omega}_3$	$\overline{\omega}_4$	$\overline{\omega}_5$
0.0	EMM	**161.8889**	**758.6370**	**885.0965**	**1191.7112**	**1764.7888**
	FEM	161.0875	746.8648	885.3082	1204.2160	1780.5730
0.025	EMM	**160.9267**	**754.8351**	**881.2515**	**1189.6656**	**1759.6329**
	FEM	160.7290	742.9200	881.4614	1201.9980	1775.2110
0.05	EMM	**159.9814**	**751.0923**	**877.4562**	**1187.6214**	**1754.5113**
	FEM	160.1158	739.2520	877.6671	1200.1320	1769.8210
0.075	EMM	**159.0526**	**747.4072**	**873.7095**	**1185.5790**	**1749.4241**
	FEM	157.2619	735.4629	873.9199	1197.8470	1764.3080
0.1	EMM	**158.1398**	**743.7782**	**870.0105**	**1183.5386**	**1744.3714**
	FEM	159.0308	731.9166	870.2184	1196.0170	1759.6920

spring-mass systems, as shown in Fig. 6, is investigated in this subsection. The physical parameters of the beam are exactly the same as those of the last example, while those of the spring-mass systems are $k^{(1)} = k^{(2)} = k^{(3)} = 6.34761 \times 10^6/3$ N/m $= 2.11587 \times 10^6$ N/m, $m^{(1)} = m^{(2)} = m^{(3)} = 7.69375/3$ kg $= 2.56458$ kg and $\alpha^{(1)} = \alpha^{(2)} = \alpha^{(3)} = 0.0$, 0.025, 0.05, 0.075 and 0.1. The locations for the attaching points of the three spring-mass systems are shown in Fig. 6. It is noted that the spring constant and lumped mass for each of the three identical spring-mass systems are equal to 1/3 of the corresponding ones for the single spring-mass system studied in section 6.1.

The first five natural frequencies of the loaded beam are listed in Table 3 and the corresponding mode shapes obtained from EMM are shown in Fig. 7. In the last figure, the solid curves with circles, crosses, triangles, rectangles and stars (—○—, —+—, —Δ—, —□—, —★—) are, respectively, for the cases with mass ratios $\alpha^{(1)} = \alpha^{(2)} = \alpha^{(3)} = 0.0$, 0.025, 0.05, 0.075 and 0.1. From Table 3, one sees that the natural frequencies obtained from the presented EMM are very close to those obtained from FEM, and the mass ratio $\alpha^{(1)}$ ($= \alpha^{(2)} = \alpha^{(3)}$) influences the first five natural frequencies of the loaded beam to some degree. However, from Fig. 7, it is found that the influence

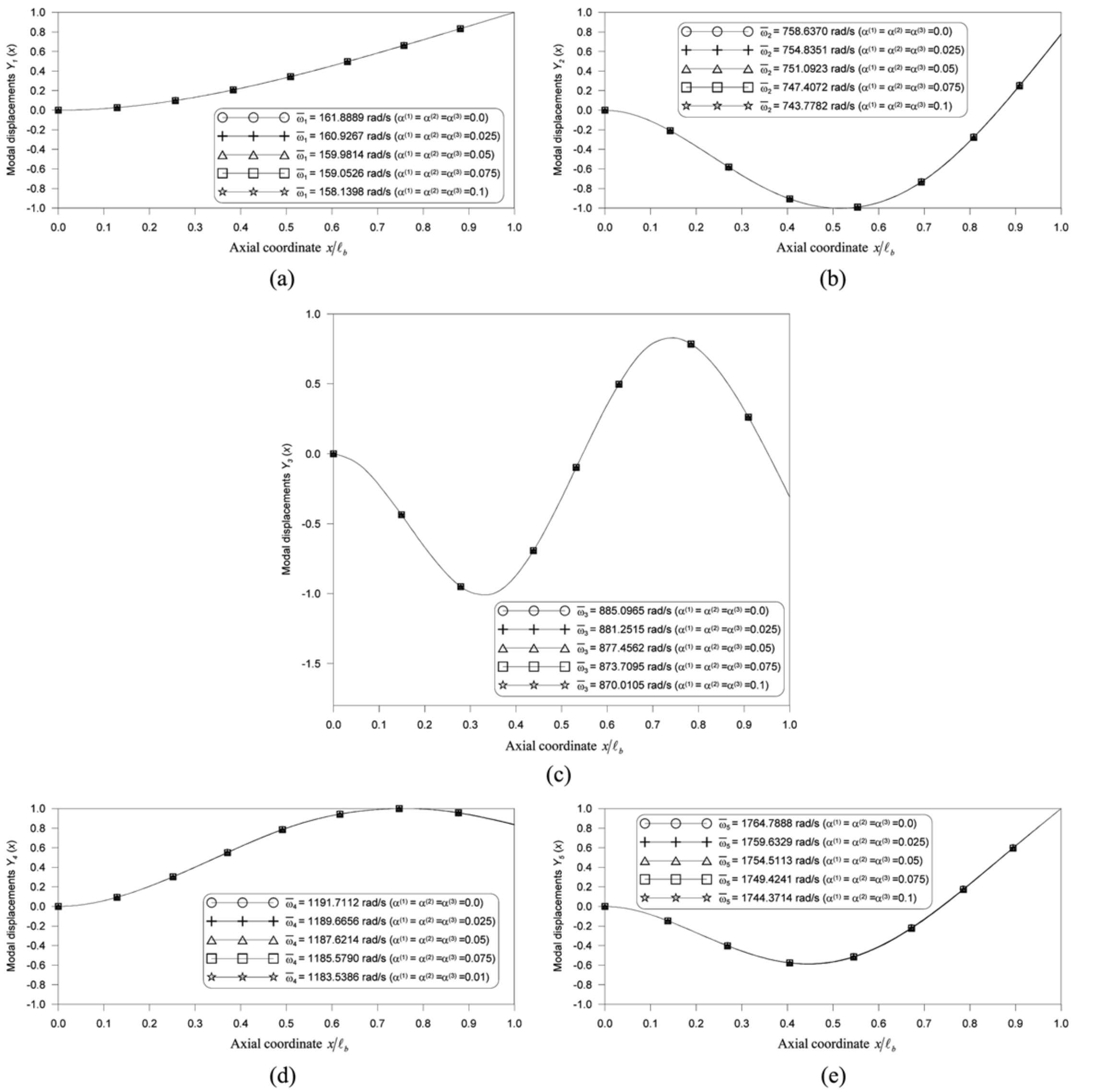

Fig. 7 Influence of mass ratios, $\alpha^{(1)} = \alpha^{(2)} = \alpha^{(3)}$, on the (a) 1st, (b) 2nd, (c) 3rd, (d) 4th and (e) 5th mode shapes of the cantilever beam carrying three spring-mass systems with spring constant $k^{(1)} = k^{(2)} = k^{(3)} = 2.11587 \times 10^6$ N/m and lumped mass $m^{(1)} = m^{(2)} = m^{(3)} = 2.56458$ kg

of mass ratio $\alpha^{(1)}$ (= $\alpha^{(2)} = \alpha^{(3)}$) on the first five mode shapes of the loaded beam is negligible. This phenomenon is different from that for the same cantilever beam carrying a spring-mass system at free end studied in the preceding sections. For a cantilever beam, its dynamic responses due to a tip load are usually much larger than the corresponding ones due to the distributed loads along the beam length, if the summation of magnitudes of the distributed loads is equal to the magnitude of the tip load. It is believed that the similar reason may be used to explain why the first five mode shapes of a cantilever beam carrying three identical spring-mass systems are different from those carrying a spring-mass system.

From the foregoing discussions, one may conclude that both the mass ratio $\alpha^{(v)}$ and the distribution of the spring-mass systems along the beam length are important factors affecting the dynamic characteristics of the loaded beam.

7. Conclusions

1. This paper presents the theory of equivalent mass method (EMM) such that the free vibration characteristics of a beam carrying any number of spring-mass systems, with inertia effect of helical springs considered, may be obtained from those of the same beam carrying the same number of rigidly attached equivalent masses. Because the magnitude for equivalent mass of a spring-mass system is dependent on the lumped mass, spring constant, mass per unit spring length and total length of the helical spring, the presented EMM also provides a technique for evaluating the overall inertia effect of a spring-mass system.
2. If $\omega^{(v)}$ and $\omega_0^{(v)}$ are the *local natural frequencies* of the v-th spring-mass system (with respect to the static beam) with mass of its helical spring considered and neglected, respectively, and $\alpha^{(v)}$ is the mass ratio of the total mass of the helical spring ($\bar{m}^{(v)}\ell^{(v)}$) to the lumped mass ($m^{(v)}$), then the influence of mass ratio $\alpha^{(v)}$ on the free vibration characteristics of a loaded beam is dependent on the magnitude of $\omega_0^{(v)}$. For a spring-mass system with lager value of $\omega_0^{(v)}$, a small change of $\alpha^{(v)}$ will lead to larger variation of $\omega^{(v)}$, and in turn, larger influence on the vibration characteristics of the loaded beam.
3. For a cantilever beam carrying a spring-mass system at its free end, the influence of mass ratio $\alpha^{(1)}$ on the 1st, 2nd and 3rd mode shapes of the loaded beam is negligible. However, this is not true for the 4th and 5th ones with larger vibration amplitudes due to larger mass ratio $\alpha^{(1)}$.
4. If the magnitudes for the physical parameters of a tip spring-mass system (such as lumped mass, spring constant, …) are equal to the summation magnitudes for the corresponding ones of multiple uniformly distributed spring-mass systems, then the influence on the dynamic characteristics of a uniform cantilever beam due to single tip spring-mass system will be much larger than that due to multiple uniformly distributed ones.

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