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Solution of randomly excited stochastic differential equations with stochastic operator using spectral stochastic finite element method (SSFEM)

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Abstract. This paper considers the solution of the stochastic differential equations (SDE^s) with random operator and/or random excitation using the spectral SFEM. The random system parameters (involved in the operator) and the random excitations are modeled as second order stochastic processes defined only by their means and covariance functions. All random fields dealt with in this paper are continuous and do not have known explicit forms dependent on the spatial dimension. This fact makes the usage of the finite element (FE) analysis be difficult. Relying on the spectral properties of the covariance function, the Karhunen-Loeve expansion is used to represent these processes to overcome this difficulty. Then, a spectral approximation for the stochastic response (solution) of the SDE is obtained based on the implementation of the concept of generalized inverse defined by the Neumann expansion. This leads to an explicit expression for the solution process as a multivariate polynomial functional of a set of uncorrelated random variables that enables us to compute the statistical moments of the solution vector. To check the validity of this method, two applications are introduced which are, randomly loaded simply supported reinforced concrete beam and reinforced concrete cantilever beam with random bending rigidity. Finally, a more general application, randomly loaded simply supported reinforced concrete beam with random bending rigidity, is presented to illustrate the method.

Keywords: stochastic differential equation; stochastic finite element method(SFEM); spectral SFEM; Karhunen-Loeve expansion; Neumann expansion; Bernoulli-beam equation; stiffness matrix; bending rigidity; Exponential covariance model; Triangular covariance model; Wiener model.

1. Introduction

Stochastic finite element methods (SFEM) have recently become an active area of research. As the name suggests, researchers in this field attempt to combine two crucial methodologies developed to deal with complex problems of modern engineering: *the finite element analysis* (Reddy 1985,

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Zienkiewicz and Taylor 1989) and *the stochastic analysis*. In the literature, various denominations have given to this field: *stochastic finite elements* (Choi and Noh 1996, Contreras 1980, Ghanem and Spanos 1991, Noh and Choi 2005, Noh and Kwak 2006, Kleiber and Hien 1992, Spanos and Ghanem 1989), *Probabilistic finite elements* (Liu *et al.* 1985, 1986, 1989), *Random field finite element* (Liu *et al.* 1986) and so on. Elishakoff *et al.* (1995, 1996) have recently proposed the terminology *finite element method for stochastic problems*, noting that the finite element discretization remains deterministic during the stochastic analysis. The finite element method can be applied to stochastic analysis by means of generating the discrete version of the stochastic field, which can be attained from the appropriate random field representation techniques (Choi and Noh 1996, 2000, Ghanem and Spanos 1991, Kleiber and Hien 1992, Spanos and Ghanem 1989, Vanmarcke and Grigoriu 1983, Yamazaki and Shinozoka 1990). In case of stochastic processes with known joint probability density functions (p.d.f.) and known explicit forms dependent on the spatial dimension, a new proposed technique introduced in El-Tawil *et al*. (2005) may be used to get the complete solution of the SDE represented by the solution p.d.f.

This paper considers the solution of the stochastic differential equations (SDE^s) with random operator and/or random excitation using the spectral SFEM. The stochastic processes involved in the problem are modeled as second order stochastic processes (random fields) defined only by their means and covariance functions. All random fields dealt with in this paper are continuous and do not have known explicit forms dependent on the spatial dimension. This fact makes the usage of the finite element (FE) analysis be difficult. Relying on the spectral properties of the covariance function, the Karhunen-Loeve expansion (Loeve 1977, Spanos and Ghanem 1989, Van Tree 1968) is used to represent these fields to overcome this difficulty. Then, a spectral approximation for the stochastic response (solution) of the SDE is obtained based on the implementation of the concept of generalized inverse defined by the Neumann expansion (Ghanem and Spanos 1991, Yamazaki *et al.* 1988). This leads to an explicit expression for the solution process as a multivariate polynomial functional of a set of uncorrelated random variables that enables us to compute the statistical moments of the solution vector. To check the validity of this method, two applications are introduced which are randomly loaded simply supported reinforced concrete beam and cantilever beam with random bending rigidity. Finally, a more general application, randomly loaded simply supported reinforced concrete beam with random bending rigidity, is presented to illustrate the method.

2. Spectral representation of the stochastic processes and solution statistics

2.1 Karhunen-Loeve (K-L) Expansion

The Karhunen-Loeve (K-L) expansion of a random field (stochastic process) is based on the spectral expansion of its covariance function. It is formally, a continuous representation method for random fields using a superposition of orthonormal random variables weighted by the eigenfunctions of the covariance of these fields.

Let us consider a random process $\alpha(x;\theta)$, function of spatial coordinate x, defined on the domain D and θ is a random outcome of a triple probability space (Ω, κ, P), where Ω is a sample space, κ is σ-algebra associated with Ω and P is a probability measure. Let $\overline{\alpha}(x)$ is the expected value of $\alpha(x;\theta)$ and $C_\alpha(x, y)$ denotes the covariance function, defined on D, which is, by definition, bounded, symmetric and positive definite. Under these assumptions, it has the following uniform convergent spectral representation (Van Tree 1968)

$$C_\alpha(x,y) = \sum_{i=1}^{\infty} \lambda_i \phi_i(x)\phi_i(y) \quad (1)$$

Here, x and y are used to denote spatial coordinates of two arbitrary observation points belong to the domain D and λ_i, ϕ_i are eigenvalues and eigenfunctions of the covariance function, respectively. Due to boundedness, symmetry and positive definiteness properties, $\{\phi_i(x)\}$ constitutes a complete orthonormal set of deterministic functions and λ_i are always real and positive (Ghanem and Spanos 1991). λ_i, ϕ_i are the solutions of the following integral equation

$$\int_D C_\alpha(x,y)\phi_i(y)dy = \lambda_i \phi_i(x) \quad (2)$$

In this work, the solution of the integral Eq. (2) is the key of the present spectral SFEM formulation. It has some useful properties reported in Van Tree (1968).

Since $\alpha(x;\theta)$ is generally second order process, we can write

$$\alpha(x;\theta) = \bar{\alpha}(x) + v(x;\theta) \quad (3)$$

where $v(x;\theta)$ is a zero mean process with the same covariance function of $\alpha(x;\theta)$, $C_\alpha(x,y)$. $v(x;\theta)$ can be expanded in terms of orthonormal (uncorrelated) random variables weighted by the orthonormal eigen functions $\phi_i(x)$ as

$$v(x;\theta) = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\xi_i(\theta)\phi_i(x) \quad (4)$$

Finally the K-L expansion of $\alpha(x;\theta)$ then takes the form

$$\alpha(x;\theta) = \bar{\alpha}(x) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\xi_i(\theta)\phi_i(x) \quad (5)$$

where, $\{\xi_i(\theta)\}$ forms a set of orthonormal random variables ($\langle \xi_i \rangle = 0$ and $\langle \xi_k(\theta)\xi_l(\theta)\rangle = \delta_{kl}$) (Ghanem and Spanos 1991).

The K-L expansion is mean square convergent irrespective of the probabilistic structure of the random field being expanded, provided that it has a finite variance. The closer the process is to the white noise process the more terms are required in its expansion, while at the other limit, a random variable can be represented by a single term. In most applications, only few terms in the K-L expansion can capture most of the uncertainty in the process. Clearly, the more accuracy is required from the predictions of a certain model, the more terms should be included in K-L expansion representing its data (Ghanem 1999).

2.2 K-L expansion in the framework of SFE formulation: (Solution Representation and Statistics)

The spectral approach is introduced through the implementation of the K-L expansion, of the underlying processes, in the Galerkin based finite element formulation after recasting it in the stochastic form. In case of SDE[s] with random operator, the need of the Neumann expansion scheme (Ghanem and Spanos 1991), which gives an explicit form of the response process in a set of uncorrelated random variables, appears. The method to be discussed is based on the direct operations on the following equation

$$\Lambda_a(x;\theta)[u(x;\theta)] = f(x;\theta), \quad x \in D, \ \theta \in \Omega \tag{6}$$

where, $\Lambda_a(x;\theta)$ is a linear differential operator with respect to space whose coefficients $a_k(x;\theta)$ can be modeled as random fields exhibiting random fluctuations in space, $u(x;\theta)$ is the solution process and $f(x;\theta)$ is the random excitation function. The spatial domain of Λ_a is denoted by D, x refers to a point in this domain, and θ is a random outcome of a triple probability space (Ω, κ, P). The aim then is to solve for the response process $u(x;\theta)$ as a function of both arguments. Since all processes dealt with are considered of second order type, $a_k(x;\theta)$ can be decomposed into purely deterministic component and a purely random component in the form

$$a_k(x;\theta) = \bar{a}_k(x) + \alpha_k(x;\theta) \tag{7}$$

where $\bar{a}_k(x)$ is the mean of the process $a_k(x;\theta)$ and $\alpha_k(x;\theta)$ is a zero-mean process, having the same covariance function as the process $a_k(x;\theta)$. Separating the random component of $\Lambda_a[\cdot]$ from its deterministic component, Eq. (6) becomes

$$[L(x) + \Pi_\alpha(x;\theta)][u(x;\theta)] = f(x;\theta) \tag{8}$$

where $L[\cdot]$ indicates a deterministic linear differential operator whose coefficients represent the average values of the random coefficients, and $\Pi_\alpha[\cdot]$ denotes a random linear operator whose coefficients $\alpha_k(x;\theta)$ are zero-mean random fields. In the above, it is assumed that the random coefficients $a_k(x;\theta)$ appear as multiplicative constants in the explicit expression of the operator $\Lambda_a[\cdot]$.

2.2.1 Stochastic finite element formulation of the problem

Let $a(x;\theta), x \in D, \theta \in \Omega$ represent the only random coefficient that appears as a multiplicative factor in the operator of the SDE, where D is the actual domain occupied by the object being investigated. Let $a(x;\theta)$ and $f(x;\theta)$ be modeled as second order random fields that permit the following representations

$$a(x;\theta) = \bar{a} + \alpha(x;\theta) \tag{9}$$

$$f(x;\theta) = \bar{f} + \alpha_1(x;\theta) \tag{10}$$

Also, it is assumed that the governing SDE is subjected to a set of deterministic boundary condition. In the view of that, Eq. (8) becomes

$$[L(x) + \alpha(x;\theta)R(x)]u(x;\theta) = \bar{f} + \alpha_1(x;\theta), \quad x \in D \tag{11}$$

subjected to

$$\Sigma(x)[u(x;\theta)] = 0, \qquad x \in \partial D \tag{12}$$

where, ∂D is the boundary of the domain D,
$R(x)$ is a deterministic differential operator on D,
$\Sigma(x)$ is a deterministic operator on ∂D.

Following the procedure of the Galerkin formulation (Zienkiewicz and Taylor 1989), we can get the stochastic version of the element equations as follows:

The method consists of expanding the solution of the differential equation along a basis of a finite dimensional subspace of an admissible Hilbert space (Oden 1979), whose functions are called "interpolation or trial functions" (interpolation displacement model), and requiring that the error resulting from taking a finite number of terms in the expansion be orthogonal to another Hilbert space, the "test space", whose functions are called "test functions". Usually, the test space is chosen to coincide with the admissible space. In other words, expanding the solution $u(x;\theta)$ in Eq. (6) in terms of the interpolation functions (Hermit cubic interpolation functions) $\{\psi_j^{(e)}(x)\}_{j=1}^{r}$ introduces an error of the form

$$\varepsilon_L = \sum_{j=1}^{r} q_j^{(e)}(\theta)\Lambda(x;\theta)[\psi_j^{(e)}(x)] - f(x;\theta) \tag{13}$$

Requiring this error to be orthogonal to the subspace spanned by the same functions $\{\psi_i^{(e)}(x)\}_{i=1}^{r}$, yields a set of the following r algebraic equations

$$(\varepsilon_L, \psi_i^{(e)}(x)) = 0, \quad i = 1, 2, \ldots, r \tag{14}$$

The previous equation is equivalent to

$$\sum_{j=1}^{r}([L(x) + \alpha(x;\theta)R(x)]\psi_i^{(e)}(x), \psi_j^{(e)}(x))q_j^{(e)}(\theta)$$
$$= (\bar{f} + \alpha_1(x;\theta), \psi_i^{(e)}(x)), \quad i = 1, 2, \ldots, r \tag{15}$$

where r is the total number of degrees of freedom (d.o.f.) per element and $q_j^{(e)}(\theta)$ are the random d.o.f. per element. Using the truncated K-L expansion of the processes involved in Eq. (15) up to order M, the following equation is obtained

$$\sum_{j=1}^{r}\left(\left[L(x) + \left(\sum_{n=1}^{M}\sqrt{\lambda_n}\xi_n\phi_n(x)\right)R(x)\right]\psi_i^{(e)}(x), \psi_j^{(e)}(x)\right)q_j^{(e)}$$
$$= (\bar{f}, \psi_i^{(e)}(x)) + \left(\sum_{n=1}^{M}\sqrt{\mu_n}\xi_n h_n(x),\ \psi_i^{(e)}(x)\right) \tag{16}$$

where, λ_n, $\phi_n(x)$ are the eigenvalues and eigenfunctions covariance kernel of the process involved in the operator, $w(x;\theta)$, and μ_n, $h_n(x)$ are the eigenvalues and eigenfunctions of the covariance kernel of the excitation process, $f(x;\theta)$.

Using integration by parts, to get the weak formulation of the problem (Reddy 1985), the operators $L(x)$ and $R(x)$ can be split into four lower order operators and Eq. (16) takes the following form

$$\sum_{j=1}^{r}\left[(L_1(x)\,\psi_i^{(e)}(x),\, L_2(x)\,\psi_j^{(e)}(x)) + \left(\left[\sum_{n=1}^{M}\sqrt{\lambda_n}\xi_n\phi_n(x)\right]R_1(x)\psi_i^{(e)}(x),\, R_2(x)\,\psi_j^{(e)}(x)\right)\right]q_j^{(e)}$$

$$= (\bar{f},\psi_i^{(e)}(x)) + \sum_{n=1}^{M}\sqrt{\mu_n}\xi_n(h_n(x),\,\psi_i^{(e)}(x)) + Q_i^{(e)}(\theta) \tag{17}$$

where, L_1, L_2 and R_1, R_2 are appropriate differential operators of lower order than L and R, respectively. $Q_i^{(e)}(\theta)$ are the random boundary terms for the element "e". Eq. (17) can be expressed in a matrix form as

$$\mathbf{K}^{(e)}\mathbf{q}^{(e)} = \mathbf{F}^{(e)} \tag{18}$$

where the element matrix $\mathbf{K}^{(e)}$ and the element excitation vector $\mathbf{F}^{(e)}$ are defined as

$$\mathbf{K}^{(e)} = \overline{\mathbf{K}}^{(e)} + \sum_{n=1}^{M}\mathbf{K}_n^{(e)}\xi_n \tag{19}$$

$$\mathbf{F}^{(e)} = \overline{\mathbf{F}}^{(e)} + \sum_{n=1}^{M}\mathbf{F}_n^{(e)}\xi_n \tag{20}$$

where

$$\overline{K}_{ij}^{(e)} = \int_D [L_1(x)\,\psi_i^{(e)}(x)][L_2(x)\,\psi_j^{(e)}(x)]dx \tag{21}$$

$$K_{nij}^{(e)} = \sqrt{\lambda_n}\int_D \phi_n(x)[R_1(x)\,\psi_i^{(e)}(x)][R_2(x)\,\psi_j^{(e)}(x)]dx \tag{22}$$

$$\overline{F}_i^{(e)} = \int_D \bar{f}\psi_i^{(e)}(x)dx + Q_i^{(e)}(\theta) \tag{23}$$

$$F_{ni}^{(e)} = \sqrt{\mu_n}\int_D h_n(x)\psi_i^{(e)}(x)dx, \qquad i,j = 1, 2, \ldots, r \tag{24}$$

Finally, by assembling the matrices $\mathbf{K}^{(e)}$ and the vectors $\mathbf{F}^{(e)}$ for all elements, the following global system of order N (N is the total number of global d.o.f ^{s}) is obtained

$$\mathbf{KU} = \mathbf{F} \tag{25}$$

where,

U is the vector of unknown random-global d.o.f ^{s} (solution vector),

K is the global stiffness matrix of the problem, which is defined as

$$\mathbf{K} = \overline{\mathbf{K}} + \sum_{n=1}^{M}\mathbf{K}_n\xi_n \tag{26}$$

$\mathbf{F}$ is the global excitation vector, which is defined as

$$\mathbf{F} = \overline{\mathbf{F}} + \sum_{n=1}^{M} \mathbf{F}_n \xi_n \tag{27}$$

and $\overline{\mathbf{K}}, \mathbf{K}_n, \overline{\mathbf{F}}, \mathbf{F}_n$ are obtained by the assembly of the elementary matrices defined in Eq. (21) through Eq. (24).

At this stage, the boundary conditions specified in Eq. (12) may be imposed on $\overline{\mathbf{K}}$ and each of $\mathbf{K}_n$ matrices individually. Also, the equilibrium conditions (Reddy 1985) must be applied on $\overline{\mathbf{F}}$ and each of $\mathbf{F}_n$ vectors individually. Finally a suitable partitioning procedure must be made to get non-singular matrices and proceed with the subsequent analysis. Now Eq. (25) may be rewritten for convenience as

$$\left[\hat{\overline{\mathbf{K}}} + \sum_{n=1}^{M} \hat{\mathbf{K}}_n \xi_n\right]\mathbf{U} = \hat{\overline{\mathbf{F}}} + \sum_{n=1}^{M} \hat{\mathbf{F}}_n \xi_n \tag{28}$$

Eq. (28) can be normalized by multiplying both sides by $\hat{\overline{\mathbf{K}}}^{-1}$ to obtain

$$\left[\mathbf{I} + \sum_{n=1}^{M} \mathbf{Q}_n \xi_n\right]\mathbf{U} = \mathbf{d} + \sum_{n=1}^{M} \mathbf{b}_n \xi_n \tag{29}$$

where,

$$\mathbf{Q}_n = \hat{\overline{\mathbf{K}}}^{-1}\hat{\mathbf{K}}_n, \mathbf{d} = \hat{\overline{\mathbf{K}}}^{-1}\hat{\overline{\mathbf{F}}}, \mathbf{b}_n = \hat{\overline{\mathbf{K}}}^{-1}\hat{\mathbf{F}}_n \tag{30}$$

Now, a straightforward scheme for obtaining the response vector $\mathbf{U}$ from Eq. (29) relies on performing a *Neumann Expansion* for the inverse operator as follows

$$\mathbf{U} = \left[\sum_{i=0}^{\infty} (-1)^i \left(\sum_{n=1}^{M} \mathbf{Q}_n \xi_n\right)^i\right]\left[\mathbf{d} + \sum_{n=1}^{M} \mathbf{b}_n \xi_n\right] \tag{31}$$

To guarantee the convergence of the series resulting from the expansion in Eq. (31), it is necessary that the following criterion must be satisfied

$$\left\|\sum_{n=1}^{M} \mathbf{Q}_n \xi_n\right\| < 1 \tag{32}$$

In general, a scaling procedure (Padovan and Guo 1989, Shinozuka and Yamazaki 1988) can be developed for $\mathbf{Q}_n$ in order to meet this convergence criterion. The explicit solution of the stochastic problem is of course impossible in a computational framework without the truncation of the infinite summation in Eq. (31). Doing that truncation to get expansion of finite order P provides the expression

$$\mathbf{U} = \left[\sum_{i=0}^{P} (-1)^i \left(\sum_{n=1}^{M} \mathbf{Q}_n \xi_n\right)^i\right]\left[\mathbf{d} + \sum_{n=1}^{M} \mathbf{b}_n \xi_n\right] \tag{33}$$

2.2.2 Statistical moments of the solution vector

One of the important methods of characterizing the stochastic solution of a SDE is computing the first and second order statistical moments of that solution. The SFEM as described in the previous section can be used efficiently to produce a good approximation to these statistics. Specifically, starting with Eq. (33), the mean vector $\bar{\mathbf{U}}$ and covariance matrix $\mathbf{COV}(\mathbf{U})$ of the response vector can be obtained as follows

$$\bar{\mathbf{U}} = <\mathbf{U}> = <\left[\sum_{i=0}^{P}(-1)^i\left(\sum_{n=1}^{M}\mathbf{Q}_n\xi_n\right)^i\right]\left[\mathbf{d} + \sum_{n=1}^{M}\mathbf{b}_n\xi_n\right]> \tag{34}$$

$$\begin{aligned}\mathbf{COV}(\mathbf{U}) &= <\mathbf{U}\mathbf{U}^T> - <\mathbf{U}><\mathbf{U}^T> \\ &= <\left[\sum_{i=0}^{P}(-1)^i\left(\sum_{n=1}^{M}\mathbf{Q}_n\xi_n\right)^i\right]\left[\mathbf{d} + \sum_{n=1}^{M}\mathbf{b}_n\xi_n\right] \\ &\quad \left[\mathbf{d}^T + \sum_{n=1}^{M}\mathbf{b}_n^T\xi_n\right]\left[\sum_{i=0}^{P}(-1)^i\left(\sum_{n=1}^{M}\mathbf{Q}_n^T\xi_n\right)^i\right]> - \bar{\mathbf{U}}\bar{\mathbf{U}}^T\end{aligned} \tag{35}$$

It should be noted that, the assumption that the random fields under consideration are Gaussian must be stated. In fact, it results from that hypothesis that the random variables of the K-L expansion are orthonormal Gaussian. Hence, the following relations can be used to find the statistical moments, of these variables, of an order higher or equal to three that appear in the expansions (34) and (35) (Imamura *et al.* 1963).

• For a Gaussian $\xi(\theta)$ of zero mean,

$$<\xi^n(\theta)> = \begin{cases} 0 & \text{if} \quad n \quad odd \\ (n-1)!! & \text{if} \quad n \quad even \end{cases} \tag{36}$$

where, $(n-1)!! = (n-1)(n-3)\ldots(5)(3)(1)$

• For uncorrelated Gaussian random variables $\{\xi_i(\theta)\}_{i=1}^{k}$,

$$<\xi_1\xi_2\xi_3\ldots\xi_k> = \begin{cases} 0 & \text{if} \quad k \quad odd \\ \sum_{p=2}^{k}<\xi_1\xi_p><\prod_{\substack{n=2 \\ n\neq p}}^{k}\xi_n> & \text{if} \quad k \quad even \end{cases} \tag{37}$$

3. Numerical applications

This section presents three numerical applications dealing with the handling of material (operator) and/or load (excitation) uncertainties of Euler-Bernolli beams[1]. The evaluation of the solution variability is obtained by the SSFEM. Reinforced concrete beams were chosen, since concrete material properties are uncertain and can be randomly spread over the beam (Lawanwisut *et al.* 2003, Pukl *et al.* 2003, Vouwenvelder 2004).

[1]The Euler-Bernolli beam model assumes that shear effects are negligible in the displacement field, which is verified as soon as the cross-sectional dimensions are much smaller than the beam length.

3.1 Simply supported reinforced concrete beam under random load represented by a random field

3.1.1 Problem description and formulation

Consider the Euler-Bernolli reinforced concrete simply supported beam of length L is subjected to a distributed random transverse load represented by a random field $f(x;\theta)$, positive upward. It is assumed that $f(x;\theta)$ is the realization of a Gaussian random process indexed over the spatial domain occupied by the beam, $x \in [0, L]$. It is also assumed $f(x;\theta)$ is represented by its mean $\bar{f}$, which is constant, and its covariance function $C_f(x, y), x, y \in [0, L]$. In this case the mathematical model is a SDE with random excitation that takes the following form

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2 w(x;\theta)}{dx^2}\right) = f(x;\theta), \quad x \in [0, L] \tag{38}$$

with

$$w(0) = w(L) = 0 \tag{39}$$

$$\left.\left(EI\frac{d^2w}{dx^2}\right)\right|_{x=0} = \left.\left(EI\frac{d^2w}{dx^2}\right)\right|_{x=L} = 0 \tag{40}$$

Comparing Eq. (38) with the general operator in Eq. (6), the operator Λ is purely deterministic and we may then drop the random part $\Pi[\cdot]$.

Discretizing the beam into N^{el} identical elements of length $l = L/N^{el}$ and following the Galerkin-based finite element formulation presented in section 2.2.1, it is found that

$$L_1(x) = EI(x)\frac{d^2}{dx^2} \quad \text{and} \quad L_2(x) = \frac{d^2}{dx^2} \tag{41}$$

$$R_1(x) = R_2(x) = 0 \tag{42}$$

and the global finite element system of order $N = 2(N^{el} + 1)$ is

$$\mathbf{KU} = \bar{\mathbf{F}} + \sum_{n=1}^{M} \xi_n \mathbf{F}_n \tag{43}$$

where the stiffness matrix **K**, purely deterministic, and the right hand side vector **F** (load vector) of Eq. (43) are computed by assembling the elementary matrices

$$K_{ij}^{(e)} = \int_0^{l_e} EI(x')\frac{d^2\psi_i^{(e)}(x')}{dx'^2}\frac{d^2\psi_j^{(e)}(x')}{dx'^2}dx', \quad i,j = 1, 2, 3, 4 \tag{44}$$

$$\bar{F}_i^{(e)} = \int_0^{l_e} \bar{f}\,\psi_i^{(e)}(x')dx' + Q_i^{(e)}(\theta) \tag{45}$$

$$F_{ni}^{(e)} = \sqrt{\mu_n}\int_0^{l_e} h_n(x')\,\psi_i^{(e)}(x')dx', \quad n = 1, 2, \ldots, M \tag{46}$$

$$i = 1, 2, 3, 4, e = 1, 2, \ldots, N^{el}$$

Table 1 Numerical values of the first six values w_n and the corresponding eigen values (Exponential covariance model; l_{cor} = 1.0, a = 0.5)

n	w_n	$\frac{\mu_n}{\alpha_\alpha^2}$
1	1.306542374189476	7.388108094159773E-001
2	3.673194406304252	1.380037753542628E-001
3	6.584620042564174	4.508848728978113E-002
4	9.631684635691391	2.132893128730332E-002
5	12.723240784131330	1.227891385451699E-002
6	15.834105369332420	7.945371034246029E-003

x' in the previous integrals denotes the local spatial coordinate along the element "e", which is related to the global coordinate x by $x' = x - x_e$ where x_e is the global coordinate of the left node of that element, $l_e = l$ is the element length. The integrals in Eq. (46) take the following form for the exponential covariance (Van Tree 1968) model

$$F_{ni}^{(e)} = \begin{cases} \sqrt{\mu_n}\int_0^{l_e} \dfrac{\cos\left[w_n\left(\dfrac{2(x'+x_e)}{L}-1\right)a\right]}{\sqrt{a+\dfrac{\sin(2w_n a)}{2w_n}}}\psi_i^{(e)}(x')dx', & \text{if } n \text{ is odd} \\ \sqrt{\mu_n}\int_0^{l_e} \dfrac{\sin\left[w_n\left(\dfrac{2(x'+x_e)}{L}-1\right)a\right]}{\sqrt{a-\dfrac{\sin(2w_n a)}{2w_n}}}\psi_i^{(e)}(x')dx', & \text{if } n \text{ is even} \end{cases} \tag{47}$$

where, $a = L/2$, and the constants w_n and the eigenvalues, $\mu_n = \dfrac{2\sigma_\alpha^2 c}{w_n^2 + c^2}, \forall n$, are given in Table 1. Also, $c = 1/l_{cor}$, α_α^2 is the process variance and l_{cor} is the correlation length of the process.

Imposing the boundary and equilibrium conditions on the system (43) and making a suitable partitioning, the following nonsingular system is obtained

$$\hat{\mathbf{K}}\mathbf{U} = \hat{\hat{\mathbf{F}}} + \sum_{n=1}^{M} \xi_n \hat{\mathbf{F}}_n \tag{48}$$

To get an explicit form of the solution vector, both sides of Eq. (48) are multiplied by $\hat{\mathbf{K}}^{-1}$ to get

$$\mathbf{U} = \mathbf{d} + \sum_{n=1}^{M} \xi_n \mathbf{b}_n \tag{49}$$

where, $$\mathbf{d} = \hat{\hat{\mathbf{K}}}\hat{\hat{\mathbf{F}}} \quad \text{and} \quad \mathbf{b}_n = \hat{\hat{\mathbf{K}}}^{-1}\hat{\mathbf{F}}_n \tag{50}$$

3.1.2 Statistical moments of the response vector

Simply from Eq. (49), we can find the first and second order moments of the response vector **U** as follows

$$\bar{\mathbf{U}} = \langle \mathbf{U} \rangle = \mathbf{d} \tag{51}$$

$$\mathbf{COV}(\mathbf{U}) = \langle \mathbf{U}\mathbf{U}^T \rangle - \mathbf{U}\bar{\mathbf{U}} \tag{52}$$

$$= \left\langle \left[\mathbf{d} + \sum_{n=1}^{M} \xi_n \mathbf{b}_n\right]\left[\mathbf{d}^T + \sum_{n=1}^{M} \xi_n \mathbf{b}_n^T\right] \right\rangle - \mathbf{d}\mathbf{d}^T$$

$$= \sum_{n=1}^{M} \mathbf{b}_n \mathbf{b}_n^T$$

3.1.3 Results

In the numerical implementation of the preceding analysis the beam is discretized into ten finite elements ($N^{el} = 10$), resulting in twenty-two ($N = 22$) degrees of freedom. The distributed random load is modeled by the exponential covariance model with correlation length, $l_{cor} = l/c = 1.0$, variance $\sigma_f^2 = 1.0$ and mean $\bar{f} = 0$. Also, it is assumed that the bending rigidity of the reinforced concrete beam, $EI = 1.0$ N.m^2 and the beam length $L = 1.0$ m.

Table 2 shows the convergence of the spectral SFEM solution along the first half of the beam, represented by the variance of the beam deflection, to the exact one, found in Elishakoff *et al.* (1999), as the order M of the K-L expansion increases. It is evident that an almost exact solution is obtained for $M = 6$. Fig. 1 shows the variance of the deflection along the beam length for $M = 4$ compared with the exact solution.

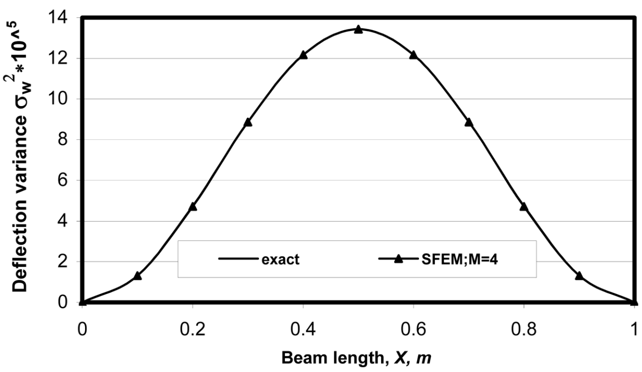

Fig. 1 Variance of the deflection along the beam, Exponential Covariance model for the load ($l_{cor} = 1.0$, $a = 0.5$, $\sigma_f = 1.0$)

Table 2 Numerical values of the variance of the beam deflection at the nodes compared with the exact values, (Exponential covariance model for the load)

x	$M = 2$ $\sigma^2_{W_{SFEM}} * 10^5$	$M = 4$ $\sigma^2_{W_{SFEM}} * 10^5$	$M = 6$ $\sigma^2_{W_{SFEM}} * 10^5$	*Exact* $\sigma^2_{W} * 10^5$
0	0	0	0	0
0.1	1.3073	1.3176	1.3178	1.317871
0.2	4.6839	4.7237	4.7245	4.724552
0.3	8.7864	8.8673	8.8685	8.868642
0.4	12.0586	12.1758	12.1772	12.1774
0.5	13.2987	13.4305	13.4319	13.43211

3.2 Cantilever reinforced concrete beam with random bending rigidity represented by a random field

3.2.1 Problem description and formulation

Consider the Euler-Bernoulli reinforced concrete beam, of length L, clamped at one end and subjected to a distributed uniform static load of intensity f_0, positive upward. It is assumed that the bending rigidity EI of the reinforced concrete beam is the realization of a Gaussian random process indexed over the spatial domain occupied by the beam, $x \in [0, L]$. It is also assumed that the random bending rigidity $EI(x;\theta)$ is represented by its mean $\overline{EI}$, which is constant, and its covariance function $C_{EI}(x, y), x, y \in [0, L]$. In this case the mathematical model is

$$\frac{d^2}{dx^2}\left(EI(x;\theta)\frac{d^2 w(x;\theta)}{dx^2}\right) = f_0, \quad x \in [0, L] \tag{53}$$

with

$$w(0) = \frac{dw}{dx}(0) = 0 \tag{54}$$

$$\left.\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right|_{x=L} = \left.\left(EI\frac{d^2w}{dx^2}\right)\right|_{x=L} = 0 \tag{55}$$

Comparing Eq. (53) with the general operator Eq. (6) and following the Galerkin-based finite element formulation, it is found that

$$L_1(x) = \overline{EI}\frac{d^2}{dx^2} \quad \text{and} \quad L_2(x) = \frac{d^2}{dx^2} \tag{56}$$

$$R_1(x) = R_2(x) = \frac{d^2}{dx^2} \tag{57}$$

where $EI(x;\theta)$ is decomposed using the K-L expansion as

$$EI(x;\theta) = \overline{EI} + \sum_{n=1}^{M} \sqrt{\lambda_n}\xi_n\phi_n(x) \tag{58}$$

and λ_n, $\phi_n(x)$ are the eigenvalues and eigenfunctions of the covariance function $C_{EI}(x, y)$.

The global finite element system of order N is

$$\left[\overline{\mathbf{K}} + \sum_{n=1}^{M} \xi_n \mathbf{K}_n\right]\mathbf{U} = \mathbf{F} \tag{59}$$

where the deterministic part, $\overline{\mathbf{K}}$ and the stochastic parts, $\mathbf{K}_n$ of the stiffness matrix $\mathbf{K}$ and the right hand side vector $\mathbf{F}$ of Eq. (59) are computed by assembling the elementary matrices

$$\overline{K}_{ij}^{(e)} = \int_0^{l_e} \overline{EI}\frac{d^2\psi_i^{(e)}(x')}{dx'^2}\frac{d^2\psi_j^{(e)}(x')}{dx'^2}dx' \quad , \quad i,j = 1, 2, 3, 4 \tag{60}$$

$$\overline{K}_{nij}^{(e)} = \sqrt{\lambda_n}\int_0^{l_e} \phi_n(x')\frac{d^2\psi_i^{(e)}(x')}{dx'^2}\frac{d^2\psi_j^{(e)}(x')}{dx'^2}dx' \qquad \begin{array}{l} i,j = 1, 2, 3, 4 \\ n = 1, 2, \dots, M \end{array} \tag{61}$$

$$\overline{F}_i^{(e)} = \int_0^{l_e} f_0 \psi_i^{(e)}(x')dx' + Q_i^{(e)}(\theta), \quad i = 1, 2, 3, 4, e = 1, 2, \ldots, N^{el} \tag{62}$$

where N^{el} is the number of finite elements taken in the FE model.

In this application the covariance kernel of the random bending rigidity is modeled using three models, the exponential model, the Triangular model and the covariance kernel of Wiener process. In view of coordinates transformation introduced in the previous application, the eigenvalues and eigenfunctions, in the element coordinate form, are:

- ***For exponential model***

As in the previous application.

- ***For Triangular model***

$$\phi_i(x') = \frac{\cos(w_i(x'+x_e)) + \tan\left(\frac{w_i a}{2}\right)\sin(w_i(x'+x_e))}{\sqrt{a + \left[\tan^2\left(\frac{w_i a}{2}\right) - 1\right]\left[\frac{a}{2} - \frac{\sin(2w_i a)}{4w_i}\right] + \frac{1}{w_i}\tan\left(\frac{w_i a}{2}\right)\sin^2(w_i a)}} \quad \text{if } i \text{ is } odd \tag{63}$$

$$\phi_i(x') = \frac{\cos(w_i(x'+x_e))}{\sqrt{\frac{a}{2} + \frac{\sin(2w_i a)}{4w_i}}}, \quad \text{if } i \text{ is } even \tag{64}$$

$a = L$, and λ_i are given in Table 3.

- ***For the Wiener process, appendix-A,***

$$\phi_i(x') = \sqrt{\frac{2}{a}}\sin\left(\frac{(2i-1)\pi(x'+x_e)}{2a}\right), \quad i \geq 1 \tag{65}$$

$$a = L \quad \text{and} \quad \lambda_i = \left(\frac{2\tau a}{(2i-1)\pi}\right)^2 \tag{66}$$

Imposing the boundary and equilibrium conditions on the system (59) and making a suitable partitioning, the following nonsingular system is obtained

Table 3 Numerical values of the first six values w_n and the corresponding eigenvalues (Triangular covariance model; $l_{cor} = 2.0$, $a = 1.0$)

i	w_i	$\frac{\lambda_i}{\alpha_\alpha^2}$
1	1.094321514520660	8.350454708026427E-001
2	3.141592741012573	1.013211780033004E-001
3	6.487974969865883	2.375645685701502E-002
4	9.424778223037720	1.125790866703338E-002
5	12.671496724511800	6.227936397184331E-003
6	15.707963705062870	4.052847120132016E-003

$$\left[\hat{\bar{\mathbf{K}}} + \sum_{n=1}^{M} \hat{\mathbf{K}}_n \xi_n\right]\mathbf{U} = \hat{\mathbf{F}} \tag{67}$$

where $\hat{\mathbf{F}}$ is the modified load vector after the imposing of conditions and partitioning.

Using truncated Neumann expansion of order P, to expand the inverse operator, the following expression for the solution vector is obtained

$$\mathbf{U} = \left[\sum_{i=0}^{P} (-1)^i \left(\sum_{n=1}^{M} \xi_n \mathbf{Q}_n\right)^i\right]\mathbf{g} \tag{68}$$

where,

$$\mathbf{Q}_n = \hat{\bar{\mathbf{K}}}^{-1}\hat{\mathbf{K}}_n \quad \text{and} \quad \mathbf{g} = \hat{\bar{\mathbf{K}}}^{-1}\hat{\mathbf{F}}_n \tag{69}$$

3.2.2 Statistical moments of the solution vector

From Eq. (68), we can find the first and second order moments of the solution vector **U** for several values of the order P of the Neumann expansion. In doing this, the recurrence relations in Eqs. (36) and (37) are used implicitly to simplify the analysis. Statistical moments for the third order Neumann expansion (P = 3) are as follows

$$\mathbf{U} = \left[\mathbf{I} - \sum_{i=1}^{M} \xi_i \mathbf{Q}_i + \sum_{i=1}^{M}\sum_{j=1}^{M} \xi_i\xi_j \mathbf{Q}_i\mathbf{Q}_j - \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{k=1}^{M} \xi_i\xi_j\xi_k \mathbf{Q}_i\mathbf{Q}_j\mathbf{Q}_k\right]\mathbf{g} \tag{70}$$

$$\bar{\mathbf{U}} = \left[\mathbf{I} + \sum_{i=1}^{M} \mathbf{Q}_i\mathbf{Q}_i\right]\mathbf{g} \tag{71}$$

$$\begin{aligned}
\mathbf{COV}(\mathbf{U}) = {} & \sum_{i=1}^{M} \mathbf{Q}_i\mathbf{G}\mathbf{Q}_i^T + \sum_{i=1}^{M}\sum_{j=1}^{M} [\mathbf{Q}_i\mathbf{G}\mathbf{Q}_i^T\mathbf{Q}_j^T\mathbf{Q}_j^T + \mathbf{Q}_i\mathbf{G}\mathbf{Q}_j^T\mathbf{Q}_j^T\mathbf{Q}_i^T + \mathbf{Q}_i\mathbf{G}\mathbf{Q}_j^T\mathbf{Q}_i^T\mathbf{Q}_j^T \\
& + \mathbf{Q}_i\mathbf{Q}_j\mathbf{G}\mathbf{Q}_j^T\mathbf{Q}_i^T + \mathbf{Q}_i\mathbf{Q}_j\mathbf{G}\mathbf{Q}_i^T\mathbf{Q}_j^T + \mathbf{Q}_i\mathbf{Q}_i\mathbf{Q}_j\mathbf{G}\mathbf{Q}_j^T + \mathbf{Q}_i\mathbf{Q}_j\mathbf{Q}_j\mathbf{G}\mathbf{Q}_i^T + \mathbf{Q}_i\mathbf{Q}_j\mathbf{Q}_i\mathbf{G}\mathbf{Q}_j^T] \\
& + \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{k=1}^{M} [\mathbf{Q}_i\mathbf{Q}_i\mathbf{Q}_j\mathbf{G}\mathbf{Q}_k^T\mathbf{Q}_k^T\mathbf{Q}_j^T + \mathbf{Q}_i\mathbf{Q}_i\mathbf{Q}_j\mathbf{G}\mathbf{Q}_k^T\mathbf{Q}_j^T\mathbf{Q}_k^T + \mathbf{Q}_i\mathbf{Q}_i\mathbf{Q}_j\mathbf{G}\mathbf{Q}_j^T\mathbf{Q}_k^T\mathbf{Q}_k^T \\
& + \mathbf{Q}_i\mathbf{Q}_j\mathbf{Q}_i\mathbf{G}\mathbf{Q}_k^T\mathbf{Q}_k^T\mathbf{Q}_j^T + \mathbf{Q}_i\mathbf{Q}_j\mathbf{Q}_i\mathbf{G}\mathbf{Q}_k^T\mathbf{Q}_j^T\mathbf{Q}_k^T + \mathbf{Q}_i\mathbf{Q}_j\mathbf{Q}_i\mathbf{G}\mathbf{Q}_j^T\mathbf{Q}_k^T\mathbf{Q}_k^T \\
& + \mathbf{Q}_i\mathbf{Q}_j\mathbf{Q}_j\mathbf{G}\mathbf{Q}_k^T\mathbf{Q}_k^T\mathbf{Q}_j^T + \mathbf{Q}_i\mathbf{Q}_j\mathbf{Q}_j\mathbf{G}\mathbf{Q}_k^T\mathbf{Q}_i^T\mathbf{Q}_k^T + \mathbf{Q}_i\mathbf{Q}_j\mathbf{Q}_j\mathbf{G}\mathbf{Q}_i^T\mathbf{Q}_k^T\mathbf{Q}_k^T \\
& + \mathbf{Q}_i\mathbf{Q}_k\mathbf{Q}_j\mathbf{G}\mathbf{Q}_k^T\mathbf{Q}_j^T\mathbf{Q}_i^T + \mathbf{Q}_i\mathbf{Q}_k\mathbf{Q}_j\mathbf{G}\mathbf{Q}_k^T\mathbf{Q}_j^T\mathbf{Q}_i^T + \mathbf{Q}_i\mathbf{Q}_k\mathbf{Q}_j\mathbf{G}\mathbf{Q}_i^T\mathbf{Q}_k^T\mathbf{Q}_j^T \\
& + \mathbf{Q}_i\mathbf{Q}_k\mathbf{Q}_j\mathbf{G}\mathbf{Q}_j^T\mathbf{Q}_k^T\mathbf{Q}_i^T + \mathbf{Q}_i\mathbf{Q}_k\mathbf{Q}_j\mathbf{G}\mathbf{Q}_j^T\mathbf{Q}_i^T\mathbf{Q}_k^T + \mathbf{Q}_i\mathbf{Q}_k\mathbf{Q}_j\mathbf{G}\mathbf{Q}_i^T\mathbf{Q}_j^T\mathbf{Q}_k^T]
\end{aligned} \tag{72}$$

where,

$$\mathbf{G} = \mathbf{g}\mathbf{g}^T \tag{73}$$

3.2.3 Results

In the numerical implementation of the preceding analysis, the reinforced concrete beam is of length $L = 1.0$ m and it is discretized into ten finite elements ($N^{el} = 10$). The random bending rigidity is modeled by the exponential covariance model with correlation length, $l_{cor} = 1/c = 1.0$, triangular covariance model with $l_{cor} = 1/c = 2.0$, and Wiener model with $\tau^2 = 1.0$. Also, it is assumed that the bending rigidity, in the three models, has unit mean, $\overline{EI} = 1.0\ \mathrm{N.m}^2$, and the uniform load intensity is also unity $f_0 = 1.0$ N/m.

» Results for Exponential covariance model

Figs. 2 and 4 show the deflection standard deviation along the beam for the bending rigidity standard deviation $\sigma_{EI} = 0.3$ and $\sigma_{EI} = 0.2$ respectively. For various values of P and M the results are plotted. These results agree with those evaluated by Ghanem and Spanos in Ghanem and Spanos (1991) in behavior sense, but they are different in magnitudes. This difference is due to the fact that the transformation of the domain of the process, $x \in [-a, a], a = L/2$, into the domain of the beam, $x \in [0, L]$, was not taken into account in the results in Ghanem and Spanos (1991), which we believe that it is not the correct procedure. In the previous analysis, if the above-mentioned transformation is dropped, the results will be identical to the results in Ghanem and Spanos (1991). Figs. 3 and 5 show the mean values of the deflection along the beam for $\sigma_{EI} = 0.2$ and $\sigma_{EI} = 0.3$

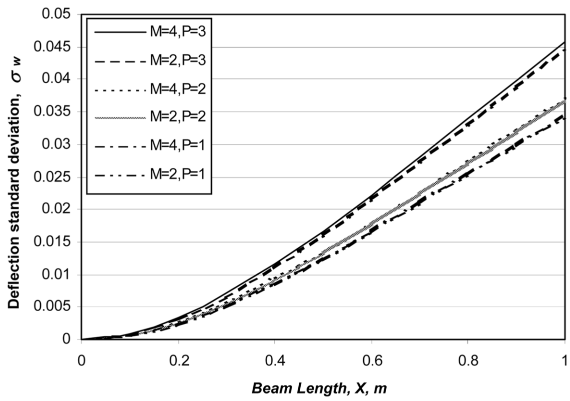

Fig. 2 Standard deviation of the deflection along the beam, Exponential covariance model of the bending rigidity, $l_{cor} = 1.0$, $\sigma_{EI} = 0.3$

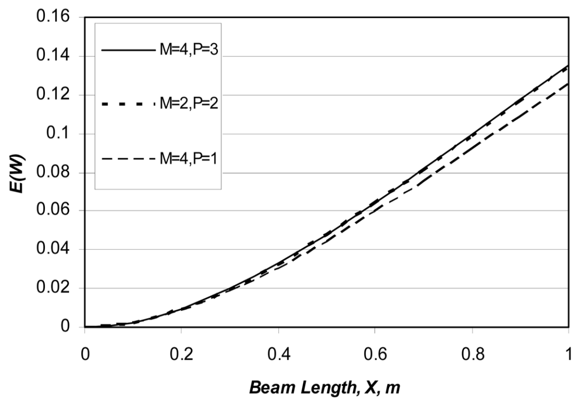

Fig. 3 Mean deflection along the beam, Exponential covariance model for the bending rigidity, ($l_{cor} = 1.0$, $\sigma_{EI} = 0.3$)

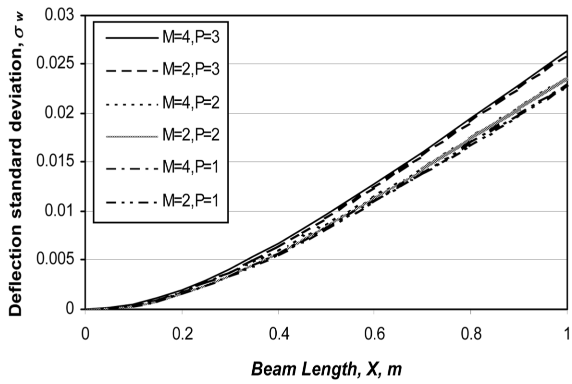

Fig. 4 Standard deviation of the deflection along the beam, Exponential covariance model for the bending rigidity, $l_{cor} = 1.0$, $\sigma_{EI} = 0.2$

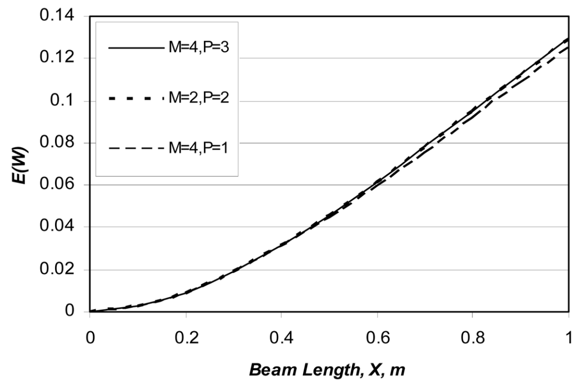

Fig. 5 Mean deflection along the beam, Exponential covariance model for the bending rigidity, ($l_{cor} = 1.0$, $\sigma_{EI} = 0.2$)

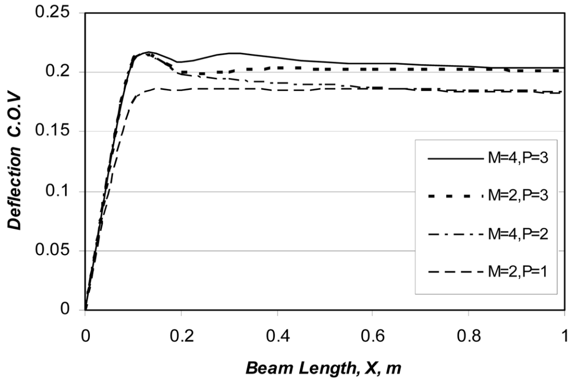

Fig. 6 Deflection C.O.V along the beam, Exponential covariance model for the bending rigidity, (l_{cor} = 1.0, σ_{EI} = 0.2)

respectively. Also the coefficient of variation (C.O.V), the standard deviation over the mean, for the deflection is plotted in Fig. 6 when σ_{EI} = 0.2.

» Results for Triangular covariance model

Figures from 7 to 11 show the corresponding results for the triangularly decaying covariance model. These results agree with those presented in Ghanem and Spanos (1991).

» Results for Wiener process

The results presented for both triangular and exponential covariance models encouraged us to apply the previous analysis for another important model, Wiener model, of the underlying random field. Fig. 12 shows the convergence of the solution, represented by the deflection standard deviation along the beam, as M and P increase. The mean deflection of the beam and deflection C.O.V in this case, are presented in Figs. 13 and 14 respectively.

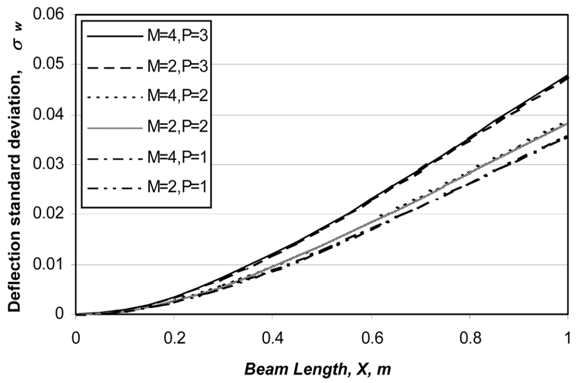

Fig. 7 Standard deviation of the deflection along the beam, Triangular covariance model for the bending rigidity, (l_{cor} = 2.0, σ_{EI} = 0.3)

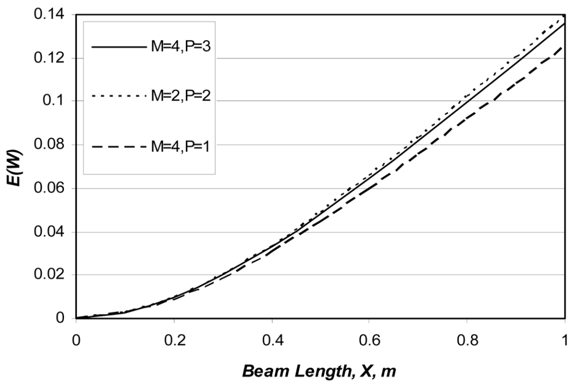

Fig. 8 Mean deflection along the beam, Triangular covariance model for the bending rigidity, (l_{cor} = 2.0, σ_{EI} = 0.3)

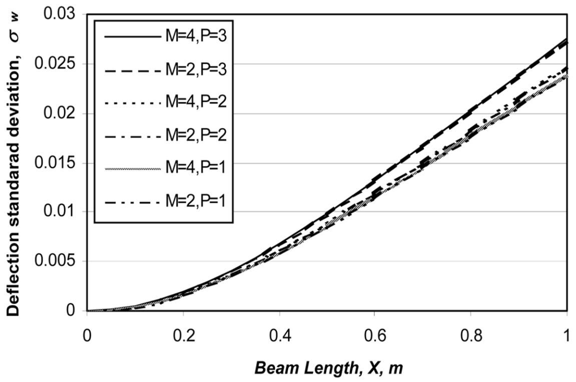

Fig. 9 Standard deviation of the deflection along the beam, Triangular covariance model for the bending rigidity, (l_{cor} = 2.0, σ_{EI} = 0.2)

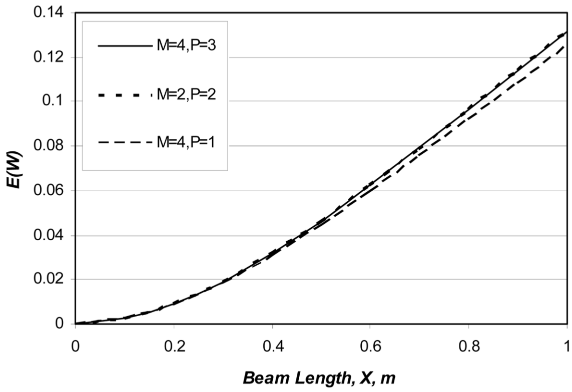

Fig. 10 Mean deflection along the beam, Triangular covariance model for the bending rigidity, (l_{cor} = 2.0, σ_{EI} = 0.2)

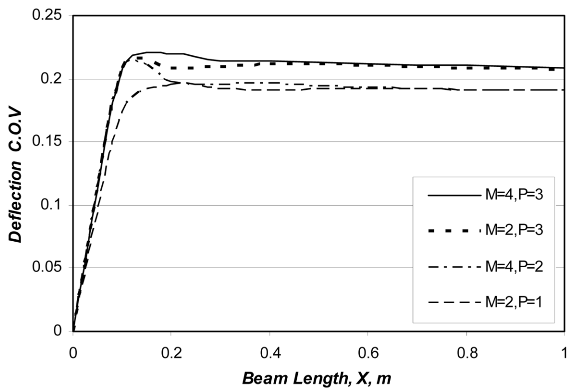

Fig. 11 Deflection C.O.V along the beam, Triangular covariance model for the bending rigidity, (l_{cor} = 2.0, σ_{EI} = 0.2)

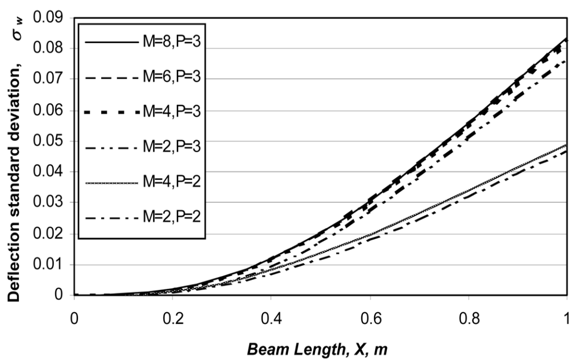

Fig. 12 Standard deviation of the deflection along the beam, Wiener model for the bending rigidity, τ^2 = 1.0

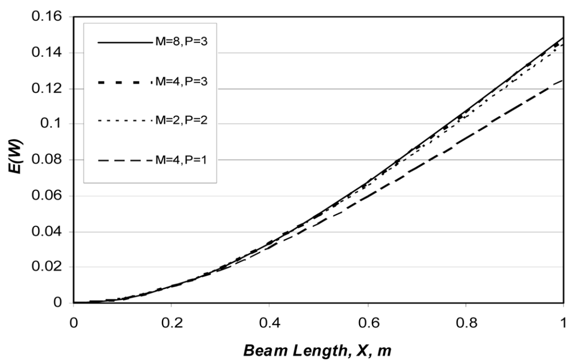

Fig. 13 Mean deflection along the beam, Wiener model for the bending rigidity, (τ^2 = 1.0)

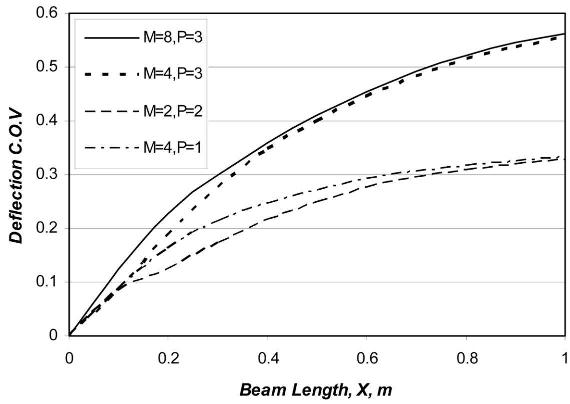

Fig. 14 Deflection C.O.V along the beam, Wiener model for the bending rigidity, (τ^2 = 1.0)

3.3 Random fields (load and bending rigidity) for the simply supported reinforced concrete beam

This is the more general case that is formulated in section 2.2.1. In realizing this work, the present application is introduced.

3.3.1 Problem description and formulation

Consider the Euler-Bernoulli reinforced concrete simple beam of length L, with random bending rigidity $EI(x;\theta)$ and subjected to a distributed random transverse load $f(x;\theta)$. It is assumed that $EI(x;\theta)$ and $f(x;\theta)$ are the realizations of Gaussian random processes indexed over the spatial domain occupied by the beam, $x \in [0, L]$. It is also assumed that they are represented by their means $\overline{EI}$ and $\bar{f}$, which are constants, and their covariance functions $C_{EI}(x, y)$ and $C_f(x, y)$, $x, y \in [0, L]$, respectively. In this case the mathematical model is

$$\frac{d^2}{dx^2}\left(EI(x;\theta)\frac{d^2w(x;\theta)}{dx^2}\right) = f(x;\theta), \quad x \in [0, L] \tag{74}$$

with

$$w(0) = w(L) = 0 \tag{75}$$

$$\left.\left(EI\frac{d^2w}{dx^2}\right)\right|_{x=0} = \left.\left(EI\frac{d^2w}{dx^2}\right)\right|_{x=L} = 0 \tag{76}$$

Expanding both processes involved in the problem using K-L expansion as follows

$$EI(x;\theta) = \overline{EI} + \sum_{n=1}^{M} \sqrt{\lambda_n}\xi_n\phi_n(x) \tag{77}$$

$$f(x;\theta) = \bar{f} + \sum_{n=1}^{M} \sqrt{\mu_n}\xi_n h_n(x) \tag{78}$$

and using the general procedures described in section 2.2, the general global system in Eq. (25) is obtained, which is

$$\left[\overline{\mathbf{K}} + \sum_{n=1}^{M} \xi_n \mathbf{K}_n\right]\mathbf{U} = \overline{\mathbf{F}} + \sum_{n=1}^{M} \xi_n \mathbf{F}_n \tag{79}$$

where the deterministic part, $\overline{\mathbf{K}}$ and the stochastic parts, $\mathbf{K}_n$ of the stiffness matrix $\mathbf{K}$ are computed by assembling the elementary matrices defined in Eqs. (60) and (61) respectively. Also, the deterministic part, $\overline{\mathbf{F}}$ and the stochastic parts, $\mathbf{F}_n$ of the right hand vector $\mathbf{F}$ are defined in Eqs. (45) and (46) respectively. After imposing the B.C.^{s} and the equilibrium conditions in the same manner as in the previous two applications, the system of order N specified in Eq. (79) becomes the system of order N-2 in Eq. (28). Using the truncated Neumann expansion, the explicit form of the solution vector is obtained as in Eq. (33).

3.3.2 Statistical moments of the response vector

Using Eq. (33), we can find the statistical moments of the response vector for various values of the orders P and M. For example, the moments in case of second order Neumann expansion the are as follows

$$\mathbf{U} = \left[\mathbf{I} - \sum_{i=1}^{M} \xi_i \mathbf{Q}_i + \sum_{i=1}^{M}\sum_{j=1}^{M} \xi_i \xi_j \mathbf{Q}_i \mathbf{Q}_j\right]\left[\mathbf{d} + \sum_{i=1}^{M} \xi_i \mathbf{b}_i\right] \tag{80}$$

$$\overline{\mathbf{U}} = \mathbf{d} + \sum_{i=1}^{M} [\mathbf{Q}_i \mathbf{Q}_i \mathbf{d} - \mathbf{Q}_i \mathbf{b}_i] \tag{81}$$

$$\begin{aligned}
\mathbf{COV}(\mathbf{U}) &= \langle \mathbf{U}\mathbf{U}^T \rangle - \overline{\mathbf{U}}\,\overline{\mathbf{U}}^T \\
&= \sum_{i=1}^{M} [\mathbf{Q}_i \mathbf{d}\mathbf{d}^T \mathbf{Q}_i^T - \mathbf{Q}_i \mathbf{d}\mathbf{b}_i^T - \mathbf{b}_i \mathbf{d}^T \mathbf{Q}_i^T + \mathbf{b}_i \mathbf{b}_i^T] \\
&\quad + \sum_{i=1}^{M}\sum_{j=1}^{M} [\mathbf{Q}_i \mathbf{Q}_j \mathbf{d}\mathbf{d}^T \mathbf{Q}_i^T \mathbf{Q}_j^T + \mathbf{Q}_i \mathbf{Q}_j \mathbf{d}\mathbf{d}^T \mathbf{Q}_j^T \mathbf{Q}_i^T + \mathbf{Q}_i \mathbf{b}_j \mathbf{d}_j^T \mathbf{Q}_i^T \\
&\quad - \mathbf{Q}_i \mathbf{d}\mathbf{b}_i^T \mathbf{Q}_j^T \mathbf{Q}_j^T - \mathbf{Q}_i \mathbf{d}\mathbf{b}_j^T \mathbf{Q}_i^T \mathbf{Q}_j^T - \mathbf{Q}_i \mathbf{d}\mathbf{b}_j^T \mathbf{Q}_j^T \mathbf{Q}_i^T - \mathbf{Q}_i \mathbf{Q}_j \mathbf{d}\mathbf{b}_i^T \mathbf{Q}_j^T \\
&\quad - \mathbf{Q}_i \mathbf{Q}_j \mathbf{d}\mathbf{b}_j^T \mathbf{Q}_i^T + \mathbf{b}_i \mathbf{b}_i^T \mathbf{Q}_j^T \mathbf{Q}_j^T + \mathbf{b}_i \mathbf{b}_j^T \mathbf{Q}_i^T \mathbf{Q}_j^T + \mathbf{b}_i \mathbf{b}_j^T \mathbf{Q}_j^T \mathbf{Q}_i^T \\
&\quad - \mathbf{Q}_i \mathbf{b}_j \mathbf{d}^T \mathbf{Q}_i^T \mathbf{Q}_j^T - \mathbf{Q}_i \mathbf{b}_j \mathbf{d}^T \mathbf{Q}_j^T \mathbf{Q}_i^T + \mathbf{Q}_i \mathbf{b}_j \mathbf{d}_i^T \mathbf{Q}_j^T + \mathbf{Q}_i \mathbf{b}_j \mathbf{b}_j^T \mathbf{Q}_i^T \\
&\quad - \mathbf{Q}_i \mathbf{Q}_i \mathbf{b}_j \mathbf{d}^T \mathbf{Q}_j^T - \mathbf{Q}_i \mathbf{Q}_j \mathbf{b}_i \mathbf{d}^T \mathbf{Q}_j^T - \mathbf{Q}_i \mathbf{Q}_j \mathbf{b}_j \mathbf{d}^T \mathbf{Q}_i^T + \mathbf{Q}_i \mathbf{Q}_i \mathbf{b}_j \mathbf{b}_j^T \\
&\quad + \mathbf{Q}_i \mathbf{Q}_j \mathbf{b}_i \mathbf{b}_j^T + \mathbf{Q}_i \mathbf{Q}_j \mathbf{b}_j \mathbf{b}_i^T] \\
&\quad + \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{k=1}^{M} [\mathbf{Q}_i \mathbf{Q}_i \mathbf{b}_j \mathbf{b}_k^T \mathbf{Q}_k^T \mathbf{Q}_j^T + \mathbf{Q}_i \mathbf{Q}_i \mathbf{b}_j \mathbf{b}_k^T \mathbf{Q}_j^T \mathbf{Q}_k^T + \mathbf{Q}_i \mathbf{Q}_i \mathbf{b}_j \mathbf{b}_j^T \mathbf{Q}_k^T \mathbf{Q}_k^T \\
&\quad + \mathbf{Q}_i \mathbf{Q}_j \mathbf{b}_i \mathbf{b}_k^T \mathbf{Q}_k^T \mathbf{Q}_j^T + \mathbf{Q}_i \mathbf{Q}_j \mathbf{b}_i \mathbf{b}_k^T \mathbf{Q}_j^T \mathbf{Q}_k^T + \mathbf{Q}_i \mathbf{Q}_j \mathbf{b}_i \mathbf{b}_j^T \mathbf{Q}_k^T \mathbf{Q}_k^T \\
&\quad + \mathbf{Q}_i \mathbf{Q}_j \mathbf{b}_j \mathbf{b}_k^T \mathbf{Q}_k^T \mathbf{Q}_j^T + \mathbf{Q}_i \mathbf{Q}_j \mathbf{b}_j \mathbf{b}_k^T \mathbf{Q}_i^T \mathbf{Q}_k^T + \mathbf{Q}_i \mathbf{Q}_j \mathbf{b}_j \mathbf{b}_i^T \mathbf{Q}_k^T \mathbf{Q}_k^T \\
&\quad + \mathbf{Q}_i \mathbf{Q}_k \mathbf{b}_j \mathbf{b}_k^T \mathbf{Q}_j^T \mathbf{Q}_i^T + \mathbf{Q}_i \mathbf{Q}_k \mathbf{b}_j \mathbf{b}_k^T \mathbf{Q}_j^T \mathbf{Q}_i^T + \mathbf{Q}_i \mathbf{Q}_k \mathbf{b}_j \mathbf{b}_i^T \mathbf{Q}_k^T \mathbf{Q}_j^T \\
&\quad + \mathbf{Q}_i \mathbf{Q}_k \mathbf{b}_j \mathbf{b}_j^T \mathbf{Q}_k^T \mathbf{Q}_i^T + \mathbf{Q}_i \mathbf{Q}_k \mathbf{b}_j \mathbf{b}_j^T \mathbf{Q}_i^T \mathbf{Q}_k^T + \mathbf{Q}_i \mathbf{Q}_k \mathbf{b}_j \mathbf{b}_i^T \mathbf{Q}_j^T \mathbf{Q}_k^T]
\end{aligned} \tag{82}$$

3.3.3 Results

In this application the covariance kernels of the random bending rigidity and the random load are modeled using the exponential model and the Triangular model. Firstly, it is considered that bending rigidity and load are both modeled by exponential model. Secondly, the load is modeled by Exponential model and the bending rigidity is modeled by Triangular model. In the numerical implementation of the preceding analysis, the beam is of length $L = 1.0$ m and it was discretized into ten finite elements $(N^{el} = 10)$. The Exponential covariance model has correlation length, $l_{cor} = 1/c = 1.0$ and the Triangular covariance model has correlation length, $l_{cor} = 1/c = 2.0$. Also, it is assumed that the bending rigidity, has unit mean, $\overline{EI} = 1.0\,\mathrm{N.m^2}$ and variance, $\sigma_{EI}^2 = .04$ and the load process has zero mean and unit variance. For the preceding numerical values the results in Figs. 15-18 are obtained. Figs. 15 and 16 show the deflection variance and the deflection C.O.V along the beam for various values of the orders M and P. It is evident that the solution is affected mainly by the change of P. As P increases we can get more accurate solution. These

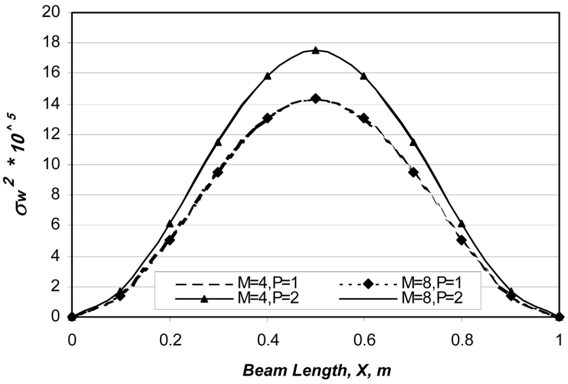

Fig. 15 Deflection variance along the beam, Exponential covariance model for the load and Triangular covariance model for the bending rigidity

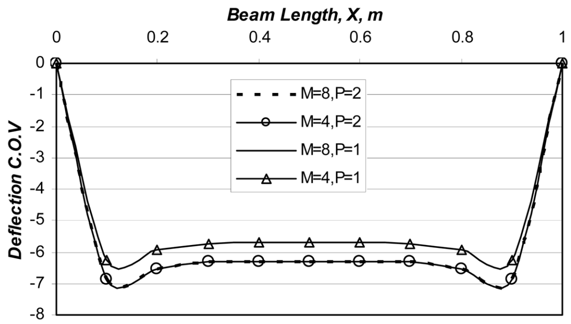

Fig. 16 Deflection C.O.V along the beam, Exponential covariance model for the load and Triangular covariance model for the bending rigidity

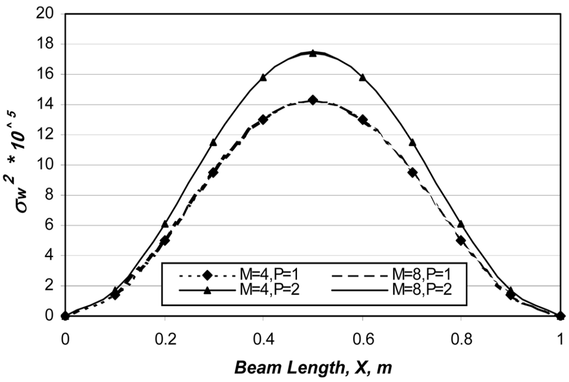

Fig. 17 Deflection variance along the beam, Exponential covariance model for both load and bending rigidity

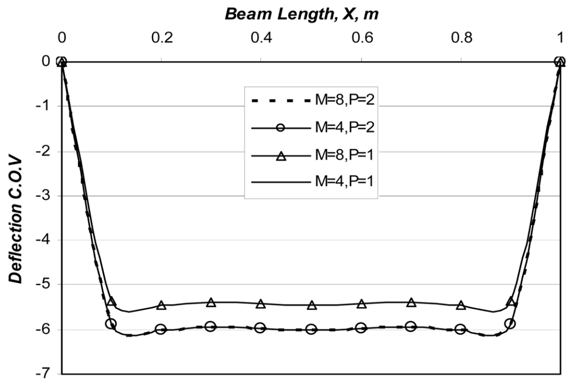

Fig. 18 Deflection C.O.V along the beam, Exponential covariance model for both load and bending rigidity

figures show the results corresponding to the Exponential model for the load and Triangular model for the bending rigidity. Figs. 17 and 18 show the corresponding results when load and bending rigidity are both modeled by exponential model.

4. Conclusions

In this paper, the spectral SFEM approach is used for solving the stochastic differential equations with random operator and/or random excitation. The work was confined to continuous-parameter processes since they constitute the majority of mathematical models used for processes encountered in practice. With no loss of generality, all random quantities are modeled as second order processes defined only by their means and covariance functions. A general SSFEM formulation of the problem is presented and three applications are introduced to check the validity of that formulation. The numerical results are compared with results available in the literature, if possible. The first application is a simply supported reinforced concrete beam under random load represented by a random field. This problem has an analytical solution presented in Elishakoff *et al.* (1999), which was taken as a validation tool for the presented spectral SFEM of this application. It is found that the SFEM solution is approximately identical to the analytical solution for relatively small number of terms in the K-L expansion of the load process. The second application is a cantilever beam with random bending rigidity (*EI*) represented by a random field. The results of this problem are compared with the results presented by Ghanem and Spanos (1991), which agreed with our presented results. The success of the presented SFEM formulation in solving the previous two problems encouraged us to present the third application, which is a new application, that is a simply supported reinforced concrete beam, whose bending rigidity is a random field, under loading being also a random field. The spectral SFEM formulation introduced in this paper succeeded in presenting an approximate solution for this problem. The convergence of the solution is approached while the orders of K-L and Neumann expansions are increased.

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Appendix-A

Derivation of the eigenfunctions and eigenvales of the covariance kernel of the Wiener Process (Nonstationary Process)

This process was developed as a model for Brownian motion and is discussed in details in (Parzen 1962, Rosenblatt 1962). It is an important process for several reasons (Loeve 1977)

1. A large class of processes can be transformed into Wiener process.
2. A large class of processes can be generated by passing a Wiener process through a linear or nonlinear system.

If $\alpha(x;\theta)$ is a Wiener process, then it is Gaussian, defined for $x > 0$ and is characterized by the following properties

$$\alpha(0;\theta) = 0 \tag{A1-a}$$

$$\langle \alpha(x;\theta)\rangle = 0 \tag{A1-b}$$

$$\langle \alpha^2(x;\theta)\rangle = \tau^2 x \tag{A1-c}$$

and its p.d.f is:

$$f_\alpha(\alpha) = \frac{1}{\sqrt{2\pi\tau^2 x}}\exp\left(-\frac{\alpha^2}{2\tau^2 x}\right) \tag{A1-d}$$

where τ^2 is a deterministic constant.

The kernel of this process is given by the equation

$$C_\alpha(x,y) = \tau^2\min(x,y) = \begin{cases} \tau^2 x & \text{if} \quad x \le y \\ \tau^2 y & \text{if} \quad x \ge y \end{cases} \tag{A2}$$

Consider realization of this process on the interval $[0, a]$. The eigenfunctions and eigenvalues of the covariance kernel are the solutions of

$$\int_D C_\alpha(x,y)\phi(y)dy = \lambda\phi(x) \tag{A3}$$

or

$$\tau^2\int_0^x y\phi(y)dy + \tau^2\int_x^a x\phi(y)dy = \lambda\phi(x) \tag{A4}$$

Differentiating Eq. (A4) twice w.r.t. x gives

$$\tau^2\int_x^a \phi(y)dy = \lambda\phi'(x) \tag{A5}$$

$$\phi''(x) + w^2\phi(x) = 0 \tag{A6}$$

where

$$w^2 = \frac{\tau^2}{\lambda} \tag{A7}$$

To find the B.C.^{s} of the D.E (A6), substitute for $x = 0$ in (A4) and $x = a$ in (A5) to get the following two conditions

$$\phi(0) = 0, \quad \phi'(a) = 0 \tag{A8}$$

The general solution of the D.E (A6) is

$$\phi(x) = b_1 \cos wx + b_2 \sin wx \tag{A9}$$

Applying the B.C.^{s} in (A8), one can get

$$b_1 = 0; \quad \cos(wa) = 0 \tag{A10}$$

The 2nd equation in (A10) gives the infinite number of solutions for w as

$$w_i = \frac{(2i-1)\pi}{2a}, \quad i \geq 1 \tag{A11}$$

For these solutions the eigenfunctions are given by

$$\phi_i(x) = b_2 \sin\left(\frac{(2i-1)\pi x}{2a}\right), \quad i \geq 1 \tag{A12}$$

Using the orthonormality condition of the eigenfunctions (Loeve 1977), the value of b_2 is found to be $\sqrt{2/a}$.

Finally,

$$\phi_i(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{(2i-1)\pi x}{2a}\right), \quad i \geq 1 \tag{A13}$$

and

$$\lambda_i = \left(\frac{2\,\tau a}{(2i-1)\pi}\right)^2 \tag{A14}$$