

## Free vibrations of anisotropic rectangular plates with holes and attached masses

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**Abstract.** Anisotropic materials are increasingly required in modern technological applications. Certainly, civil, mechanical and naval engineers frequently deal with the situation of analyzing the dynamical behaviour of structural elements being composed of such materials. For example, panels of anisotropic materials must sometimes support electromechanical engines, and besides, holes are performed in them for operational reasons e.g., conduits, ducts or electrical connections. This study is concerned with the natural frequencies and normal modes of vibration of rectangular anisotropic plates supported by different combinations of the classical boundary conditions: clamped, simply – supported and free, and with additional complexities such holes of free boundaries and attached concentrated masses. A variational approach (the well known Ritz method) is used, where the displacement amplitude is approximated by a set of beam functions in each coordinate direction corresponding to the sides of the rectangular plate. Consequently each coordinate function satisfies the essential boundary conditions at the outer edge of the plate. The influence of the position and magnitude of both hole and mass, on the natural frequencies and modal shapes of vibration are studied for a generic anisotropic material. The classical Ritz method with beam functions as spatial approximation proved to be a suitable procedure to solve a problem of such analytical complexity.

**Keywords:** vibration of plates; anisotropic plates; concentrated mass; holes of free edge; Ritz method.

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### 1. Introduction

The present study deals with the analysis of transverse vibrations of thin rectangular plates of anisotropic materials carrying concentrated masses rigidly attached and rectangular holes of free edges.

The proposed mechanical system is of great interest in many technological situations since it is quite common in a large variety of engineering fields: from plates supporting machinery with holes

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to printed circuit boards with electronic elements attached to them. A plate – like chassis or a printed circuit can be approximated as flat rectangular plates carrying concentrated masses with holes, subjected to vibration.

As it is known it does not appear possible to obtain an exact analytical solution for the mode shapes and natural frequencies of transverse vibration of such a complex structural system.

It is important to point out that the thorough treatise due to Lekhnitskii (1968) does not solve any problem of vibration of anisotropic plates. Nevertheless, there are several textbooks on anisotropic plates where the vibration problems are included (Reddy 1997, Whitney 1987).

The variational Ritz method (pointed out by Leissa 2005, Mikhlin 1964 as being incorrectly called the Rayleigh-Ritz method by some persons), is employed to perform the analysis.

The displacement amplitude is approximated by a set of beam functions in each principal coordinate direction as it has been done by pioneering works on the vibration of solid anisotropic plates (Ashton 1969, Ashton and Waddoups 1969, Ashton and Anderson 1969, Bert and Mayberry 1969, Mohan and Kingsbury 1971).

Unfortunately at least one of those “almost classical” works, the paper by Mohan and Kingsbury, published thirty five years ago commits a mathematical error since the eigenvalues are determined by the Galerkin method. In view of the fact that the coordinate functions do not satisfy, generally, the natural boundary conditions the methodology is not admissible and the eigenvalues are not, in general, valid.

In the treatment of anisotropic vibrating rectangular plates with additional complexities, Avalos *et al.* (1991) considered doubly connected domains for the simply – supported case and Cincio *et al.* (2006) studied the cantilever anisotropic plate with a rigidly attached mass.

The first five natural frequency coefficients are obtained for plates of different combinations of the classical boundary conditions with a centered orifice, and varying the position and magnitude of the concentrated mass. The considered structural systems are shown in Fig. 1.

The corresponding modal shapes are also studied.

Due to the quantity and variability of the parameters involved in the description of the dynamical behaviour of these kinds of structures, just a few representative cases will be considered to demonstrate the convenience of the procedure.

## 2. Approximate analytical solution

According to the classical thin anisotropic plate theory, (Lekhnitskii 1968), the energy functional corresponding to the vibrating described system is given by

$$J(W) = \frac{1}{2} \iint_{A_p} \left[ D_{11} \left[ \frac{\partial^2 W}{\partial \bar{x}^2} \right]^2 + 2D_{12} \frac{\partial^2 W}{\partial \bar{x}^2} \frac{\partial^2 W}{\partial \bar{y}^2} + D_{22} \left[ \frac{\partial^2 W}{\partial \bar{y}^2} \right]^2 + 4D_{66} \left[ \frac{\partial^2 W}{\partial \bar{x} \partial \bar{y}} \right]^2 + \right. \\ \left. + 4 \left[ D_{16} \frac{\partial^2 W}{\partial \bar{x}^2} + D_{26} \frac{\partial^2 W}{\partial \bar{y}^2} \right] \frac{\partial^2 W}{\partial \bar{x} \partial \bar{y}} \right] d\bar{x} d\bar{y} - \frac{1}{2} \rho h \omega^2 \iint_{A_p} W^2 d\bar{x} d\bar{y} + \frac{1}{2} m \omega^2 [W(\bar{x}_m, \bar{y}_m)]^2 \quad (1)$$

where  $W = W(\bar{x}, \bar{y})$  is the deflection amplitude of the middle plane of the plate.  $D_{ij}$  are the well known flexural rigidities of the anisotropic plate.

$A_p$  is the net area of the plate plan form:  $A_p = A - A_h$  where  $A$  is the area of the whole rectangle:  $a \times b$  and  $A_h$  is the area of the hole:  $a_1 \times b_1$ .

$\rho$ ,  $h$  are the density and the thickness of the plate, respectively,  $m$  is the magnitude of the concentrated mass,  $W(\bar{x}_m, \bar{y}_m)$  is the plate displacement amplitude at the mass position  $(\bar{x}_m, \bar{y}_m)$  and  $\omega$  is the natural circular frequency of the system.

The rotatory inertia of the concentrated mass is neglected in the present analysis.

As the length of the sides of the rectangular plate are  $a$  and  $b$  in the  $\bar{x}$  and  $\bar{y}$  directions respectively, the coordinates can be written in the dimensionless form

$$\begin{aligned} x &= \bar{x}/a, y = \bar{y}/b \\ x_m &= \bar{x}_m/a, y_m = \bar{y}_m/b \end{aligned} \quad (2)$$

and the aspect ratio of the plate

$$\lambda = a/b$$

For simplicity, holes of the same aspect ratio of the plate are just considered:  $a_1/b_1 = a/b = \lambda$ .

The expression of the deflection of the plate is approximated in the form of a truncated series of beam functions  $X_m(x)$  and  $Y_n(y)$ .

$$W(x, y) = \sum_{m=1}^M \sum_{n=1}^N A_{mn} X_m(x) Y_n(y) \quad (3)$$

$X_m(x)$  and  $Y_n(y)$  are the characteristic functions for the normal modes of vibration of beams with end conditions nominally similar to those of the opposite edges of the plate in each coordinate direction.

When the configuration of the plate leads to beams with both ends free, for example in the case of a cantilever plate, the first two characteristic functions correspond to rigid motions: translation and rotation.

Obviously  $X_m(x)$  and  $Y_n(y)$  do not satisfy the natural boundary conditions at outer and inner edges, as previously stated but this is legitimate when using the Ritz method (Nallim and Grossi 2003).

Substituting Eq. (3) into Eq. (1) and, requiring that  $J(W)$  be a minimum with respect to the  $A_{mn}$ 's coefficients

$$\frac{\partial J[W]}{\partial A_{mn}} = 0 \quad m = 1, 2, \dots, M; \quad n = 1, 2, \dots, N \quad (4)$$

one obtains a homogeneous linear system of equation in terms of the  $A_{mn}$ 's parameters.

From the non-triviality conditions, one can get natural frequency coefficients:

$\Omega_i = \omega_i a^2 \sqrt{\rho h / D_{11}}$  as eigenvalues, and vibration modes as eigenvectors of the secular determinant.

The present study is concerned with the determination of the first five natural frequency coefficients  $\Omega_1$  to  $\Omega_5$  in the case of anisotropic rectangular plate, and their respective modal shapes.

### 3. Numerical results

The natural frequencies and modal shapes of the described plates are analyzed.

The plates are simply supported, clamped or free at their external edges.

The results of previous investigations show that the plate modal shapes and natural frequency

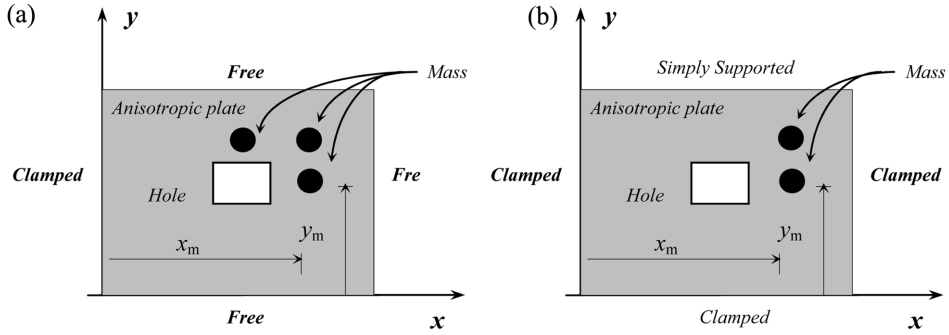


Fig. 1 Vibrating systems under consideration: (a) Cantilever anisotropic plate and (b) C-C-C-SS anisotropic plate

Table 1 Frequency coefficients values for a cantilever (CFFF) anisotropic, doubly connected plate with a concentrated mass attached at  $(x_m = 0.5, y_m = 0.75)$ 

| $\lambda = a/b$ | $a_1/a$ | $M = m/m_p$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|-----------------|---------|-------------|------------|------------|------------|------------|------------|
| 2/3             | 0       | 0           | 3.0663     | 4.8750     | 12.6397    | 18.8565    | 22.4569    |
|                 |         | 0           | 3.0331     | 4.8073     | 12.6503    | 18.6547    | 22.3895    |
|                 |         | 0.1         | 3.0289     | 4.5176     | 12.4955    | 18.1002    | 21.1687    |
|                 |         | 0.5         | 2.9987     | 3.7125     | 11.9093    | 15.7691    | 19.7586    |
|                 | 0.1     | 1           | 2.8595     | 3.2408     | 11.4299    | 14.9039    | 19.5961    |
|                 |         | 0           | 2.6362     | 4.4472     | 12.3105    | 16.7700    | 21.7948    |
|                 |         | 0.1         | 2.6356     | 4.1703     | 12.1403    | 16.2680    | 20.7922    |
|                 |         | 0.5         | 2.6315     | 3.3914     | 11.4729    | 14.6065    | 18.8206    |
|                 |         | 1           | 2.6096     | 2.8444     | 10.9450    | 14.1058    | 18.4814    |
|                 | 0.2     | 0           | 2.8285     | 5.5269     | 18.9016    | 20.0922    | 27.5157    |
|                 |         | 0           | 2.7539     | 5.3895     | 18.9804    | 20.0111    | 27.4192    |
|                 |         | 0.1         | 2.7505     | 5.0583     | 18.4618    | 19.0017    | 26.1460    |
|                 |         | 0.5         | 2.7322     | 4.1367     | 15.4203    | 18.9893    | 24.9564    |
| 1               | 0       | 1           | 2.6911     | 3.4958     | 14.1884    | 18.9887    | 24.6828    |
|                 |         | 0           | 1.0822     | 4.5794     | 18.2573    | 19.4234    | 26.2727    |
|                 |         | 0.1         | 1.0781     | 4.3012     | 17.6624    | 18.5167    | 24.9961    |
|                 |         | 0.5         | 1.0614     | 3.5420     | 14.7672    | 18.3846    | 23.8515    |
|                 | 0.1     | 1           | 1.0401     | 3.0096     | 13.5522    | 18.3756    | 23.5981    |
|                 |         | 0           | 2.4493     | 6.1930     | 19.4740    | 24.7333    | 44.1457    |
|                 |         | 0           | 2.2369     | 5.8656     | 19.4341    | 24.9712    | 43.8940    |
|                 |         | 0.1         | 2.2357     | 5.4784     | 19.0262    | 22.7165    | 43.5189    |
|                 |         | 0.5         | 2.2296     | 4.4233     | 17.0592    | 20.7932    | 43.0314    |
|                 | 0.2     | 1           | 2.2196     | 3.6832     | 15.9173    | 20.5385    | 42.8685    |
|                 |         | 0           | 2.3585     | 6.3847     | 19.3754    | 28.0044    | 44.7723    |
|                 |         | 0.1         | 2.3563     | 5.9087     | 18.5621    | 26.0624    | 44.0196    |
|                 |         | 0.5         | 2.3464     | 4.6692     | 16.6710    | 24.0524    | 43.1147    |
| 3/2             | 0       | 1           | 2.3299     | 3.8486     | 15.8055    | 23.5755    | 42.8383    |

coefficients are strongly affected by the characteristic of anisotropic material and that such structures do not exhibit easily predictable behavior.

Therefore, and in view of the fact that the principal aim of the present work is to show the flexibility of the proposed procedure, just a generic arbitrary anisotropic material is considered ( $D_{22} = D_{12} = D_{66} = D_{11}/2$ ,  $D_{16} = D_{26} = D_{11}/3$ ), and two situations including the different boundary conditions are analyzed (Fig. 1).

In the first place, a cantilever anisotropic plate is studied.

Table 1 to Table 3 contain the first five frequency coefficients for a cantilever plate with a centered free edge hole and different locations and magnitudes of the concentrated mass. The variation of the aspect ratio of the plate and the magnitude of hole and mass are also taken into account.

Table 2 Frequency coefficients values for a cantilever (CFFF) anisotropic, doubly connected plate with a concentrated mass attached at ( $x_m = 0.75$ ,  $y_m = 0.5$ )

| $\lambda = a/b$ | $a_1/a$ | $M = m/m_p$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|-----------------|---------|-------------|------------|------------|------------|------------|------------|
| 2/3             | 0       | 0           | 3.0663     | 4.8750     | 12.6397    | 18.8565    | 22.4569    |
|                 |         | 0           | 3.0331     | 4.8073     | 12.6503    | 18.6547    | 22.3895    |
|                 |         | 0.1         | 2.8516     | 4.6959     | 11.4026    | 18.6263    | 22.3887    |
|                 |         | 0.5         | 2.3075     | 4.4910     | 9.6593     | 18.5904    | 22.3828    |
|                 |         | 1           | 1.9075     | 4.4055     | 9.0745     | 18.5784    | 32.1185    |
|                 | 0.2     | 0           | 2.6362     | 4.4472     | 12.3105    | 16.7700    | 21.7948    |
|                 |         | 0.1         | 2.5278     | 4.2678     | 10.8575    | 16.6717    | 21.7074    |
|                 |         | 0.5         | 2.1416     | 3.8984     | 8.9721     | 16.5860    | 21.3488    |
|                 |         | 1           | 1.8069     | 3.7386     | 8.3851     | 16.5639    | 21.1958    |
|                 | 0       | 0           | 2.8285     | 5.5269     | 18.9016    | 20.0922    | 27.5157    |
|                 |         | 0           | 2.7539     | 5.3895     | 18.9804    | 20.0111    | 27.4192    |
|                 |         | 0.1         | 2.6012     | 5.2302     | 17.7760    | 19.7613    | 26.7249    |
|                 |         | 0.5         | 2.1488     | 4.9191     | 15.4208    | 19.6812    | 25.9909    |
|                 |         | 1           | 1.8049     | 4.7761     | 14.4980    | 19.6689    | 25.7901    |
| 1               | 0       | 0           | 1.0822     | 4.5794     | 18.2573    | 19.4234    | 26.2727    |
|                 |         | 0.1         | 1.0590     | 4.2847     | 17.0637    | 18.8364    | 25.5745    |
|                 |         | 0.5         | 0.9764     | 3.5964     | 14.4855    | 18.7110    | 24.9838    |
|                 |         | 1           | 0.8928     | 3.1995     | 13.5450    | 18.6960    | 24.8417    |
|                 | 0.2     | 0           | 2.4493     | 6.1930     | 19.4740    | 24.7333    | 44.1457    |
|                 |         | 0           | 2.2369     | 5.8656     | 19.4341    | 24.9712    | 43.8940    |
|                 |         | 0.1         | 2.1264     | 5.6379     | 19.2929    | 24.2895    | 41.4174    |
|                 |         | 0.5         | 1.7942     | 5.1715     | 18.8761    | 22.9645    | 37.3624    |
|                 |         | 1           | 1.5308     | 4.9407     | 18.5845    | 22.4117    | 36.0353    |
| 3/2             | 0       | 0           | 2.3585     | 6.3847     | 19.3754    | 28.0044    | 44.7723    |
|                 |         | 0.1         | 2.2352     | 6.1529     | 19.2357    | 26.6257    | 42.4859    |
|                 |         | 0.5         | 1.8726     | 5.6765     | 18.8161    | 24.1201    | 39.356     |
|                 |         | 1           | 1.5913     | 5.4396     | 18.5145    | 23.1757    | 38.4687    |
|                 | 0.2     | 0           | 2.4493     | 6.1930     | 19.4740    | 24.7333    | 44.1457    |
|                 |         | 0           | 2.2369     | 5.8656     | 19.4341    | 24.9712    | 43.8940    |
|                 |         | 0.1         | 2.1264     | 5.6379     | 19.2929    | 24.2895    | 41.4174    |
|                 |         | 0.5         | 1.7942     | 5.1715     | 18.8761    | 22.9645    | 37.3624    |
|                 |         | 1           | 1.5308     | 4.9407     | 18.5845    | 22.4117    | 36.0353    |

All the values are determined taking  $M = N = 10$  in Eq. (3).

Fig. 2 to Fig. 8 show the modal shapes for the cantilever anisotropic plate with the free edge hole and the concentrated mass.

Then an anisotropic rectangular plate with three outer edges clamped and the remaining simply supported, is analyzed.

Table 4 and Table 5 show values for similar features of this situation, as those considered for the cantilever plate.

As well Fig. 9 to Fig. 13 show the modal shapes of some particular cases of this plate.

In order to evaluate the accuracy of the expounded procedure, in Table 6 a comparison is made with the results obtained by Cupial (1997) for a highly anisotropic simply supported plate, by means of the Ritz method using orthogonal polynomials.

Table 3 Frequency coefficients values for a cantilever (CFFF) anisotropic, doubly connected plate with a concentrated mass attached at  $(x_m = 0.75, y_m = 0.75)$

| $\lambda = a/b$ | $a_1/a$ | $M = m/m_p$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|-----------------|---------|-------------|------------|------------|------------|------------|------------|
| 2/3             | 0       | 0           | 3.0663     | 4.8750     | 12.6397    | 18.8565    | 22.4569    |
|                 |         | 0           | 3.0331     | 4.8073     | 12.6503    | 18.6547    | 22.3895    |
|                 |         | 0.1         | 2.9938     | 4.0409     | 12.2414    | 17.9541    | 18.2756    |
|                 |         | 0.5         | 2.3474     | 3.4122     | 12.0826    | 13.8717    | 17.7570    |
|                 | 0.1     | 1           | 1.9228     | 2.9447     | 11.2692    | 11.9367    | 16.7336    |
|                 |         | 0           | 2.6362     | 4.4472     | 12.3105    | 16.7700    | 21.7948    |
|                 |         | 0.1         | 2.6349     | 3.7253     | 12.2740    | 15.8876    | 21.6851    |
|                 |         | 0.5         | 2.4753     | 2.6610     | 12.2110    | 14.9249    | 21.5809    |
|                 |         | 1           | 1.9157     | 2.6436     | 12.1840    | 14.6491    | 21.5528    |
|                 | 0.2     | 0           | 2.8285     | 5.5269     | 18.9016    | 20.0922    | 27.5157    |
|                 |         | 0           | 2.7539     | 5.3895     | 18.9804    | 20.0111    | 27.4192    |
|                 |         | 0.1         | 2.7501     | 4.2228     | 15.1750    | 17.6949    | 23.1222    |
|                 |         | 0.5         | 2.1519     | 3.6119     | 14.9921    | 17.4857    | 22.7534    |
| 1               | 0.1     | 1           | 1.7593     | 3.1616     | 14.7835    | 17.2516    | 19.9787    |
|                 |         | 0           | 1.0822     | 4.5794     | 18.2573    | 19.4234    | 26.2727    |
|                 |         | 0.1         | 1.0795     | 3.8509     | 18.0962    | 19.3740    | 25.9693    |
|                 |         | 0.5         | 1.0677     | 2.6277     | 17.8462    | 19.3196    | 25.4568    |
|                 | 0.2     | 1           | 1.0497     | 2.0603     | 17.7509    | 19.3034    | 25.2551    |
|                 |         | 0           | 2.4493     | 6.1930     | 19.4740    | 24.7333    | 44.1457    |
|                 |         | 0           | 2.2369     | 5.8656     | 19.4341    | 24.9712    | 43.8940    |
|                 |         | 0.1         | 2.2069     | 5.0441     | 19.4287    | 24.8912    | 43.1459    |
|                 |         | 0.5         | 2.0641     | 3.6974     | 19.4211    | 24.7707    | 41.6088    |
|                 | 0.1     | 1           | 1.8642     | 3.1739     | 19.4183    | 24.7244    | 40.9251    |
|                 |         | 0           | 2.3585     | 6.3847     | 19.3754    | 28.0044    | 44.7723    |
|                 |         | 0.1         | 2.31768    | 5.4787     | 19.3575    | 27.8792    | 43.9428    |
|                 |         | 0.5         | 2.1358     | 4.0422     | 19.3333    | 27.6864    | 42.1260    |
| 3/2             | 0.2     | 1           | 1.9065     | 3.5002     | 19.3247    | 27.6113    | 41.3487    |

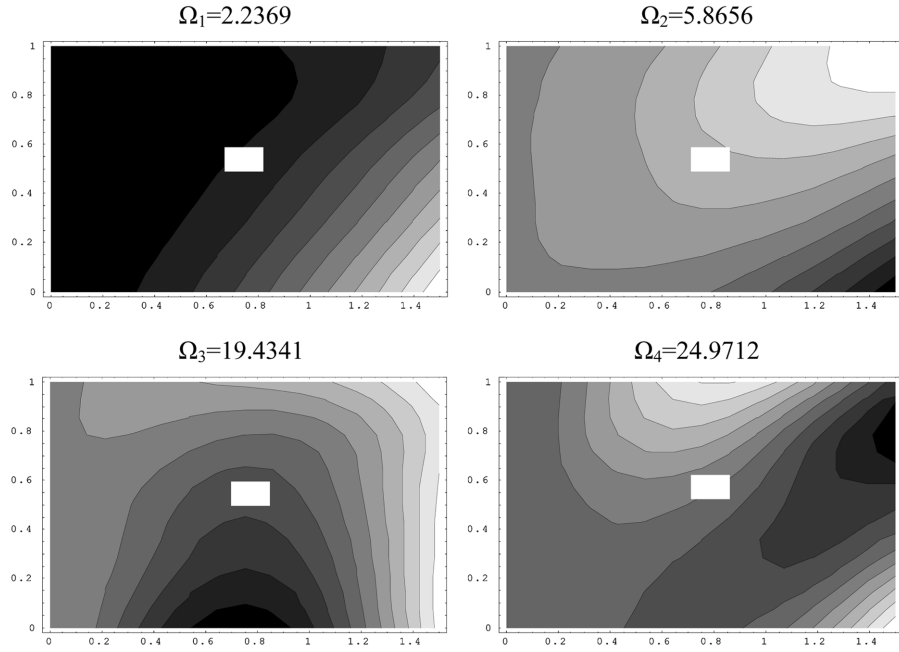


Fig. 2 First four modal shapes of vibration of a cantilever anisotropic plate of  $\lambda = 3/2$ , hole of  $a_1/a = 0.1$  and without attached mass

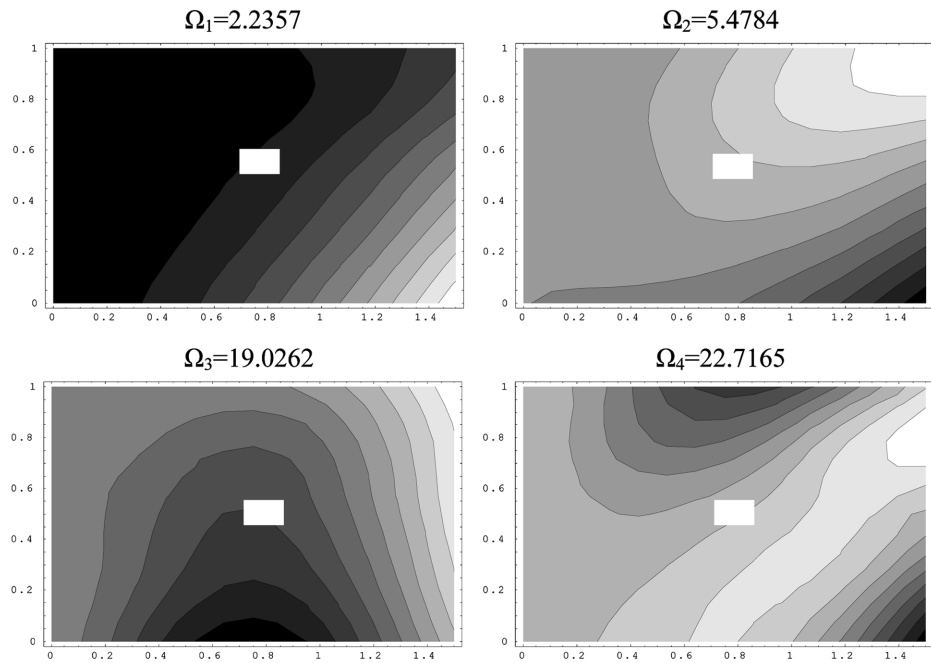


Fig. 3 First four modal shapes of vibration of a cantilever anisotropic plate of  $\lambda = 3/2$ , hole of  $a_1/a = 0.1$  and a concentrated mass ( $M = 0.1$ ) attached at  $(x_m, y_m) = (0.5, 0.75)$

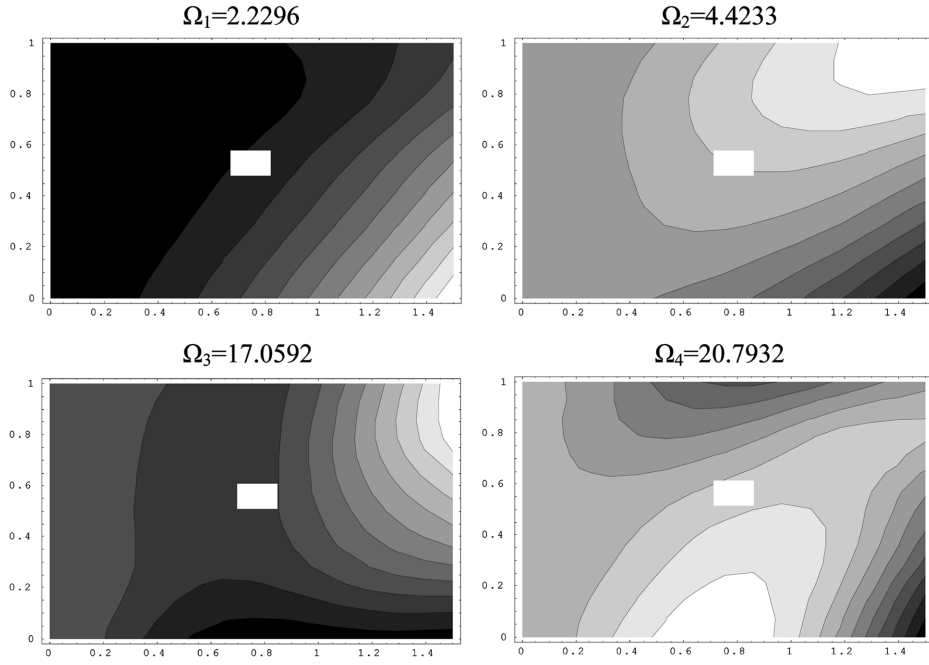


Fig. 4 First four modal shapes of vibration of a cantilever anisotropic plate of  $\lambda = 3/2$ , hole of  $a_1/a = 0.1$  and a concentrated mass ( $M = 0.5$ ) attached at  $(x_m, y_m) = (0.5, 0.75)$

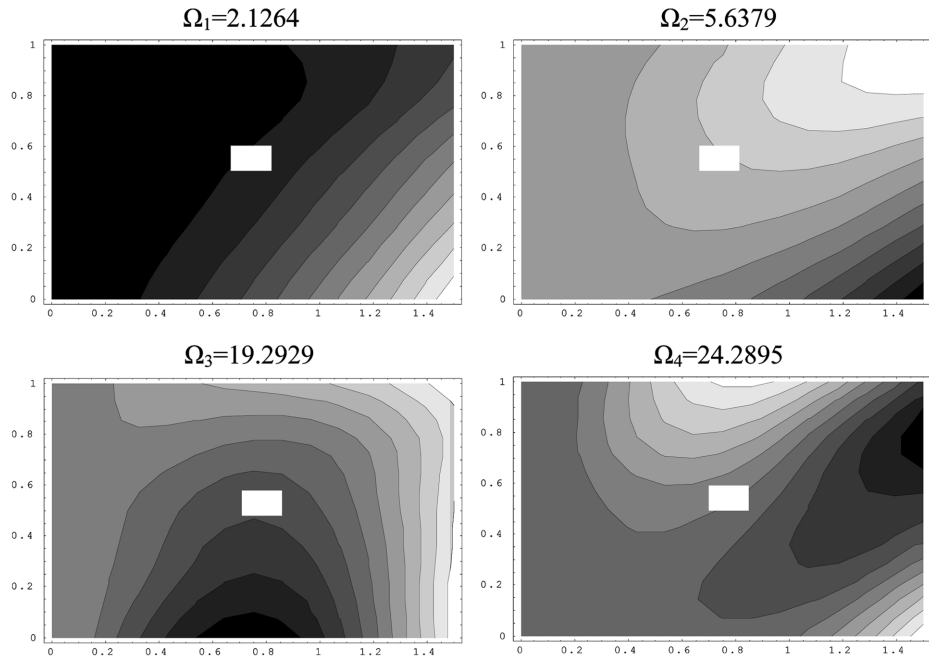


Fig. 5 First four modal shapes of vibration of a cantilever anisotropic plate of  $\lambda = 3/2$ , hole of  $a_1/a = 0.1$  and a concentrated mass ( $M = 0.1$ ) attached at  $(x_m, y_m) = (0.75, 0.5)$



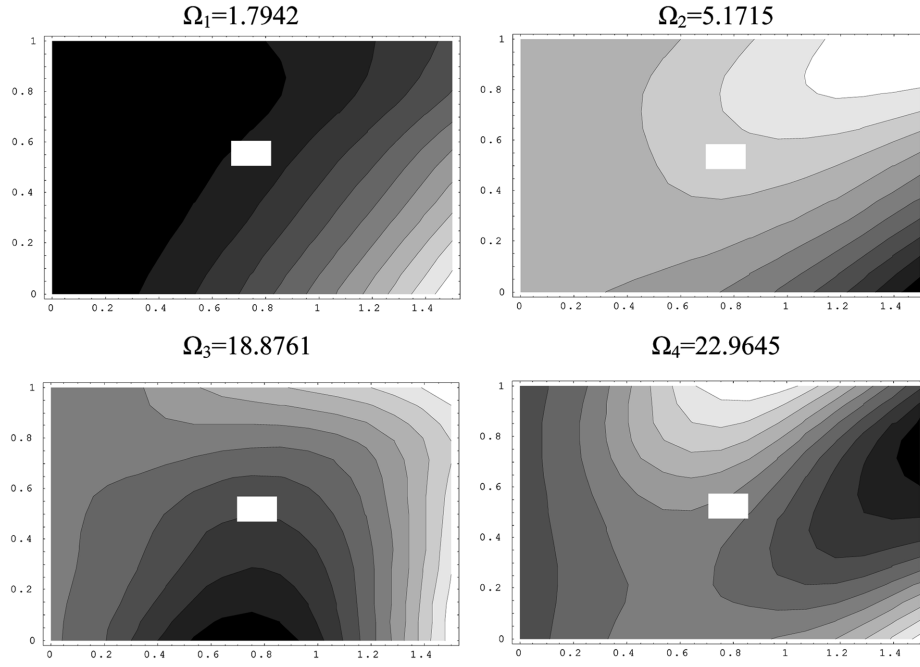


Fig. 6 First four modal shapes of vibration of a cantilever anisotropic plate of  $\lambda = 3/2$ , hole of  $a_1/a = 0.1$  and a concentrated mass ( $M = 0.5$ ) attached at  $(x_m = 0.75, y_m = 0.5)$

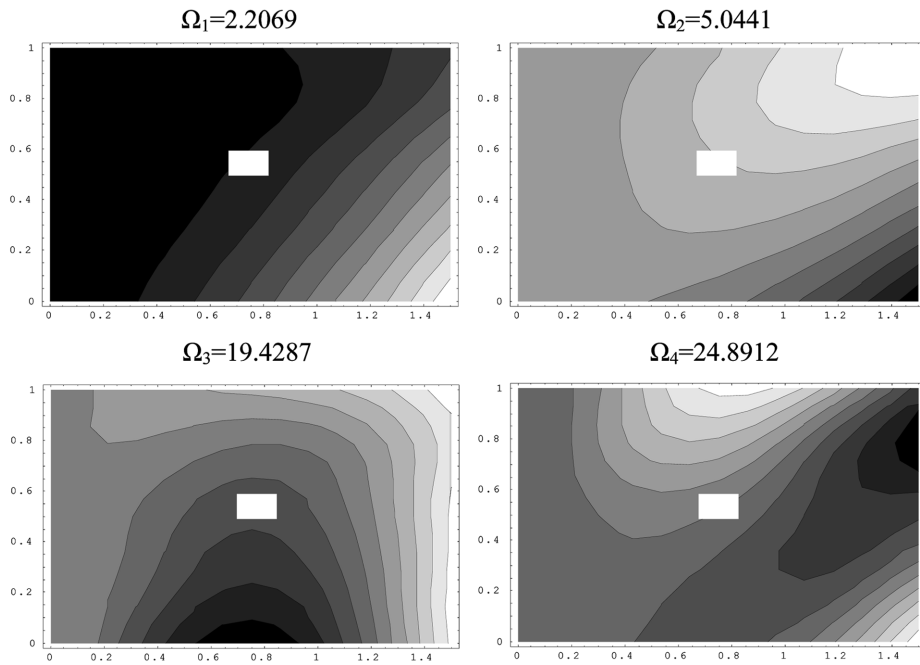


Fig. 7 First four modal shapes of vibration of a cantilever anisotropic plate of  $\lambda = 3/2$ , hole of  $a_1/a = 0.1$  and a concentrated mass ( $M = 0.1$ ) attached at  $(x_m = 0.75, y_m = 0.75)$

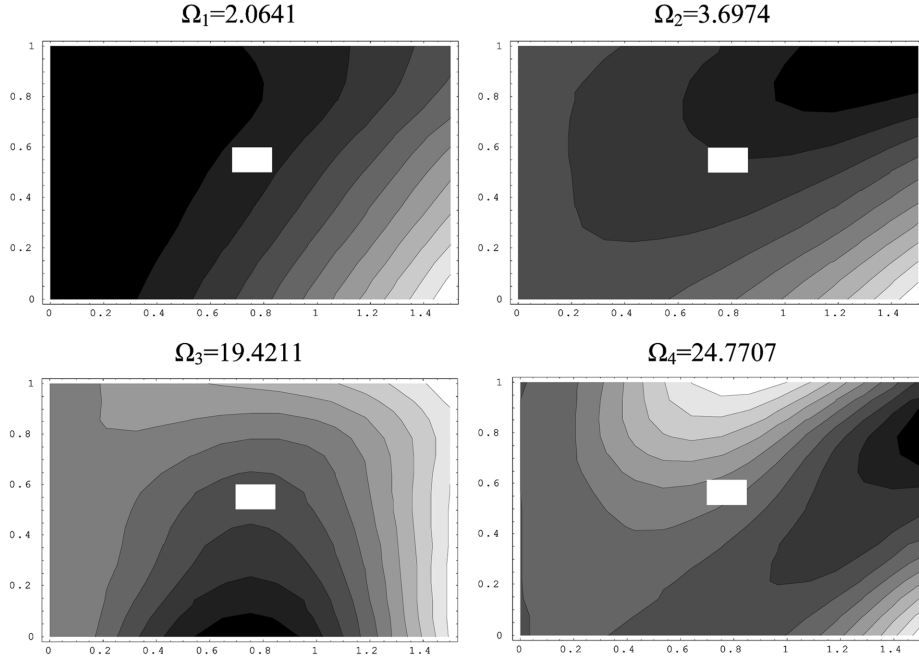


Fig. 8 First four modal shapes of vibration of a cantilever anisotropic plate of  $\lambda = 3/2$ , hole of  $a_1/a = 0.1$  and a concentrated mass ( $M = 0.5$ ) attached at  $(x_m = 0.75, y_m = 0.75)$

Table 4 Frequency coefficients values for a CCCS anisotropic, doubly connected plate with a concentrated mass attached at  $(x_m = 0.75, y_m = 0.75)$

| $\lambda = a/b$ | $a_1/a$ | $M = m/m_p$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|-----------------|---------|-------------|------------|------------|------------|------------|------------|
| 2/3             | 0       | 0           | 25.7724    | 35.2778    | 49.9407    | 65.525     | 70.1787    |
|                 |         | 0           | 24.8513    | 32.0440    | 45.7753    | 62.6881    | 65.6947    |
|                 |         | 0.1         | 23.1764    | 29.9772    | 43.9352    | 50.9795    | 62.8010    |
|                 |         | 0.5         | 15.9805    | 27.7657    | 39.1383    | 46.0149    | 62.7838    |
|                 | 0.1     | 1           | 12.0293    | 27.4321    | 39.9398    | 45.9639    | 62.7820    |
|                 |         | 0           | 26.8278    | 35.2345    | 52.3965    | 66.5107    | 72.0796    |
|                 |         | 0.1         | 24.3510    | 33.4069    | 51.6343    | 52.6782    | 71.3595    |
|                 |         | 0.5         | 16.2107    | 31.2550    | 43.2288    | 52.4258    | 70.8826    |
|                 |         | 1           | 12.1436    | 30.8584    | 42.1582    | 52.4228    | 70.8141    |
|                 | 0.2     | 0           | 30.5162    | 51.7154    | 72.4358    | 81.633     | 102.175    |
|                 |         | 0           | 28.0672    | 43.7729    | 68.6633    | 70.8784    | 100.769    |
|                 |         | 0.1         | 26.0828    | 42.3615    | 56.8117    | 68.7228    | 92.5347    |
|                 |         | 0.5         | 19.0186    | 37.4857    | 47.9728    | 68.7166    | 88.0908    |
| 1               | 0.1     | 1           | 14.6184    | 36.0325    | 47.1737    | 68.7159    | 87.4502    |
|                 |         | 0           | 31.5729    | 51.1067    | 73.8302    | 81.3427    | 101.600    |
|                 |         | 0.1         | 28.6852    | 49.1300    | 60.2238    | 79.7175    | 96.5330    |
|                 |         | 0.5         | 19.8401    | 42.6949    | 54.1184    | 79.3450    | 93.9024    |
|                 | 0.2     | 1           | 15.0443    | 41.1919    | 53.6819    | 79.2907    | 93.4821    |

Table 4 Continued

| $\lambda = a/b$ | $a_1/a$ | $M = m/m_p$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|-----------------|---------|-------------|------------|------------|------------|------------|------------|
| 3/2             | 0       | 0           | 42.1101    | 78.4318    | 99.5379    | 124.335    | 150.353    |
|                 |         | 0           | 35.6861    | 56.8301    | 93.4766    | 102.985    | 148.993    |
|                 |         | 0.1         | 32.5645    | 56.1075    | 78.3357    | 97.6760    | 131.609    |
|                 |         | 0.5         | 23.5599    | 53.0553    | 62.9248    | 96.9603    | 125.301    |
|                 | 0.1     | 1           | 18.2394    | 51.3574    | 61.0942    | 96.8765    | 124.417    |
|                 |         | 0           | 42.9540    | 76.2962    | 101.445    | 120.163    | 150.184    |
|                 |         | 0.1         | 38.6180    | 72.2201    | 82.2312    | 116.612    | 137.545    |
|                 |         | 0.5         | 26.5340    | 61.0356    | 79.8809    | 115.118    | 133.292    |
|                 |         | 1           | 20.1489    | 58.7002    | 79.4949    | 114.878    | 132.702    |

Table 5 Frequency coefficients values for a CCCS anisotropic, doubly connected plate with a concentrated mass attached at ( $x_m = 0.75$ ,  $y_m = 0.5$ )

| $\lambda = a/b$ | $a_1/a$ | $M = m/m_p$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|-----------------|---------|-------------|------------|------------|------------|------------|------------|
| 2/3             | 0       | 0           | 25.7724    | 35.2778    | 49.9407    | 65.525     | 70.1787    |
|                 |         | 0           | 24.8513    | 32.0440    | 45.7753    | 62.6881    | 65.6947    |
|                 |         | 0.1         | 22.1185    | 30.9576    | 43.9352    | 51.2905    | 64.6980    |
|                 |         | 0.5         | 14.6141    | 29.6753    | 39.1383    | 47.7237    | 64.6016    |
|                 | 0.1     | 1           | 10.9771    | 29.3880    | 38.2088    | 47.8887    | 64.5890    |
|                 |         | 0           | 26.8278    | 35.2345    | 52.3965    | 66.5107    | 72.0796    |
|                 |         | 0.1         | 24.0822    | 34.3506    | 46.2095    | 57.4780    | 69.9709    |
|                 |         | 0.5         | 15.9229    | 32.6360    | 39.5466    | 55.5072    | 69.6295    |
|                 |         | 1           | 11.9210    | 32.1790    | 38.7735    | 55.3244    | 69.5856    |
|                 | 0.2     | 0           | 30.5162    | 51.7154    | 72.4358    | 81.633     | 102.175    |
|                 |         | 0           | 28.0672    | 43.7729    | 68.6633    | 70.8784    | 100.769    |
|                 |         | 0.1         | 25.3999    | 40.4442    | 61.1848    | 69.4842    | 88.8229    |
|                 |         | 0.5         | 17.5011    | 36.1917    | 54.6164    | 69.3986    | 83.9458    |
| 1               | 0.1     | 1           | 13.2953    | 35.3238    | 53.5916    | 69.3885    | 83.2865    |
|                 |         | 0           | 31.5729    | 51.1067    | 73.8302    | 81.3427    | 101.600    |
|                 |         | 0.1         | 28.3847    | 47.2353    | 64.4369    | 81.0546    | 89.3167    |
|                 |         | 0.5         | 19.2178    | 41.5076    | 58.6972    | 80.8308    | 85.5898    |
|                 | 0.2     | 1           | 14.5130    | 40.3806    | 57.9792    | 80.7803    | 85.1403    |
|                 |         | 0           | 42.1101    | 78.4318    | 99.5379    | 124.335    | 150.353    |
|                 |         | 0           | 35.6861    | 56.8301    | 93.4766    | 102.985    | 148.993    |
|                 |         | 0.1         | 32.4520    | 50.8013    | 83.3812    | 101.480    | 136.766    |
|                 |         | 0.5         | 22.4512    | 44.8990    | 76.3090    | 101.026    | 129.671    |
|                 | 0.1     | 1           | 17.0762    | 43.8567    | 75.0657    | 100.954    | 128.459    |
|                 |         | 0           | 42.9540    | 76.2962    | 101.445    | 120.163    | 150.184    |
|                 |         | 0.1         | 38.6887    | 64.2182    | 99.8588    | 111.626    | 139.620    |
|                 |         | 0.5         | 26.0067    | 54.8097    | 97.8596    | 107.335    | 135.749    |
| 3/2             | 0.1     | 1           | 19.5929    | 53.3858    | 97.3649    | 106.754    | 135.185    |

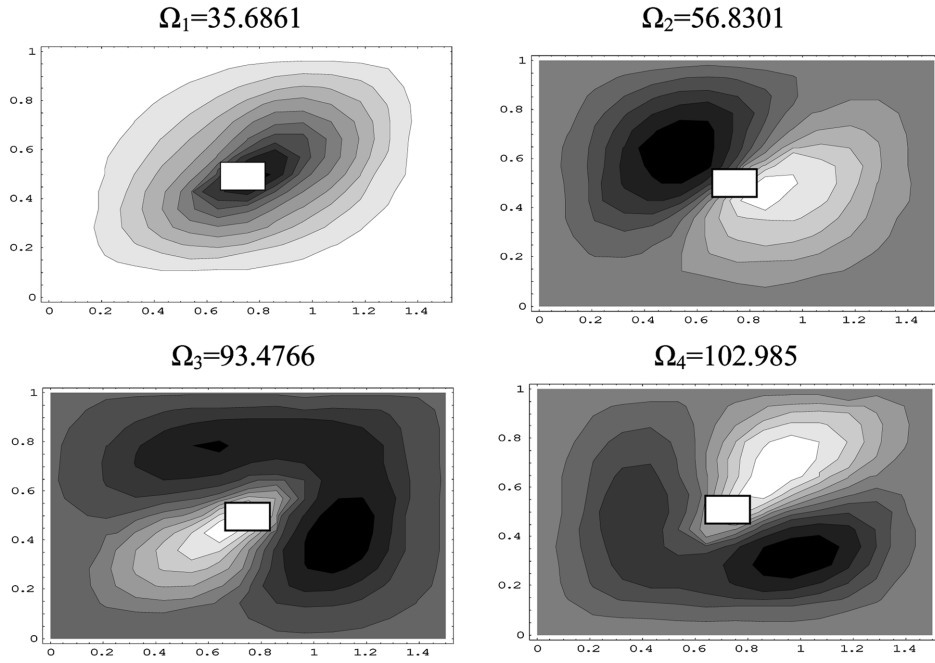


Fig. 9 First four modal shapes of vibration of a CCCS anisotropic plate of  $\lambda = 3/2$ , hole of  $a_1/a = 0.1$  and without attached mass

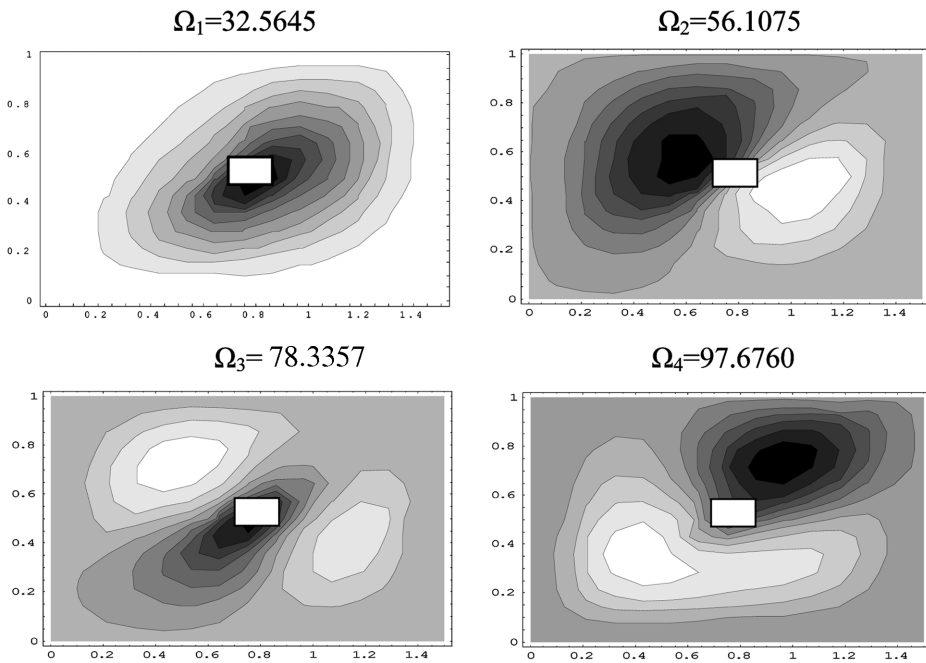


Fig. 10 First four modal shapes of vibration of a CCCS anisotropic plate of  $\lambda = 3/2$ , hole of  $a_1/a = 0.1$  and a concentrated mass ( $M = 0.1$ ) attached at  $(x_m = 0.75, y_m = 0.75)$

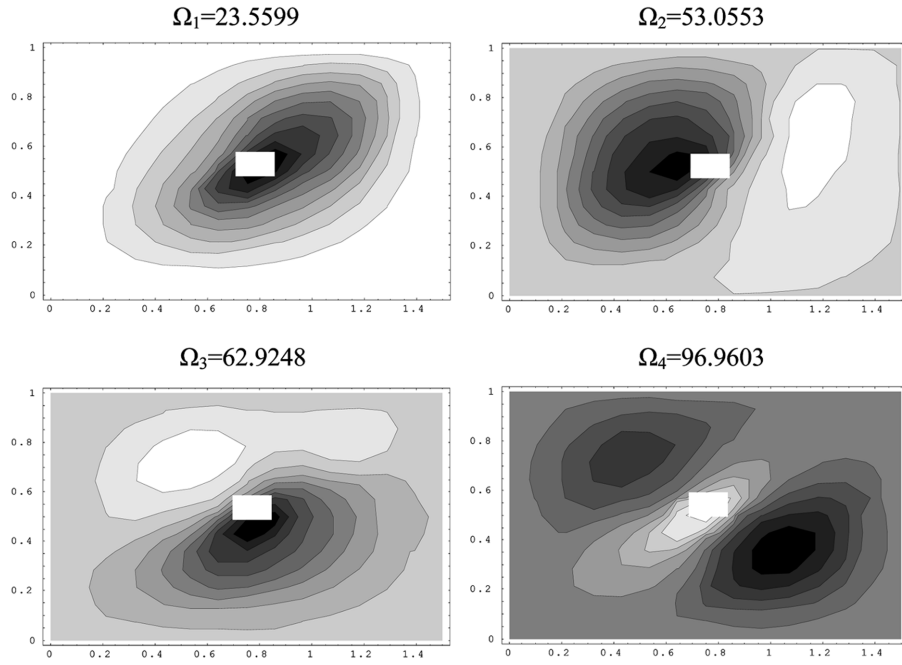


Fig. 11 First four modal shapes of vibration of a CCCS anisotropic plate of  $\lambda = 3/2$ , hole of  $a_1/a = 0.1$  and a concentrated mass ( $M = 0.5$ ) attached at  $(x_m = 0.75, y_m = 0.75)$

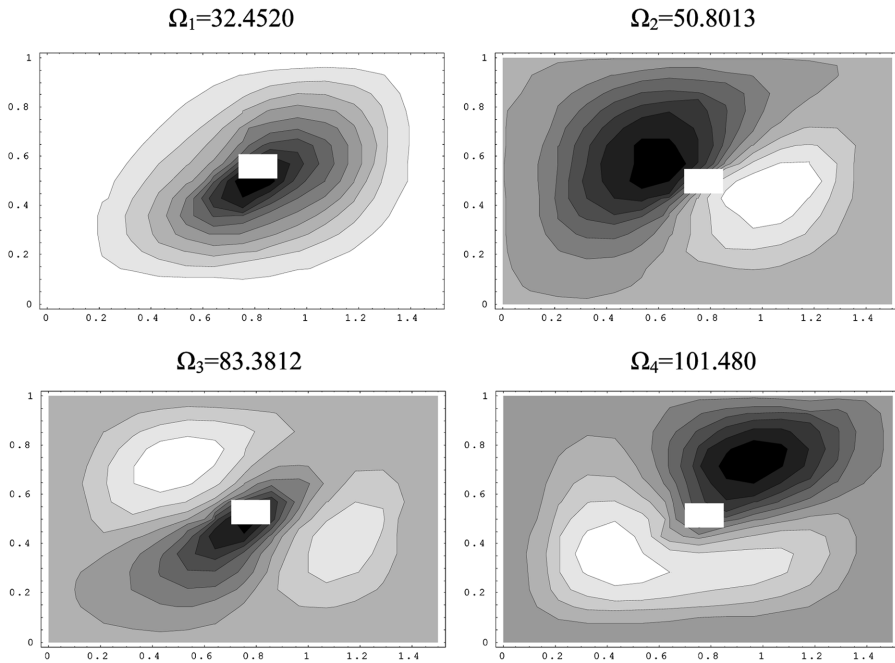


Fig. 12 First four modal shapes of vibration of a CCCS anisotropic plate of  $\lambda = 3/2$ , hole of  $a_1/a = 0.1$  and a concentrated mass ( $M = 0.1$ ) attached at  $(x_m = 0.75, y_m = 0.5)$

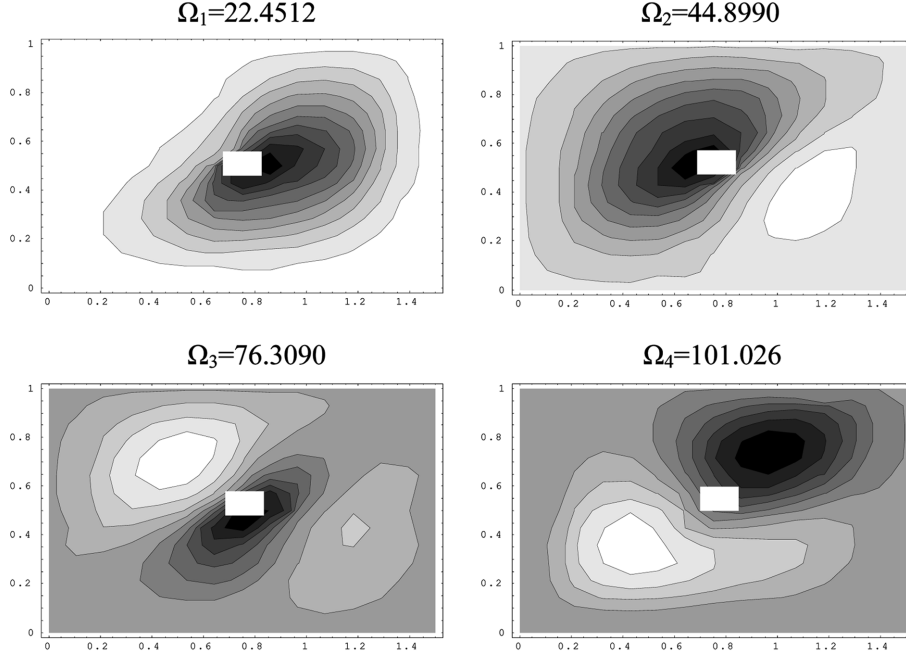


Fig. 13 First four modal shapes of vibration of a CCCS anisotropic plate of  $\lambda = 3/2$ , hole of  $a_1/a = 0.1$  and a concentrated mass ( $M = 0.5$ ) attached at  $x_m = 0.75$ ,  $y_m = 0.5$

Table 6 The non-dimensional frequency  $\omega a^2 [12(1 - \nu_{12}\nu_{21})\rho/E_1 h^3]^{1/2}$  of a single-layer ( $30^\circ$ ) simply supported square plate

| Mode             | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ | $\Omega_6$ |
|------------------|------------|------------|------------|------------|------------|------------|
| Cupial, 1997     | 11.233     | 20.282     | 32.995     | 34.866     | 47.133     | 48.283     |
| Present approach | 11.372     | 20.485     | 33.234     | 35.085     | 47.592     | 48.659     |

The analysis is done for the composite material properties:  $E_1 = 138$  [GPa],  $E_2 = 8.96$  [GPa],  $G_{12} = 7.1$  [GPa], and  $\nu_{12} = 0.30$ .

This case is chosen because, as Cupial (1997) himself and Whitney (1972) stated, the convergence of the Ritz method using beam functions may be slow for the free vibration frequencies of highly anisotropic plates with simply supported edges.

Present computing facilities and a convenient and straightforward algorithm (Felix *et al.* 2004) make possible to increase the number in terms in Eq. (3) without difficulty.

For this particular case  $M = N = 30$  is taken and the obtained values show good accuracy from an engineering viewpoint.

#### 4. Conclusions

As a general conclusion, one may say that the Ritz method, using beam function provides an accurate and convenient procedure to attack a difficult elastodynamics problem: the vibration of thin rectangular plates with structural and mechanical complexities like for the present situation where

anisotropic material characteristics, doubly connected domain and attached masses are present.

The obtained values are the outcome of an algorithm, relatively simple to implement, (Felix *et al.* 2004) which allows studying those situations where the plates possess additional complexities, with the only assistance of a P.C.

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