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Symmetrically loaded beam on a two-parameter tensionless foundation

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Abstract. Static response of an elastic beam on a two-parameter tensionless foundation is investigated by assuming that the beam is symmetrically subjected to a uniformly distributed load and concentrated edge loads. Governing equations of the problem are obtained and solved by pointing out that a concentrated edge foundation reaction in addition to a continuous foundation reaction along the beam axis in the case of complete contact and a discontinuity in the foundation reactions in the case of partial contact come into being as a direct result of the two-parameter foundation model. The numerical solution of the complete contact problem is straightforward. However, it is shown that the problem displays a highly non-linear character when the beam lifts off from the foundation. Numerical treatment of the governing equations is accomplished by adopting an iterative process to establish the contact length. Results are presented in figures to demonstrate the linear and non-linear behavior of the beam-foundation system for various values of the parameters of the problem comparatively.

Keywords: elastic beam; two-parameter foundation; lift-off.

1. Introduction

The response of elastic beams resting on an elastic foundation is a structural engineering problem of theoretical and practical interest. Since the soil exhibits a very complex behavior, its response is modeled in various ways in the analysis of structural elements on soil, as given, e.g., by Kerr (1964). The simplest representation has been given by Winkler assuming that the pressure exerted at a specific point by the soil is proportional to the displacement of the soil at the same point. A Winkler model can be considered as a system of closely spaced independent and unconnected linear springs of stiffness K. This single parameter model has been the object of some criticism because of the discontinuities in the displacement of the foundation surface between its loaded and unloaded parts and because of the assumption that the foundation reacts not only in compression but also in

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tension, which is questionable in many practical situations.

Because of the absence of interaction between adjacent springs in a Winkler foundation, the foundation surface outside the loading area does not contribute to the foundation response. The model fails to reproduce the characteristics of a continuous medium. This shortcoming is eliminated by using a two-parameter model such as the Pasternak or Wieghardt foundation models. These models each can be considered as a system of closely spaced linear springs of a stiffness K coupled to each other with elements which transmit a shear force proportional to the slope of the foundation surface having a shear stiffness G or a membrane having a tensile force G on the linear springs (Kerr 1964). The second parameter of the foundation represents shear behavior of the foundation. The two-parameter model produces two types of foundation reaction. One of them is a distributed reaction under the beam due to these two stiffnesses. The second one is a concentrated reaction at the point where a discontinuity of the slope of the foundation comes into being.

Numerous linear studies for beams resting on a two-parameter foundation can be found. Nogami and O'Neill (1985) performed a study dealing with beam on a two-parameter foundation and Ylinen and Mikkola (1967) investigated a beam on this kind of foundation by taking into account the effect of shear on the curvature of the beam. They pointed out that a concentrated foundation reaction comes into being at the points where the slope of the foundation displacement has a discontinuity as is the case at the end points of the beam and where distribution of the shearing force has a discontinuity, when the effect of shear force on the deformation is taken into account. Buckling of a beam on a two-parameter elastic foundation is studied by Smith (1969). Razaqpur and Shah (1991) derived a new finite element and solved a continuous beam having supports at its two ends and a free-free beam. In the later example two cases are considered. Firstly, the foundation does not extend beyond the ends of the beam by avoiding the free end conditions partially. Secondly, the foundation is assumed to be of infinite extent, where the shear force at the ends is assumed to be proportional with the displacement at the same point, but with a different spring constant. Gülkan and Alemdar (1999) obtained analytical expressions for the shape functions of a beam supported by a two-parameter foundation and compared them with their conventional counterparts. A similar study is carried out by Morfidis and Avamidis (2002) by deriving a generalized beam element on a two-parameter elastic foundation. Eisenberger and Bielak (1992) investigated finite beams supported by a infinite two-parameter elastic foundation.

These studies are extended to vibration and buckling problems of beams. Narasimba (1973) and Matsunaga (1999) investigated buckling of beam on a two-parameter elastic foundation. Various studies can be found dealing with vibration of Timoshenko beams given by Wang and Stephens (1977), Wang and Gagnon (1978), Filipich and Rosales (1988), Yokoyama (1991), De Rosa (1995) and El-Mously (1999). Further studies are carried out by Radeş (1970), Celep (1984), Franciosi and Masi (1993), Onu (2000), Filipich and Rosales (2002), Rao (2003) and Mallik, Chandra and Singh (2006). Recently, Çatal (2006) investigated free vibrations of a simply supported beam subjected to an axial force on an elastic foundation. In these studies the problems are analyzed by assuming that the foundation reacts in tension as well as in compression. Generally the numerical examples assume that either the beam has supports at its two ends or the foundation is defined under the beam only and does not extend beyond the end of the beam. In these ways generally two types of simplification are accomplished. Firstly, the governing equation of the problem is solved under the beam only and the differential equation for the foundation beyond the ends of the beam is omitted. Secondly, the complexity of boundary conditions of the beam ends is reduced (Onu 2000, Rao 2003). However, Kerr and Coffin (1991) pointed out that the omission of solution for base layer beyond the point of separation is a major error in the case of the completely free beam.

Numerous solutions of problems can be found involving beams on Winkler foundation by assuming that the foundation is attached to the beam. However, lift-off problems are much more plausible to avoid tension reactions from the foundation (Weisman 1970). When separation develops, analysis involving the tensionless Winkler foundation displays a non-linear character and gets complicated, due to the difficulty that the extent of the contact region is not known in advance. As a result of this difficulty, only a relatively limited number of studies dealing with the tensionless Winkler foundation are published. Various lift-off problems involving beams resting on a tensionless Winkler foundation are analyzed by Tsai and Westmann (1967), Weisman (1971), Lin and Adams (1987), Celep, Malaika and Abu-Hussein (1989), Celep (1990), Kerr and Soicher (1996) and Zhang and Murphy (2004).

A literature survey has revealed that only a limited number of studies have been done dealing the two-parameter tensionless foundation. The major difficulty is the definition of the contact region and the formulation of the corresponding boundary conditions. By using a variational approach, Kerr (1976) derived the proper boundary conditions at the ends of the beam and the matching conditions at the point where the contact and lift-off regions are separated. Furthermore, he pointed out that intuitive approaches in the formulation of the boundary conditions may lead to incorrect formulations by giving specific examples. Kerr and Coffin (1991) studied beams subjected to symmetrically distributed load and a concentrated load at the middle. In addition to the analyses of the problem, the authors pointed out two incorrectly formulated analyses. Çoşkun (2003) discussed the response of a beam of finite length subjected to a harmonic load, and Güler (2004) studied a circular plate under uniformly distributed load and central concentrated load on a two-parameter tensionless foundation. Nonlinear bending behavior of a Reissner-Mindlin free plate on this type of foundation is investigated by Shen and Yu (2004). They formulated the lift-off condition of a plate from a Pasternak-type foundation by stating that the lift-off develops when the displacement is negative. As it will be discussed below, this requirement holds only for Winkler type of foundations and it is not true for a two-parameter type foundation. Furthermore, Shen and Yu (2004) do not consider the concentrated foundation reaction due to the membrane stiffness of the foundation as well. Celep and Demir (2005) dealt with a circular rigid ring beam on a tensionless foundation surface.

The aim of the present paper is to study the behavior of an elastic beam on the two-parameter tensionless foundation under uniformly distributed load and two edge loads. The study is carried out by considering the symmetry of the problem. The analysis is done in two separate parts; first assuming that complete contact exists between the beam and the foundation, and second assuming that a partial contact develops and the beam lifts off the foundation. Special attention is paid to the boundary and matching conditions and global force equilibrium. Numerical results are presented to verify the solution procedure. It is noted that iterations are required in the numerical solutions in case of partial contact, because the lift-off and the contact regions are not known in advance. They depend on the parameters of the problem, and the governing equation of the problem is highly non-linear. The paper gives various parametric numerical results and discuses them comparatively. The analysis presented is equally valid whether the foundation reacts in compression or in tension.

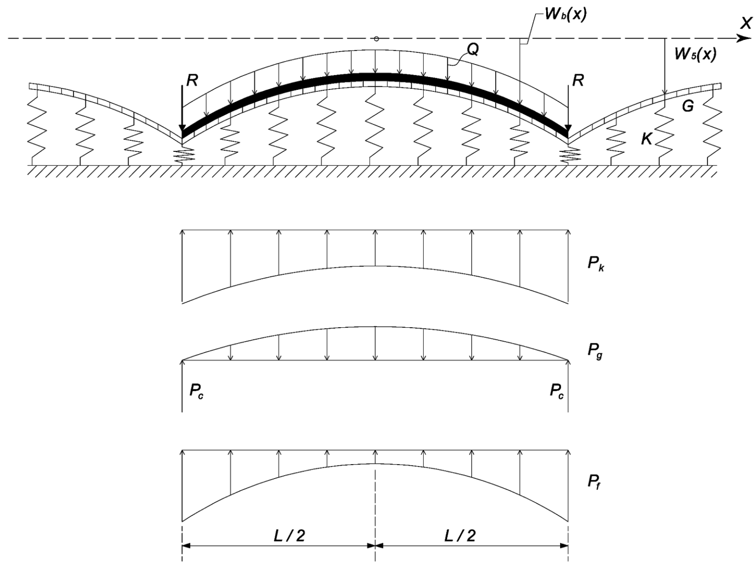

Fig. 1 Elastic beam on a two-parameter foundation having complete contact

2. Statement of the problem

Consider a completely free elastic beam of length L and bending stiffness EI, on a two-parameter elastic foundation having stiffnesses K and G, as Fig. 1 shows. The figure can be seen as a combined footing which supports two columns. The beam is assumed to be subjected to two concentrated loads R applied at its two ends and a uniformly distributed load Q. Since the geometry and the loading of the problem is symmetric with respect to the middle axis, all the parameters of the problem reflect the same symmetry, including the vertical displacement of the beam and that of the elastic foundation. For the two-parameter foundation, it is assumed that foundation reaction $P_f(X)$ and the foundation displacement $W_s(X)$ are related to each other according to

$$P_f(X) = P_k(X) + P_g(X) = KW_s(X) - GW_s''(X) \tag{1}$$

where the prime denotes differentiation with respect to X. As the inspection of Eq. (1) reveals, the two-parameter foundation can be seen as closely spaced linear springs of stiffness K which are connected to each other by a membrane having a surface tension G. Furthermore, Eq. (1) displays that the foundation reaction consists of two parts. The first one is related to the displacement directly, i.e., vertical reactions of the linear springs. The second part is proportional to the second derivative of the surface displacement, i.e., the vertical component of the surface tension.

2.1 Complete contact

Assuming the complete contact between the beam and the foundation, the displacement of the foundation is expressed in two regions. The first one is the free surface of the foundation, where no pressure is exerted on the foundation and its displacement $W_s(X)$ is controlled by the equation

$$GW_s'' - KW_s = 0 \qquad \text{for} \qquad X \geq L/2 \tag{2}$$

In formation of the governing equations the symmetry of the problem with respect to the middle axis of the beam is used. The second region corresponds to the displacement of the foundation under the beam and is governed by the equation assuming the validity of beam bending theory

$$EIW_b^{iv} - GW_b'' + KW_b = Q \qquad \text{for} \qquad 0 \leq X \leq L/2 \tag{3}$$

where the foundation displacement and the beam displacement are equal, i.e., $W_s(X) = W_b(X)$. By introducing the following non-dimensional parameters

$$x = X/L \qquad w_s(x) = W_s(X)/L \qquad w_b(x) = W_b(X)/L$$
$$g = GL^2/EI \qquad k = KL^4/EI \qquad r = RL^2/EI \qquad q = QL^3/EI \tag{4}$$

the two governing equations of the problem (2, 3) can be expressed as

$$w_s'' - \lambda^2 w_s = 0 \qquad w_b^{iv} - 4\alpha_o\beta_o^2 w_b'' + 4\beta_o^4 w_b = q \tag{5a,b}$$

where

$$4\beta_o^4 = k \qquad 4\alpha_o\beta_o^2 = g \qquad \lambda^2 = \beta_o^2/\alpha_o = k/g$$

The solution of the first Eq. (5a) can be expressed as

$$w_s(x) = A_1 e^{-\lambda x} \tag{6}$$

where the condition $w_s(x \to \infty) = 0$ is used to eliminate the second integration constant. The solution of the second Eq. (5b) assumes different forms depending on the value of α_o.

(i) when $\alpha_o < 1$, then

$$w_b(x) = B_1 \sin\beta x \sinh\alpha x + B_2 \cos\beta x \cosh\alpha x + q/(4\beta_o^4) \tag{7a}$$

where

$$\alpha = \beta_o\sqrt{1 + \alpha_o} \qquad \beta = \beta_o\sqrt{1 - \alpha_o}$$

(ii) when $\alpha_o > 1$, then

$$w_b(x) = B_1 \cosh\gamma x + B_2 \cosh\delta x + q/(4\beta_o^4) \tag{7b}$$

where

$$\gamma = \beta_o\sqrt{2(\alpha_o + \sqrt{\alpha_o^2 - 1})} \qquad \delta = \beta_o\sqrt{2(\alpha_o - \sqrt{\alpha_o^2 - 1})}$$

In formation of these solutions, the symmetry condition of the problem with respect to the middle axis of the beam is used. It is possible to express a separate solution $w_b(x)$ for $\alpha_o = 1$ as well; however, sufficiently accurate numerical results can be obtained by just using one the above solutions for $\alpha_o = 1 \pm 0$. For this reason the solution for $\alpha_o = 1$ is not presented here. The solutions (6) and (7) have three integration constants A_1, B_1 and B_2 to be determined from the boundary conditions. Since the symmetry of the problem is already used in the derivations of these solutions, the two remaining boundary conditions are the continuity of the deflections and one of the free end conditions of the beam,

$$w_s(1/2) = w_b(1/2) \qquad w_b''(1/2) = 0 \tag{8}$$

As Fig. 1 shows, the slope of the deflection of the foundation displays discontinuity at the ends of the beam, although the displacement itself is continuous. A concentrated foundation reaction load comes into being due to this discontinuity

$$P_c = G[W_s'(L-0) - W_s'(L+0)] \tag{9}$$

or in a non-dimensional form

$$p_c = P_c L^2/EI = 4\alpha_o\beta_o^2[w_b'(1/2) - w_s'(1/2)] \tag{10}$$

The third boundary condition is the force equilibrium at the end of the beam between the foundation concentrated reaction, the external edge load and the shearing force of the beam

$$4\alpha_o\beta_o^2[w_b'(1/2) - w_s'(1/2)] = w_b'''(1/2) + r \tag{11}$$

The distributed foundation reaction $p_f(x)$ given in Eq. (1) can be expressed as two non-dimensional parts as well,

$$\begin{gathered} p_f(x) = P_f(X)L^3/EI = p_k(x) + p_g(x) \\ p_k = P_k L^3/EI = 4\beta_o^4 w_b \qquad p_g = P_g L^3/EI = -4\alpha_o\beta_o^4 w_b'' \end{gathered} \tag{12}$$

In the present formulation, non-dimensionlization is carried out for the parameters of the problem, so that a wide range of numerical values of the parameters can be covered in the numerical evaluation. Having applied the boundary conditions (8) and (11), the following system of algebraic equations are obtained

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ A_1 \end{bmatrix} = \begin{bmatrix} -q/(4\beta_o^4) \\ 0 \\ -r \end{bmatrix} \tag{13}$$

where

(i) when $\alpha_o < 1$, than

$$a_{11} = \sin\beta/2\sinh\alpha/2 \qquad a_{12} = \cos\beta/2\cosh\alpha/2 \qquad a_{13} = -e^{-\lambda/2}$$

$$a_{21} = (\alpha^2 - \beta^2)\sin\beta/2\sinh\alpha/2 + 2\alpha\beta\cos\beta/2\cosh\alpha/2$$

$$a_{22} = (\alpha^2 + \beta^2)\cos\beta/2\cosh\alpha/2 - 2\alpha\beta\sin\beta/2\sinh\alpha/2 \qquad a_{23} = 0$$

$$a_{31} = \beta(3\alpha^2 - \beta^2 - 4\alpha_o\beta_o^2)\cos\beta/2\sinh\alpha/2 + \alpha(\alpha^2 - 3\beta^2 - 4\alpha_o\beta_o^2)\sin\beta/2\cosh\alpha/2$$

$$a_{32} = \beta(\beta^2 - 3\alpha^2 + 4\alpha_o\beta_o^2)\sin\beta/2\cosh\alpha/2 + \alpha(\alpha^2 - 3\beta^2 - 4\alpha_o\beta_o^2)\cos\beta/2\sinh\alpha/2$$

$$a_{33} = -4\alpha_o\beta_o^2\lambda e^{-\lambda/2}$$

(ii) when $\alpha_o > 1$, than

$$a_{11} = \cosh\gamma/2 \qquad a_{12} = \cosh\delta/2 \qquad a_{13} = -e^{-\lambda/2}$$

$$a_{21} = \gamma^2\cos\gamma/2 \qquad a_{22} = \delta^2\cosh\delta/2 \qquad a_{23} = 0$$

$$a_{31} = \gamma(\gamma^2 - 4\alpha_o\beta_o^2)\sinh\gamma/2 \qquad a_{32} = \delta(\delta^2 - 4\alpha_o\beta_o^2)\sinh\delta/2$$

$$a_{33} = -4\alpha_o\beta_o^2\lambda e^{-\lambda/2}$$

By considering the entire beam the global force equilibrium can be written as

$$2R + QL = 2P_c + P_{fs} \tag{14}$$

where

$$P_{fs} = \int_{-L/2}^{L/2} P_f dX = P_{ks} + P_{gs} = K\int_{-L/2}^{L/2} W_s dX - G\int_{-L/2}^{L/2} W_s'' dX$$

or in the non-dimensional form

$$2r + q = 2p_c + p_{fs} = 2p_c + p_{ks} + p_{gs} \tag{15}$$

where

$$p_{fs} = \int_{-1/2}^{1/2} p_f dx = p_{ks} + p_{gs} = 4\beta_o^4\int_{-1/2}^{1/2} w_s dx - 4\alpha_o\beta_o^2\int_{-1/2}^{1/2} w_s'' dx$$

In the global equilibrium Eqs. (14), (15) p_c denotes the concentrated foundation reaction at the ends of the beam; P_{ks} and p_{ks}, the sum of the foundation reaction exerted by the linear springs and its non-dimensional form and P_{gs} and p_{gs}, the sum of the foundation reaction applied by the vertical component of the surface tension and its non-dimensional form. As seen in Eq. (15), the global equilibrium of forces can be satisfied only by including the edge force p_c. It is strictly not admissible to neglect it, as it is done by various authors (Shen and Yu 2004, Onu 2000). As it will be discussed below, the edge force will not develop when the end of the beam lifts off the foundation. This fact indicates that the definition of the lift-off condition is of prime importance for satisfaction of global force equilibrium.

The foundation pressure $p_k(x)$ is proportional to the displacement and displays a positive variation under the beam, when the beam penetrates into the foundation. It has a maximum value $p_k(x = \pm 1/2)$ at the two ends of the beam and a minimum value $p_k(x = 0)$ under the middle of the beam. However, the foundation pressure $p_g(x)$ is proportional to the second derivative of the displacement and displays a negative variation. It is $p_g(x = \pm 1/2) = 0$ at the two ends of the beam due to the free boundary condition of the beam (8). Thus, the total foundation reaction $p_f(x)$ gets its smallest value under the middle of the beam

$$p_f(x = 0) = 4\beta_o^4 w_s(x = 0) - 4\alpha_o \beta_o^2 w_s''(x = 0) \tag{16}$$

Numerical solution of the problem can be carried out in a straightforward manner by using the equations above, where complete contact is assumed between the beam and the foundation surface. This case is possible only when the foundation pressure is positive $p_k(x) \geq 0$ or for the present problem $p_k(x = 0) \geq 0$. However, beyond the case $p_k(x = 0) \geq 0$, complete contact is lost, and a partial lift-off appears in the middle beam.

2.2 Partial contact

Fig. 2 shows the same beam on the two-parameter foundation. The beam has a lift-off region for $-A \leq X \leq A$ and a contact region for $A \leq X \leq L/2$. The displacement of the beam and the foundation are denoted as $W_{b1}(-A \leq X \leq A)$, $W_{b2}(A \leq X \leq L/2)$ and $W_{s1}(-A \leq X \leq A)$, $W_s(A \leq X \leq L/2) = W_{b2}(A \leq X \leq L/2)$ and $W_{s2}(X \geq L/2)$, respectively. The corresponding governing equations of these displacements can be expressed as follows

$$\begin{aligned} w_{b1}^{iv} &= q, \qquad w_{b2}^{iv} - 4\alpha_o\beta_o^2 w_{b2}'' + 4\beta_o^4 w_{b2} = q \\ w_{s1}'' - \lambda^2 w_{s1} &= 0 \qquad w_{s2}'' - \lambda^2 w_{s2} = 0 \end{aligned} \tag{17}$$

where the above defined non-dimensional parameters are used

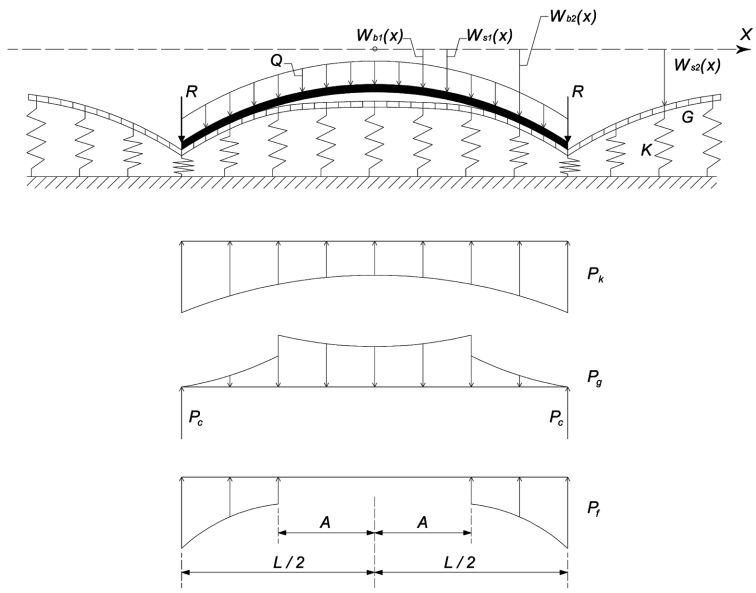

Fig. 2 Elastic beam on a two-parameter foundation having partial contact

$$w_{s1}(x) = W_{s1}(X)/L \qquad w_{s2}(x) = W_{s2}(X)/L$$
$$w_{b1}(x) = W_{b1}(X)/L \qquad w_{b2}(x) = W_{b2}(X)/L$$

The following solutions are obtained for the displacement functions

$$w_{s1}(x) = D_1 \cosh\lambda x \qquad w_{s2}(x) = A_1 e^{-\lambda x} \qquad w_{b1}(x) = qx^4/24 + C_1x^2 + C_2 \tag{18}$$

(i) when $\alpha_o < 1$, then

$$w_{b2}(x) = B_1 \sin\beta x \sinh\alpha x + B_2 \sin\beta x \cosh\alpha x + B_3 \cos\beta x \sinh\alpha x + B_4 \cos\beta x \cosh\alpha x + q/(4\beta_o^4) \tag{19a}$$

(ii) when $\alpha_o > 1$, then

$$w_b(x) = B_1 \cosh\gamma x + B_2 \cosh\delta x + B_3 \sinh\gamma x + B_4 \sinh\delta x + q/(4\beta_o^4) \tag{19b}$$

where the condition $w_{s2}(x \to \infty) = 0$ and the symmetry property of the problem are used. The most critical step in the analysis of contact problems dealing with a two-parameter tensionless foundation is the statement of the boundary conditions. The solutions (18) and (19) have eight integration constants $A_1, B_1, B_2, B_3, B_4, C_1, C_2$ and D_1 to be determined on the basis of the boundary conditions. The ninth unknown is the coordinate of the point which separates the contact and the lift-off regions, i.e., $A = aL$. These nine constants require the nine boundary or continuity conditions

$$w_{b2}(1/2) = w_{s2}(1/2) \quad w_{b1}(a) = w_{b2}(a) \quad w'_{b1}(a) = w'_{b2}(a) \quad w'_{b2}(a) = w'_{s1}(a)$$
$$w''_{b1}(a) = w''_{b2}(a) \quad w''_{b1}(1/2) = 0 \quad w'''_{b1}(a) = w'''_{b2}(a)$$
$$4\alpha_o\beta_o^2[w'_{b2}(1/2) - w'_{s2}(1/2)] = w'''_{b2}(1/2) + r \qquad w_{b2}(a) = w_{s1}(a) \tag{20a~i}$$

The above boundary conditions require continuity of the displacements and their slope at the separation point ($x = a$) (Eqs. (20b,c,d,i)) and that of the shearing force and bending moment of the beam at the separation section ($x = a$) (Eqs. (20e,g)) and continuity of the displacements at the end of the beam ($x = 1/2$) (Eq. (20a)) and the moment-free end condition (Eq. (20f)). Force equilibrium at the end of the beam between the foundation concentrated force, the edge load and the shearing force of the beam is another condition to be satisfied (Eq. (20h)). Since the lift-off region develops at the middle of the beam, not at the end, a concentrated reaction exists at the end of the beam as in the complete contact case. This once more indicates the importance of the definition of the contact condition. At the separation point the slopes of the beam and the foundation are equal. It is not correct to require that the distributed foundation reaction should be zero at the separation point. The nine boundary conditions (20) lead to the following system of nine linear algebraic equations for the eight unknowns $B_1, B_2, B_3, B_4, C_1, C_2, A_1$ and D_1

$$[a_{ij}][B_1 \;\; B_2 \;\; B_3 \;\; B_4 \;\; C_1 \;\; C_2 \;\; A_1 \;\; D_1]^T = [c_i] \tag{21}$$

where the non-zero terms of the matrix a_{ij} and the vector c_i are

(i) when $\alpha_o < 1$, then

$$a_{11} = \sin\beta/2\sinh\alpha/2 \qquad a_{12} = \sin\beta/2\cosh\alpha/2 \qquad a_{13} = \cos\beta/2\sinh\alpha/2$$

$$a_{14} = \cos\beta/2\cosh\alpha/2 \qquad a_{17} = -e^{-\lambda/2} \qquad c_1 = -q/(4\beta_o^4)$$

$$a_{21} = \sin\beta a\sinh\alpha a \qquad a_{22} = \sin\beta a\cosh\alpha a \qquad a_{23} = \cos\beta a\sinh\alpha a$$

$$a_{24} = \cos\beta a\cosh\alpha a \qquad a_{25} = -a^2 \qquad a_{26} = -1$$

$$c_2 = q(a^4/6 - 1/\beta_o^4)/4$$

$$a_{31} = \beta\cos\beta a\sinh\alpha a + \alpha\sin\beta a\cosh\alpha a$$

$$a_{32} = \beta\cos\beta a\cosh\alpha a + \alpha\sin\beta a\sinh\alpha a$$

$$a_{33} = -\beta\sin\beta a\sinh\alpha a + \alpha\cos\beta a\cosh\alpha a$$

$$a_{34} = -\beta\sin\beta a\cosh\alpha a + \alpha\cos\beta a\sinh\alpha a \qquad a_{35} = -2a \qquad c_3 = qa^3/6$$

$$a_{45} = -2a \qquad a_{48} = \delta\sin\delta a \qquad c_4 = qa^3/6$$

$$a_{51} = (\alpha^2 - \beta^2)\sin\beta a\sinh\alpha a + 2\alpha\beta\cos\beta a\cosh\alpha a$$

$$a_{52} = (\alpha^2 - \beta^2)\sin\beta a\cosh\alpha a + 2\alpha\beta\cos\beta a\sinh\alpha a$$

$$a_{53} = (\alpha^2 - \beta^2)\cos\beta a\sinh\alpha a - 2\alpha\beta\sin\beta a\cosh\alpha a$$

$$a_{54} = (\alpha^2 - \beta^2)\cos\beta a\cosh\alpha a - 2\alpha\beta\sin\beta a\sinh\alpha a \qquad a_{55} = -2 \qquad c_5 = qa^2/2$$

$$a_{61} = (\alpha^2 - \beta^2)\sin\beta/2\sinh\alpha/2 + 2\alpha\beta\cos\beta/2\cosh\alpha/2$$

$$a_{62} = (\alpha^2 - \beta^2)\sin\beta/2\cosh\alpha/2 + 2\alpha\beta\cos\beta/2\sinh\alpha/2$$

$$a_{63} = (\alpha^2 - \beta^2)\cos\beta/2\sinh\alpha/2 - 2\alpha\beta\sin\beta/2\cosh\alpha/2$$

$$a_{64} = (\alpha^2 - \beta^2)\cos\beta/2\cosh\alpha/2 - 2\alpha\beta\sin\beta/2\sinh\alpha/2$$

$$a_{71} = \beta(3\alpha^2 - \beta^2)\cos\beta a\sinh\alpha a + \alpha(\alpha^2 - 3\beta^2)\sin\beta a\cosh\alpha a$$

$$a_{72} = \beta(3\alpha^2 - \beta^2)\cos\beta a\cosh\alpha a + \alpha(\alpha^2 - 3\beta^2)\sin\beta a\sinh\alpha a$$

$$a_{73} = -\beta(3\alpha^2 - \beta^2)\sin\beta a\sinh\alpha a + \alpha(\alpha^2 - 3\beta^2)\cos\beta a\cosh\alpha a$$

$$a_{74} = -\beta(3\alpha^2 - \beta^2)\sin\beta a\cosh\alpha a + \alpha(\alpha^2 - 3\beta^2)\cos\beta a\sinh\alpha a \qquad c_7 = qa$$

$$a_{81} = (4\alpha_o\beta_o^2 + \beta^2 - 3\alpha^2)\beta\cos\beta/2\sinh\alpha/2 + (4\alpha_o\beta_o^2 - \alpha^2 + 3\beta^2)\alpha\sin\beta/2\cosh\alpha/2)$$

$$a_{82} = (4\alpha_o\beta_o^2 + \beta^2 - 3\alpha^2)\beta\cos\beta/2\cosh\alpha/2 + (4\alpha_o\beta_o^2 - \alpha^2 + 3\beta^2)\alpha\sin\beta/2\sinh\alpha/2)$$

$$a_{83} = (-4\alpha_o\beta_o^2 - \beta^2 + 3\alpha^2)\beta\sin\beta/2\sinh\alpha/2 + (4\alpha_o\beta_o^2 - \alpha^2 + 3\beta^2)\alpha\cos\beta/2\cosh\alpha/2)$$

$$a_{84} = (-4\alpha_o\beta_o^2 - \beta^2 + 3\alpha^2)\beta\sin\beta/2\cosh\alpha/2 + (4\alpha_o\beta_o^2 - \alpha^2 + 3\beta^2)\alpha\cos\beta/2\sinh\alpha/2)$$

$$a_{87} = 4\alpha_o\beta_o^2\lambda e^{-\lambda/2} \qquad c_8 = r \qquad a_{95} = -a^2 \qquad a_{96} = -1 \qquad a_{97} = e^{-\lambda a} \qquad c_9 = qa^4/24$$

(ii) when $\alpha_o > 1$, then

$$a_{11} = \cosh\gamma/2 \qquad a_{12} = \cosh\delta/2 \qquad a_{13} = \sinh\gamma/2$$

$$a_{14} = \sinh\delta/2 \qquad a_{17} = -e^{-\lambda/2} \qquad c_1 = -q/(4\beta_o^4)$$

$$a_{21} = \cosh\gamma a \qquad a_{22} = \cosh a\delta \qquad a_{23} = \sinh\gamma a$$

$$a_{24} = \sinh\delta a \qquad a_{25} = -a^2 \qquad a_{26} = -1 \qquad c_2 = q(a^4/6 - 1/\beta_o^4)/4$$

$$a_{31} = \gamma\sinh\gamma a \qquad a_{32} = \delta\sinh\delta a \qquad a_{33} = \gamma\cosh\gamma a \qquad a_{34} = \delta\cosh\delta a$$

$$a_{35} = -2a \qquad c_3 = qa^3/6$$

$$a_{45} = -2a \qquad a_{48} = \lambda\sin\lambda a \qquad c_4 = qa^3/6$$

$$a_{51} = \gamma^2\cosh\gamma a \qquad a_{52} = \delta^2\cosh\delta a \qquad a_{53} = \gamma^2\sinh\gamma a \qquad a_{54} = \delta^2\sinh\delta a$$

$$a_{55} = -2 \qquad c_5 = qa^2/2$$

$$a_{61} = \gamma^2\cosh\gamma/2 \qquad a_{62} = \delta^2\cosh\delta/2 \qquad a_{63} = \gamma^2\sinh\gamma/2 \qquad a_{64} = \delta^2\sinh\delta/2$$

$$a_{71} = \gamma^3\sinh\gamma a \qquad a_{72} = \delta^3\sinh\delta a \qquad a_{73} = \gamma^3\cosh\gamma a \qquad a_{74} = \delta^3\cosh\delta a$$

$$c_7 = qa$$

$$a_{81} = (4\alpha_o\beta_o^2 - \gamma^2)\gamma\sinh\gamma/2 \qquad a_{82} = (4\alpha_o\beta_o^2 - \delta^2)\delta\sinh\delta/2$$

$$a_{83} = (4\alpha_o\beta_o^2 - \gamma^2)\gamma\cosh\gamma/2 \qquad a_{84} = (4\alpha_o\beta_o^2 - \delta^2)\delta\cosh\delta/2 \qquad c_8 = r$$

$$a_{95} = -a^2 \qquad a_{96} = -1 \qquad a_{97} = e^{-\lambda a} \qquad c_9 = qa^4/24$$

In case of partial contact, the sum of the foundation reactions to be used in the global force equilibrium (15) yields

$$p_{fs} = 2\int_a^{1/2} p_f dx = p_{ks} + p_{gs} = 8\beta_o^4\int_a^{1/2} w_s dx - 8\alpha_o\beta_o^2\int_a^{1/2} w_s'' dx \tag{22}$$

where the integration will be carried along the contact region $a \le x \le 1/2$.

The governing equations of the problem (13) where the complete contact is maintained are linear and can be solved numerically in a straightforward manner without requiring any iteration. However, in case of the partial contact, it is worth noting the following points:

a. The governing Eq. (21) of this case consist of nine linear equations having eight unknowns $B_1, B_2, B_3, B_4, C_1, C_2, A_1$ and D_1. In fact there is one additional unknown to be determined, i.e., the extent of the contact region a which appears in the coefficients of the matrix in a highly non-linear manner by to virtue of unilateral constraints. It requires an iterative process for the numerical solution.

b. A concentrated foundation reaction force comes into being at the end of the beam due to the discontinuity in the slope of the foundation displacement at the end of the beam ($x = \pm 1/2$) independent of the lift-off. However at the separation points of the beam and the foundation surface ($x = \pm a$), there is no concentrated reaction force due to the continuity of the slope of the foundation surface as stated in Eq. (20d).

c. The distributed foundation reaction

$$p_f(x) = p_k(x) + p_g(x) = 4\beta_o^4 w_s(x) - 4\alpha_o\beta_o^2 w_s''(x) \tag{23}$$

consists of two parts as seen, and it vanishes obviously for the free surface of the foundation

$-a < x < +a$ and $|x| > 1/2$ due to Eq. (5a). Since the foundation displacement is continuous at the separation point, so is the first part of the foundation reaction p_k in Eq. (23). However, the second derivative of the displacement of the beam is continuous at the separation point Eq. (20e), whereas the second derivative of the displacement of the foundation is not continuous which causes a discontinuity of the second part of the foundation reaction p_g and consequently in the total foundation reaction p_f as well. This result will be discussed further in the numerical solutions.

3. Numerical solutions and discussion

Various numerical examples are solved and their results are presented in figures in order to illustrate the effects of the parameters of the problem on the behavior of the beam. Fig. 3 shows

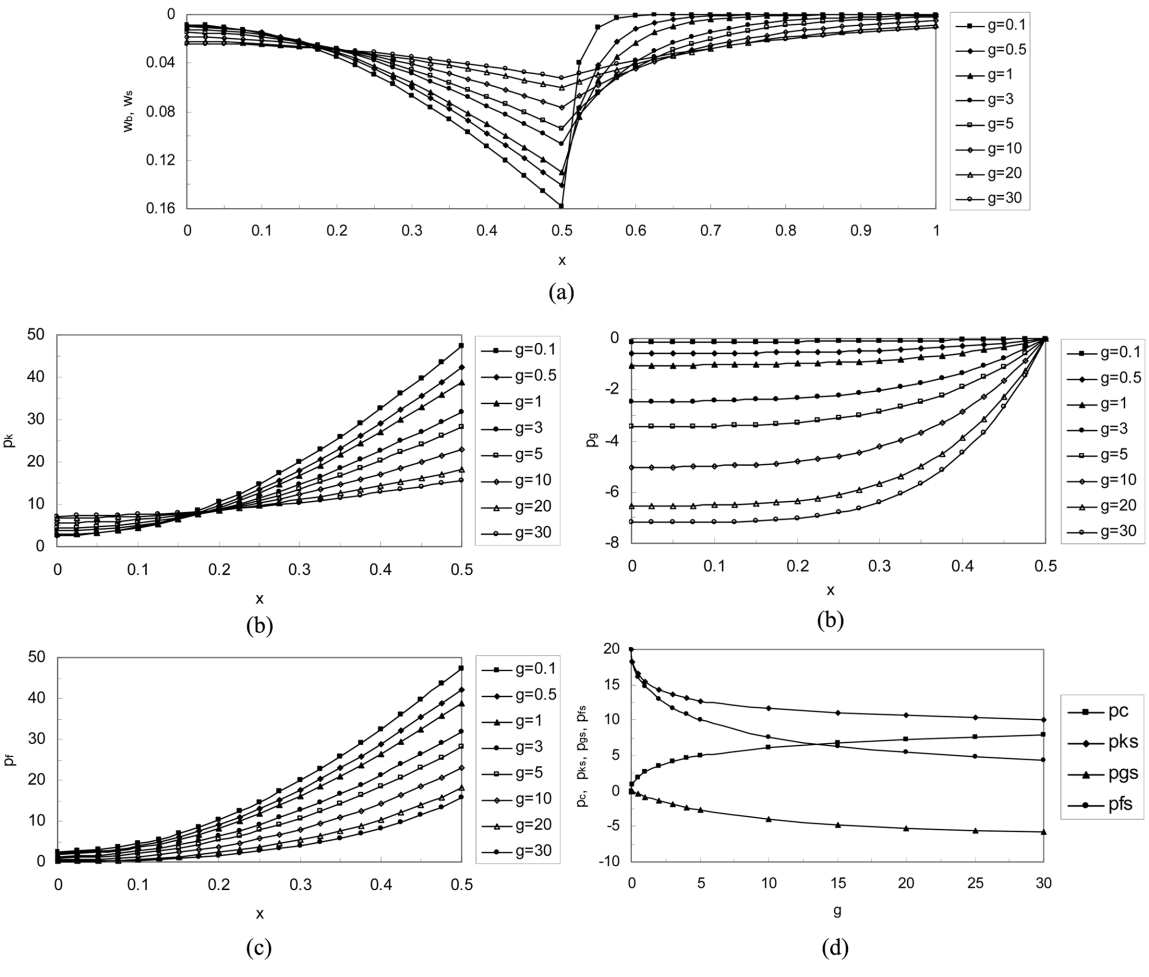

Fig. 3 (a), (b), (c), (d) and (e) Variation of the beam and foundation displacement w_b, w_s; foundation reactions p_k, p_g, p_f and the sum of the foundation reactions p_{ks}, p_{gs}, p_{fs} for q = 0, r = 10, k = 300 and various values of g

variations of the displacement of the beam and the foundation under the beam $w_b(0 \le x \le 0.5) = w_s(0 \le x \le 0.5)$ and the foundation beyond the end of the beam $w_s(0.5 \le x \le 1.0)$, the foundation reactions $p_k(x), p_g(x)$ and $p_f(x) = p_k(x) + p_g(x)$ for various values of the foundation stiffness g for $q = 0, r = 10$ and $k = 300$. Since all of the foundation reaction is compression $p_f(0 \le x \le 0.5) \ge 0$ for the numerical values of the parameters used in Fig. 3, complete contact develops between the beam and the foundation, i.e., $w_b(0 \le x \le 0.5) = w_s(0 \le x \le 0.5)$. As Fig. 3 shows, an increase in the stiffness g causes a decrease in the displacements $w_b(x)$ and $w_s(x)$ near the free end of the beam, whereas the displacements display the opposite trend under the middle of the beam and beyond the edge of the beam.

The stiffness g represents the tension of the membrane in the foundation model. When the stiffness of the membrane g decreases, the effect of the stiffness of the spring k becomes evident. Consequently, the behavior of the two-parameter foundation resembles a Winkler model and a discontinuity of the foundation displacement starts to appear at the edge of the beam, as Fig. 3(a) shows. However, when g gets larger, the membrane behavior of the foundation becomes effective

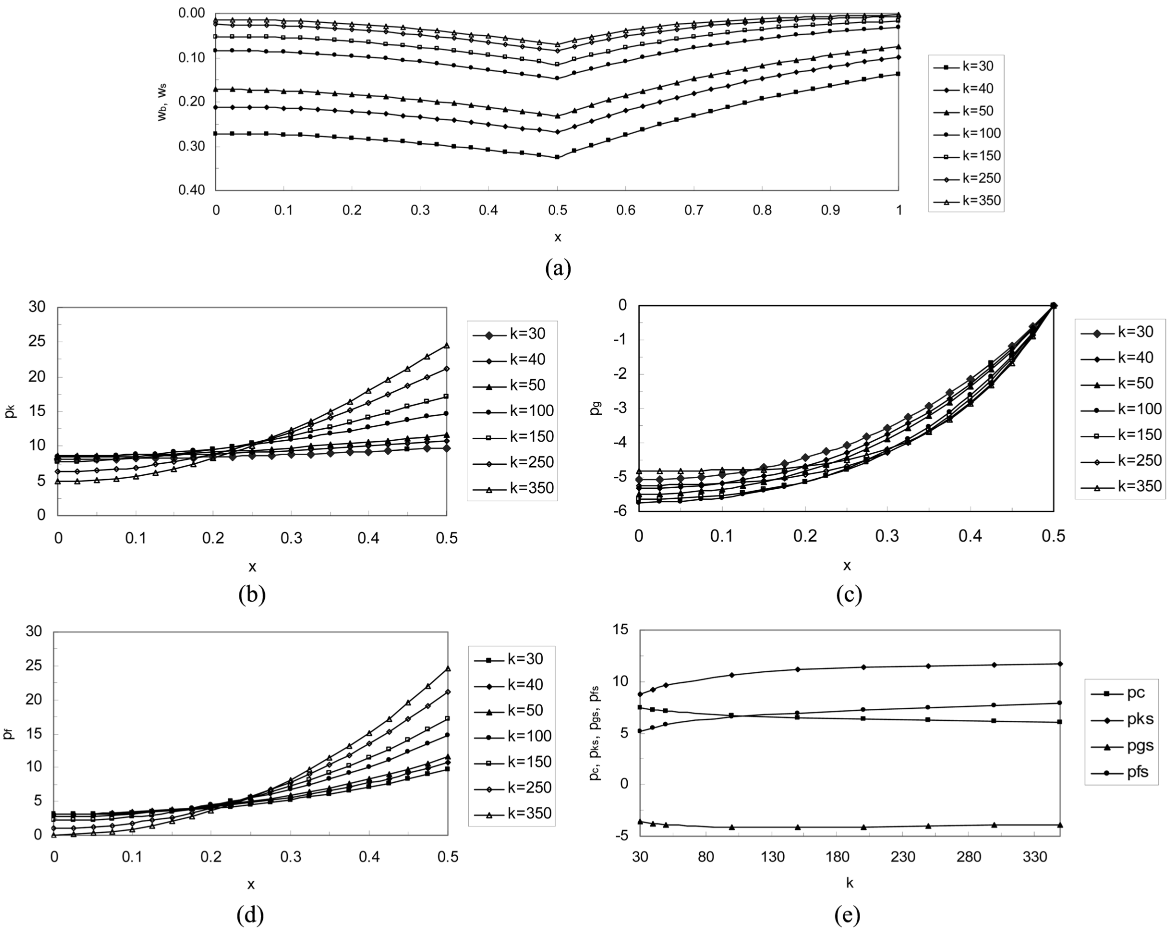

Fig. 4 (a), (b), (c), (d) and (e) Variation of the beam and foundation displacement w_b, w_s; foundation reactions p_k, p_g, p_f and the sum of the foundation reactions p_{ks}, p_{gs}, p_{fs} for q = 0, r = 10, g = 10 and various values of k

and its cooperation with the springs becomes pronounced. Consequently a smooth variation of displacement extending beyond of the beam edge appears. The foundation reaction $p_k(x)$ displays a similar variation. Furthermore, an increase in the stiffness g produces a decrease in the total foundation reaction $p_f(x)$, and a corresponding increase in the foundation reaction $p_g(x)$. When complete contact is maintained, the problem is linear and all displacements and foundation reactions depend on the loads r and q linearly. Fig. 3(e) shows variations of the concentrated foundation reaction p_c and the sum of the distributed reactions of the foundation p_{ks} and p_{gs} for various values of the stiffness g. The foundation reactions p_c and p_{gs} come into being due to the stiffness g. Consequently they get larger, as the stiffness g increases. Of course, then the foundation reactions decrease to satisfy global force equilibrium (15), i.e., $2p_c + p_{ks} + p_{gs} = 2r + q = 20$. Fig. 4 shows variations of the displacement of the beam and the foundation under the beam $w_b(0 \le x \le 0.5) = w_s(0 \le x \le 0.5)$ and the foundation beyond the beam ends $w_s(0.5 \le x \le 1.0)$, as well as variations of the foundation reactions $p_k(x)$, $p_g(x)$ and $p_f(x)$ for various values of the foundation stiffness k

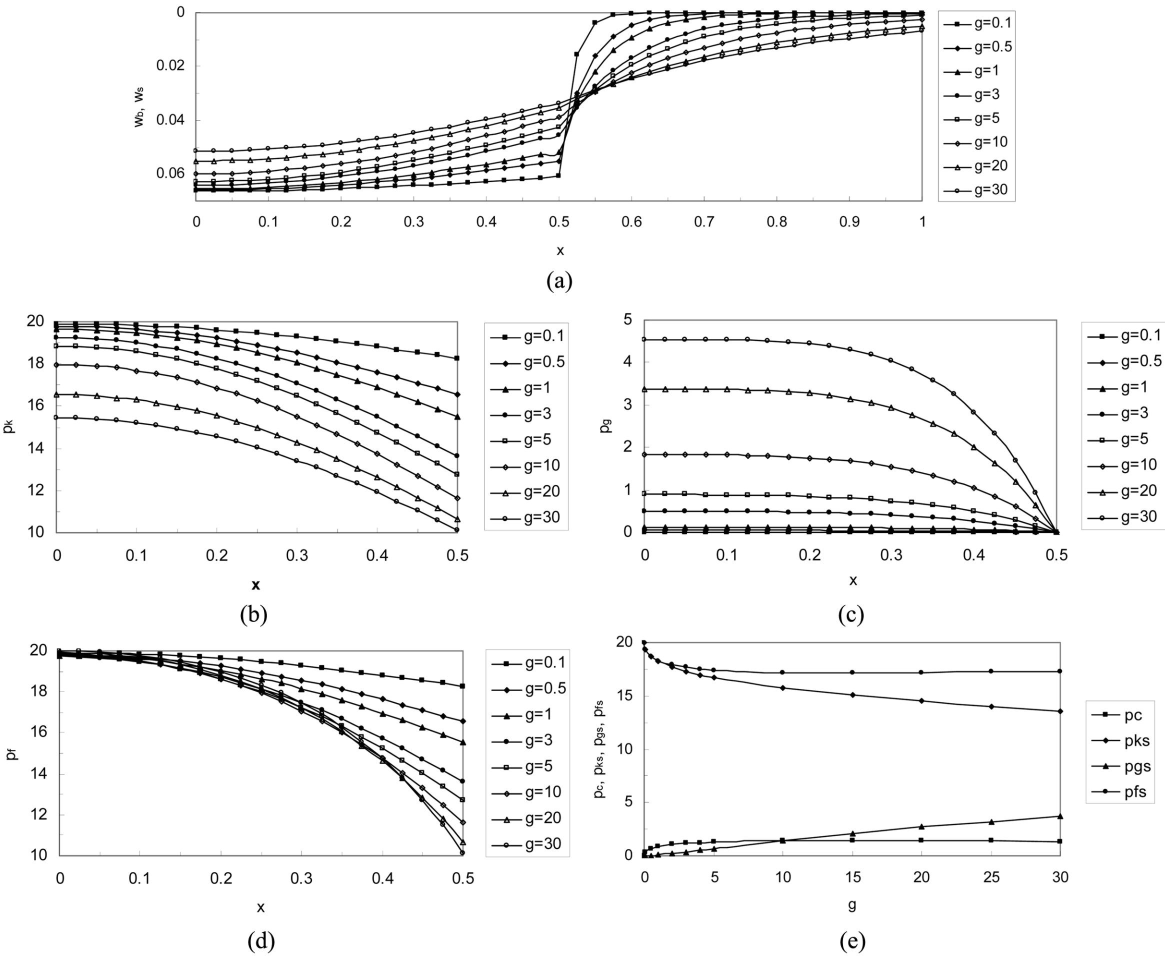

Fig. 5 (a), (b), (c), (d) and (e) Variation of the beam and foundation displacement w_b, w_s; foundation reactions p_k, p_g, p_f and the sum of the foundation reactions p_{ks}, p_{gs}, p_{fs} for q = 20, r = 0, k = 300 and various values of g

for $q = 0, r = 10$ and $g = 10$. Since the variation of the total foundation reaction under the beam is $p_f(x)$ is positive for the numerical values adopted, complete contact of the beam develops here as well. As seen, increase in the stiffness k causes uniform decrease in the displacements $w_b(x)$ and $w_s(x)$ and in the foundation reaction $p_k(x)$, the latter offset by increase in the foundation reaction $p_g(x)$. Fig. 4(e) shows the variations of the foundation reactions p_c, p_{ks} and p_{gs} for various values of the stiffness g. Since the foundation reactions p_c and p_{gs} directly depend on the stiffness g and indirectly on k, they display a smooth variation in Fig. 4(e). Since $w_b''(x = 0.5) = w_s''(x = 0.5) = 0$, the foundation reaction p_g vanishes under the free end of the beam, i.e., $p_g(x = 0.5) = 0$. As Fig. 3 and Fig. 4 show, the foundation pressure p_k displays positive variation, whereas the foundation pressure p_g displays negative variation, due to negative values of the second derivative of the displacement. Fig. 5 and Fig. 6 show the corresponding results for the beam subjected to a distributed load. As it is seen, the sum of the foundation reactions p_k and p_g both display positive variations.

In the above figures, complete contact has been established due to the numerical combination of

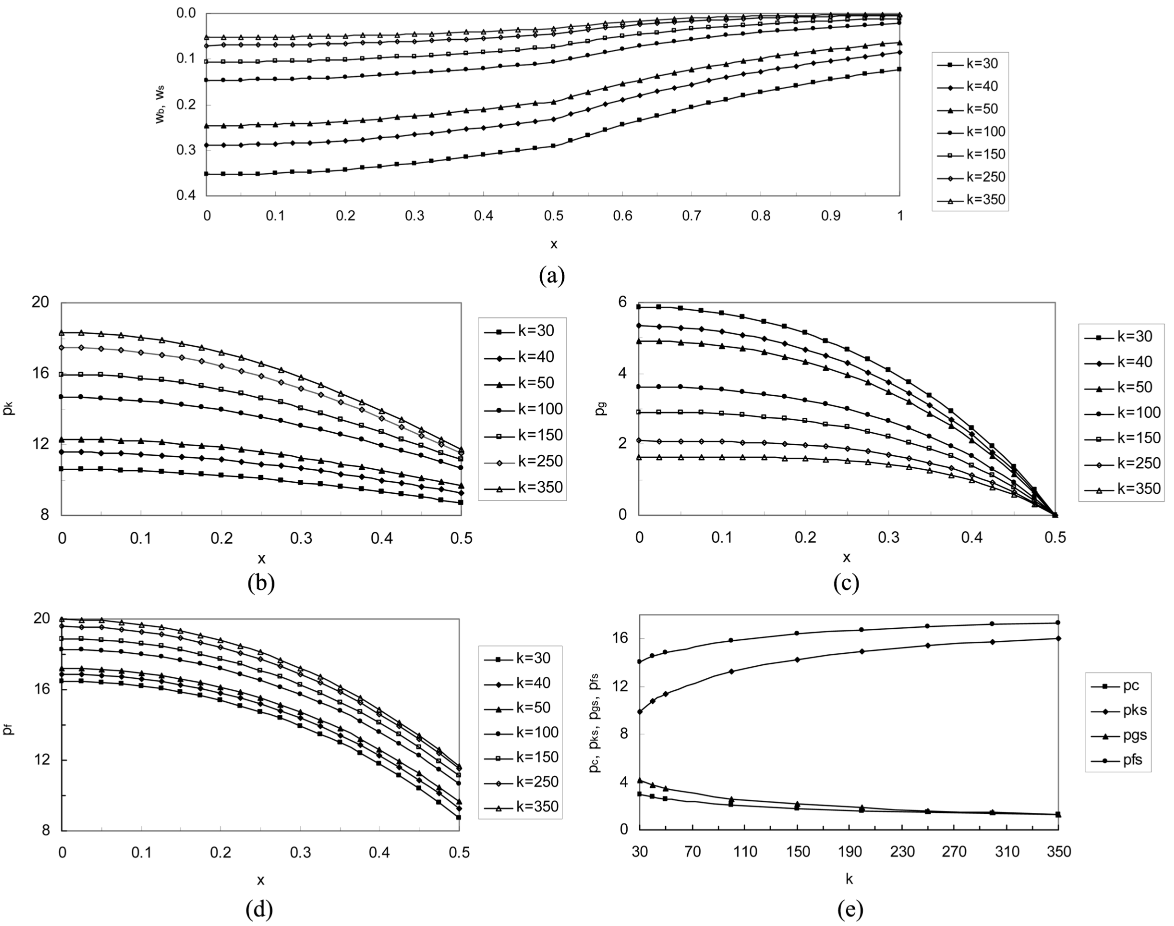

Fig. 6 (a), (b), (c), (d) and (e) Variation of the beam and foundation displacement w_b, w_s; foundation reactions p_k, p_g, p_f and the sum of the foundation reactions p_{ks}, p_{gs}, p_{fs} for $q = 20$, $r = 0$, $g = 10$ and various values of k

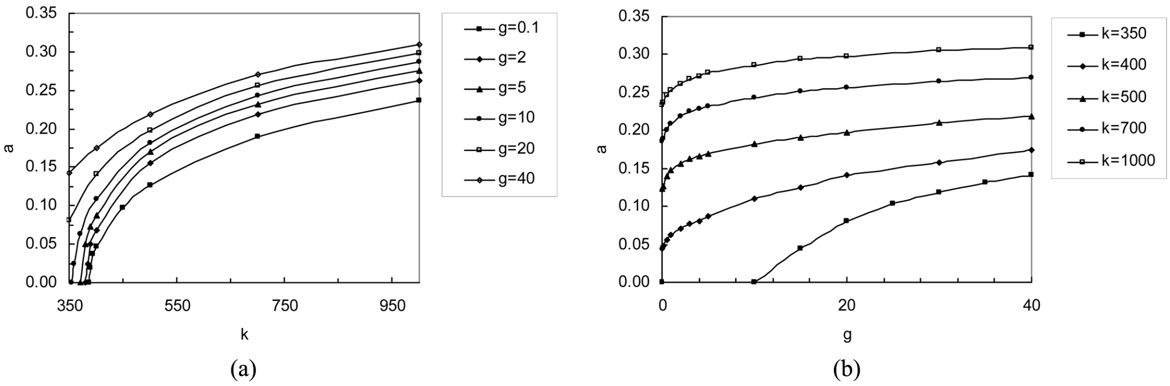

Fig. 7 (a) and (b) Variation of the contact length a as a function of the foundation stiffnesses k and g for $q = 0$

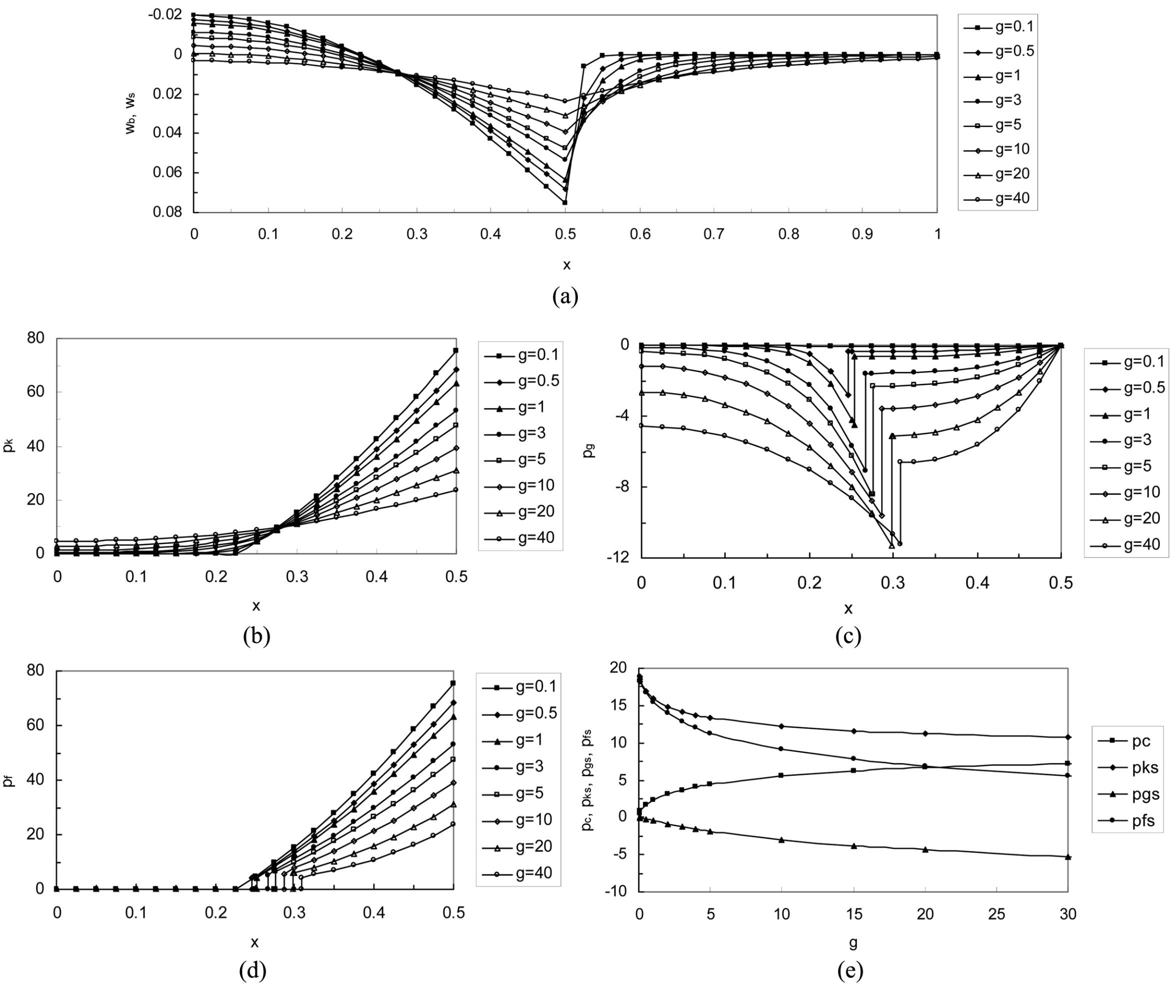

Fig. 8 (a), (b), (c), (d) and (e) Variation of the beam and foundation displacement w_b, w_s; foundation reactions p_k, p_g, p_f and the sum of the foundation reactions p_{ks}, p_{gs}, p_{fs} for $q = 0$, $r = 10$, $k = 1000$ and various values of g

the parameters of the beam and the foundation, i.e., $p_f(x) = p_k(x) + p_g(x) \geq 0$ for $0 \leq x \leq 0.5$. Fig. 3(d) shows that the total foundation reaction approaches zero at the middle of the beam, i.e., $p_f(x = 0) \to 0$, for larger values of g. Since a negative value, i.e., tensile reaction for $p_f(x = 0)$ cannot be supported by the foundation, a lift-off region enters the picture, as it is shown in Fig. 2. In this case non-linearity arises due to the fact that the foundation cannot support tension and the beam lifts off the foundation. Fig. 7 shows the lift-off length a for various values of the foundation stiffnesses k and g for a beam subjected to two end loads r, but no uniformly distributed load q. When the external loads increase linearly, the location of the point of separation of the beam and the foundation does not change, as indicated by Kerr and Soicher (1996). As expected, the beam lifts off the foundation for large values of the foundation stiffnesses k and g. Fig. 8 and Fig. 9 show variations of the displacement of the beam and the foundation under the beam $w_b(0 \leq x \leq 0.5)$ and the foundation beyond the end of the beam $w_s(0.5 \leq x \leq 1.0)$, and the foundation reactions $p_k(x), p_g(x)$ and $p_f(x)$ for various values of the foundation stiffness g and k, respectively, for $q = 0$

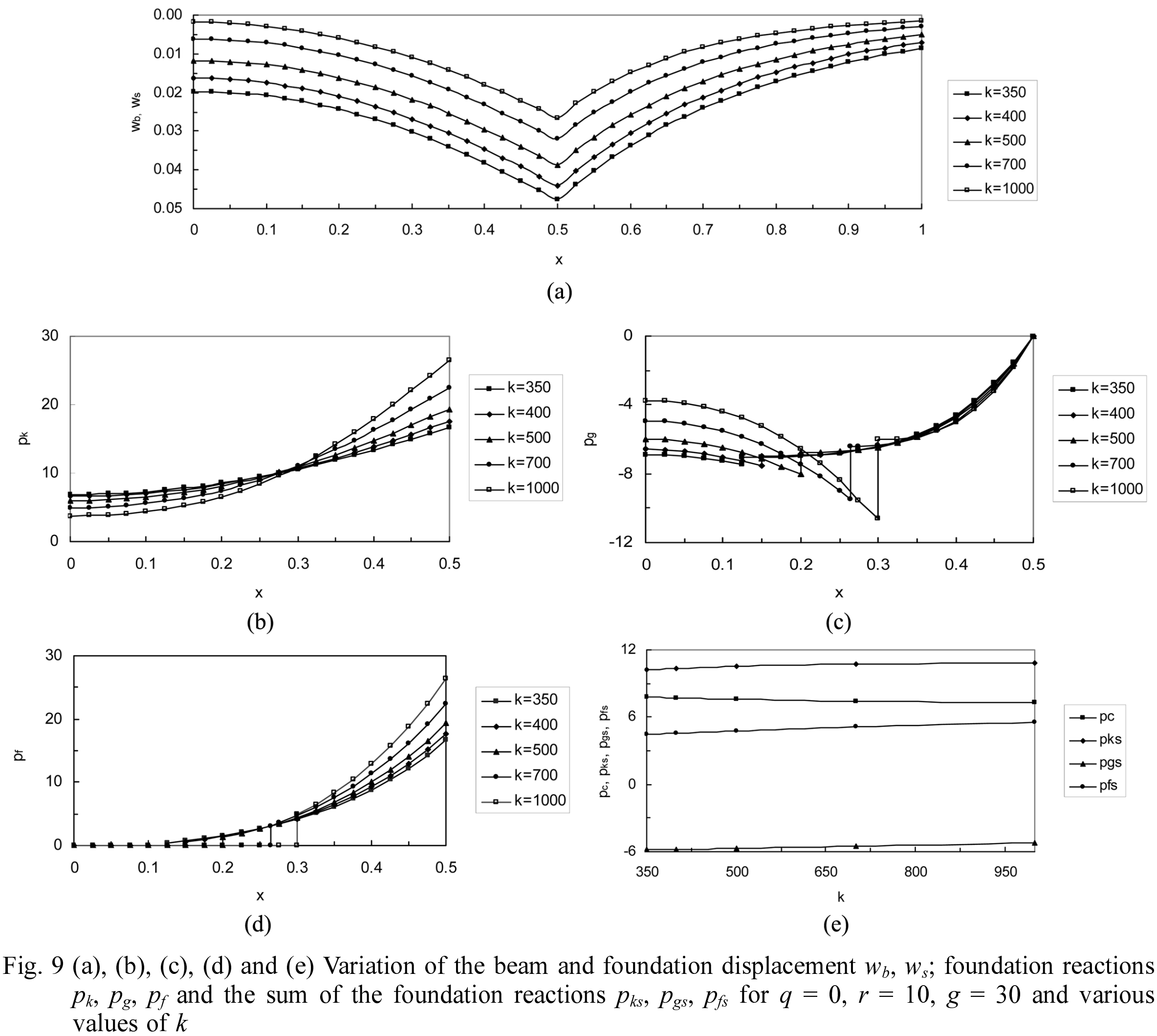

Fig. 9 (a), (b), (c), (d) and (e) Variation of the beam and foundation displacement w_b, w_s; foundation reactions p_k, p_g, p_f and the sum of the foundation reactions p_{ks}, p_{gs}, p_{fs} for q = 0, r = 10, g = 30 and various values of k

and r = 10. These two figures correspond to the lift-off case. The lift-off region develops where the total foundation reaction $p_f(0 \le x \le a)$ vanishes. The corresponding contact length a can be found in Fig. 7. As explained, the problem becomes highly non-linear and its numerical solution requires iteration. The displacements of the beam and the foundation as well as their derivatives are equal at the point which separates the contact and lift-off regions, as stated in Eqs. (20d) and (20i). This result complicates the numerical procedure and increases the number of iterations. Since there is not any condition which requires the continuity of the second derivative of the foundation displacement at the separation point, i.e., $x = a$, a discontinuity appears at this point. This discontinuity appears in the foundation reaction $p_g(x)$ and consequently in $p_f(x)$ as well, as seen in Figs. 8(c), 8(d), 9(c) and 9(d). These results show once more that the assumption of zero contact reaction at the separation point is not correct (Kerr and Coffin 1991). Due to the nature of the problem, the difference between the displacements of the beam and the foundation is very small in the lift-off region.

4. Conclusions

The paper presents an analysis for the lift-off problem of a beam subjected to concentrated end loads and uniformly distributed load supported by a two-parameter foundation. Special attention is paid to the conditions which define the onset of the lift-off and to the boundary conditions in cases of either complete or partial contact between the beam and the foundation, including the foundation reaction at the end of the beam and the discontinuity of the foundation pressure at the contact point. The formulation and the numerical results presented contribute to the understanding of the interaction between the completely free beam and the two-parameter tensionless foundation. The problem is linear when complete contact between the beam and the foundation exists. However, it is highly non-linear when it is assumed that the foundation cannot support tension and part of the beam lifts off the foundation. The numerical solution is accomplished by using iteration, and comparative results are presented for various values of the parameters of the problem. The global vertical force equilibrium is checked for each case as well. From the analyses presented, the following conclusions can be drawn:

a. Although small displacements for the beam and the foundation are assumed in the problem, the governing equations of the problem are highly non-linear when the beam lifts off the foundation. However, when complete contact develops, the problem is linear and its numerical solution is straightforward.

b. Since the problem is highly non-linear, evaluation of the contact region can be accomplished by using an iterative procedure in the case of partial contact. Usually numerical iteration requires an initial estimation of the contact length, so that the iteration can be accomplished easily. Consequently, it is recommended that the numerical solution should start from the complete contact case, by gradual changing one of the parameters, so that the partial contact case can be dealt with relatively easily.

c. When the external loads of any type increase linearly, the displacements of the beam will increase linearly, the position of the point of separation will not change and the overall equilibrium between the external loads and the foundation reaction will be established.

d. In a two-parameter foundation model, the separation point is to be determined by requiring the continuity of the displacement and the slope of the displacement of both the beam and the foundation. However, a discontinuity of the second derivative of the foundation displacement appears at this point, which causes a discontinuity in the foundation reactions as well. On the other hand, the second derivative of the displacement of the beam displays continuity, provided that there is no concentrated external moment (there is no such moment in the present problem).

e. When complete contact is established, an edge reaction comes into being a result of discontinuity of the slope of the foundation displacement. In the presented formulation, the reaction is included in the governing equations of the problem as a boundary condition.

f. As it is shown, the free end condition of the beam depends on whether the contact is established or not and an intuitive approach in the formulation of the boundary conditions may lead to incorrect formulation.

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