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Non linear vibrations of stepped beam system under different boundary conditions

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Abstract. In this study, the nonlinear vibrations of stepped beams having different boundary conditions were investigated. The equations of motions were obtained using Hamilton's principle and made non dimensional. The stretching effect induced non-linear terms to the equations. Forcing and damping terms were also included in the equations. The dimensionless equations were solved for six different set of boundary conditions. A perturbation method was applied to the equations of motions. The first terms of the perturbation series lead to the linear problem. Natural frequencies for the linear problem were calculated exactly for different boundary conditions. Second order non-linear terms of the perturbation series behave as corrections to the linear problem. Amplitude and phase modulation equations were obtained. Non-linear free and forced vibrations were investigated in detail. The effects of the position and magnitude of the step, as well as effects of different boundary conditions on the vibrations, were determined.

Keywords: stepped beam; nonlinear vibration; perturbation method.

1. Introduction

In real life, many engineering problems can be modeled as stepped beams. Examples of these structures include bridges, rails, automotive industries and machine elements. The most important aspect of vibration analysis is that the natural frequency of their can be estimated. If the system is forced with a frequency close to its natural frequencies, the system comes to resonance state and the amplitudes increase dangerously. While computing the natural frequencies of the systems, assuming the systems linear makes the calculations easier but the results are usually not reliable. Because no system moves linearly obtained linear results may deceive us. Therefore, nonlinear effects originated from the stretching during the vibration of the beam should be included in the computations as well.

Many studies on beam vibrations, both linear and nonlinear, have previously been performed.

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Particularly, the nonlinear behavior caused by the immobility of beam-ends has been analyzed by various researchers (Hou and Yuan 1998, McDonald 1991, Pakdemirli and Nyfeh, 1994, Öz *et al.* 1998). Qaisi (1997) obtained the nonlinear vibration of beams with simply and clamped supports by using a power series approach and compared the results with existing solutions. Özkaya *et al.* (1997) analyzed mass beam system for different boundary conditions. By considering the effects of stretching, they solved the obtained problem with the method of multiple scales, a perturbation technique. Özkaya (2002) considered a beam-masses under simply supported system end conditions. The effect of positions, magnitudes and number of the masses was investigated.

Studies on stepped beam systems are usually linear. Balasubramanian *et al.* (1990) analyzed vibrations for beams stepped in the middle and acquired natural frequencies for high mode structures. Jang and Bert (1989) obtained the frequency equation for stepped beam under various boundary conditions and computed the smallest natural frequencies for a circular cross-section beam. They compared the results they obtained with the results used a finite element analysis. In another study, Jang and Bert (1989) obtained natural frequencies for high mode structures using the frequency equation they acquired from the study by Jang and Bert (1989). In a study performed by Naguleswaran (2002), motion equations of three different Euler-Bernoulli stepped beams with all states of boundary conditions were obtained and three natural frequencies were computed using motion equation. His other study, he (Naguleswaran 2002) considered three different types of stepped beams and investigated vibration of Euler-Bernoulli beam with up to three step changes. The first three frequency parameters of beams with one, two and three step changes were tabulated. The dynamic stability of a stepped beam carrying mass was studied by Aldraihem and Baz (2002). The stepped beam equations of motion developed a discrete parameter form and a finite element form. Aydoğdu and Taskin (2006), free vibration of simply supported FG beam explored and also they found the equations by applying Hamilton's principle. In order to obtain frequencies used Navier type solution method. Kwon and Park (2002), focused on the effect of the position of the stepped point and thickness ratio on the dynamic characteristics of the system. The equation of motion and boundary were analytically obtained by using Hamilton principle. The exact solutions were compared with the results obtained by FEM. Naguleswaran (2003) the vibration of beams with up to three step changes in cross section and in which the axial force in portion was contented. The frequency equation for classical boundary was expressed and the first three frequency parameters for the three types of beams were displayed. Krishnan *et al*. (1998), the analysis of stepped beams using finite difference method studied by using of a single differential equation.

In this study, nonlinear vibration analysis for stepped beams was performed and the contributions of nonlinear terms on natural frequency were investigated. Phase modulation equations were acquired and frequency amplitude graphs were plotted using these equations.

2. Equation of motion

The considered system is a stepped beam with a single step located at $x = x_s$, where x is the spatial co-ordinate along the beam length. Six different cases of support at the ends of the beam are treated, as shown in Fig. 1.

The Lagrangian for the system can be written as

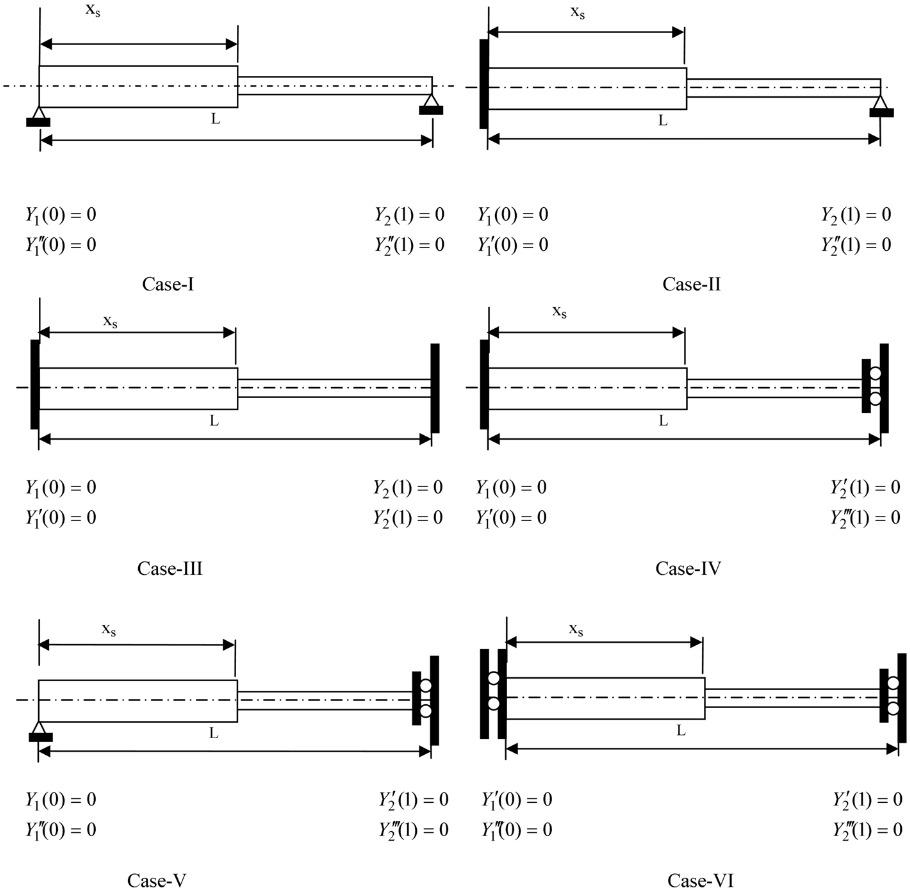

Fig. 1 The support end conditions for six different cases

$$
\begin{aligned}
\pounds \;=\; & (1/2)\int_0^{x_s} \rho A_1 \dot{w}_1^{*2} dx^* + (1/2)\int_{x_s}^{L} \rho A_2 \dot{w}_2^{*2} dx^* - (1/2)\int_0^{x_s} EI_1 w_1''^{*2} dx^* - (1/2)\int_{x_s}^{L} EI_2 w_2''^{*2} dx^* \\
& - (1/2)\int_0^{x_s} EA_1 (u_1'^{*} + (1/2) w_1'^{*2})^2 dx^* - (1/2)\int_{x_s}^{L} EA_2 (u_2'^{*} + (1/2) w_2'^{*2})^2 dx^* \qquad (1)
\end{aligned}
$$

where L is the length, ρ is the density, A_1 and A_2 are cross-sectional areas, E is Young's modulus, I_1 and I_2 are the moment of inertia of the beam cross-section with respect to the neutral axis of the beam, u_1 and u_2 are the left and right axial displacements, w_1 and w_2 are the left and right transverse displacements, $(\cdot)$ and $(\ )'$ denote differentiations with respect to time t^* and the spatial variable x^* respectively. The terms in Eq. (1) are the kinetics energies due to transverse motion of beam, elastic energies due to bending and stretching of the beam, respectively. Invoking Hamilton's principle

$$\delta \int_{t_1}^{t_2} \pounds dt^* \;=\; 0 \qquad (2)$$

and substituting the Lagrangian from Eq. (1), performing the necessary algebra and eliminating the axial displacements, one finally obtains the following non-linear coupled integro-differential equations

$$\rho A_1 \ddot{w}_1^* + EI_1 w_1^{iv*} = \frac{EA_1}{2[x_s + (L - x_s)/\alpha^2]}\left[\int_0^{x_s} w_1'^{*2}\, dx^* + \int_{x_s}^{L} w_2'^{*2}\, dx^*\right] w_1''^* \tag{3}$$

$$\rho A_2 \ddot{w}_2^* + EI_2 w_2^{iv*} = \frac{EA_1}{2[x_s + (L - x_s)/\alpha^2]}\left[\int_0^{x_s} w_1'^{*2}\, dx^* + \int_{x_s}^{L} w_2'^{*2}\, dx^*\right] w_2''^* \tag{4}$$

In Eqs. (3) and (4) the dimensionless parameter α is defined as the ratio of the diameter of the second portion to the diameter of the first portion ($\alpha = r_2/r_1$). Note that viscous damping coefficient μ^*, external excitation with amplitude $F_{1,2}^*$ and frequency Ω^* are be added to the equations. The boundary conditions equations can be written for this as follows

$$w_1^*(x_s, t^*) = w_2^*(x_s, t^*), \qquad w_1'^*(x_s, t^*) = w_2'^*(x_s, t^*) \tag{5}$$

$$EI_1 w_1''^*(x_s, t^*) - EI_2 w_2''^*(x_s, t^*) = 0, \quad EI_1 w_1'''^*(x_s, t^*) - EI_2 w_2'''^*(x_s, t^*) = 0 \tag{6}$$

The equations are made dimensionless through the definitions

$$x = x^*/L, \quad w_{1,2} = w_{1,2}^*/R_{1,2}, \quad \eta = x_s/L, \quad t = (1/L^2)(EI_1/\rho A_1)^{1/2} t^* \tag{7}$$

where $R_{1,2}$ is the radius of gyration of the beam cross-section with respect to the neural axis. Substituting the dimensionless parameters into the equations of motion yield for the general case

$$\ddot{w}_1 + w_1^{iv} = \frac{1}{2[\eta + (1-\eta)/\alpha^2]}\left[\int_0^{\eta} w_1'^2\, dx + \alpha^2 \int_{\eta}^{1} w_2'^2\, dx\right] w_1'' \tag{8}$$

$$\ddot{w}_2 + \alpha^2 w_2^{iv} = \frac{1}{2\alpha^2[\eta + (1-\eta)/\alpha^2]}\left[\int_0^{\eta} w_1'^2\, dx + \alpha^2 \int_{\eta}^{1} w_2'^2\, dx\right] w_2'' \tag{9}$$

and boundary conditions are

$$w_1 = \alpha w_2, \quad w_1' = \alpha w_2', \quad w_1'' - \alpha^5 w_2'' = 0, \quad w_1''' - \alpha^5 w_2''' = 0 \quad \text{at} \quad x = \eta \tag{10}$$

The equation of motion including damping and forcing is given below

$$\ddot{w}_1 + w_1^{iv} = \frac{1}{2[\eta + (1-\eta)/\alpha^2]}\left[\int_0^{\eta} w_1'^2\, dx + \alpha^2 \int_{\eta}^{1} w_2'^2\, dx\right] w_1'' - 2\mu^* \dot{w}_1 + F_1^* \cos\Omega t \tag{11}$$

$$\ddot{w}_2 + \alpha^2 w_2^{iv} = \frac{1}{2\alpha^2[\eta + (1-\eta)/\alpha^2]}\left[\int_0^{\eta} w_1'^2\, dx + \alpha^2 \int_{\eta}^{1} w_2'^2\, dx\right] w_2'' - 2\mu^* \dot{w}_2 + F_2^* \cos\Omega t \tag{12}$$

The solutions and results for different parameters will be presented in the next section.

3. Approximate analytical solution

In this section, we search for the approximate solutions of Eqs. (11) and (12) with the associated boundary conditions. We apply the method of multiple scales (a perturbation technique) to the partial differential system and boundary conditions directly. This direct treatment of partial differential systems (the direct perturbation method) has some advantages over the more common method of discrediting the partial differential system and then applying perturbation (the discretization perturbation method) (Pakdemirli and Boyacı 1995). In our case, however, both methods may yield identical results, since we are not considering a higher order perturbation scheme. Due to the absence of quadratic non-linearities, we assume expansions of the forms

$$w_1(w, t; \varepsilon) = \varepsilon w_{11}(x, T_0, T_2) + \varepsilon^3 w_{13}(x, T_0, T_2) + \dots \tag{13}$$

$$w_2(w, t; \varepsilon) = \varepsilon w_{21}(x, T_0, T_2) + \varepsilon^3 w_{23}(x, T_0, T_2) + \dots \tag{14}$$

where ε is a small book-keeping parameter artificially inserted into the equations. This parameter can be taken as 1 at the end upon keeping in mind, however, that deflections are small. We therefore investigate a weakly non-linear system. $T_0 = t$ and $T_2 = \varepsilon^2 t$ are the fast and slow time scales. We consider only the primary resonance case and hence, the forcing and damping terms are ordered so that they counter the effect of non-linear terms: that is

$$\mu^* = \varepsilon^2 \mu, \quad F^*_{1,2} = \varepsilon^3 F_{1,2} \tag{15}$$

The time derivatives can be written as

$$(\dot{\ }) = D_0 + \varepsilon^2 D_2, \quad (\ddot{\ }) = D_0^2 + 2\varepsilon^2 D_0 D_2, \quad D_n = \frac{\partial}{\partial T_n} \tag{16}$$

Inserting Eqs. (13)-(16) into Eqs. (10)-(12) and equating coefficients of like powers of ε, one obtains, to order ε

$$D_0^2 w_{11} + w_{11}^{iv} = 0; \qquad D_0^2 w_{21} + \alpha^2 w_{21}^{iv} = 0 \tag{17,18}$$

$$w_{11} = \alpha w_{21}, \quad w'_{11} = \alpha w'_{21}, \quad w''_{11} = \alpha^5 w''_{21}, \quad w'''_{11} = \alpha^5 w'''_{21} \quad \text{at} \quad x = \eta \tag{19}$$

$$w_{11} = w''_{11} = 0 \quad \text{at} \quad x = 0, \quad w_{21} = w''_{21} = 0 \quad \text{at} \quad x = 1 \tag{20}$$

and, to order ε^3

$$D_0^2 w_{13} + w_{13}^{iv} = -2D_0 D_2 w_{11} - 2\mu D_0 w_{11} + \frac{1}{2[\eta + (1-\eta)/\alpha^2]}\left[\int_0^\eta w_{11}'^2\, dx + \alpha^2 \int_\eta^1 w_{21}'^2\, dx\right] w''_{11} + F_1 \cos\Omega T_0 \tag{21}$$

$$D_0^2 w_{23} + \alpha^2 w_{23}^{iv} = -2D_0 D_2 w_{21} - 2\mu D_0 w_{21} + \frac{1}{2\alpha^2[\eta + (1-\eta)/\alpha^2]}\left[\int_0^\eta w_{11}'^2\, dx + \alpha^2 \int_\eta^1 w_{21}'^2\, dx\right] w''_{21} + F_2 \cos\Omega T_0 \tag{22}$$

$$w_{13} = \alpha w_{23}, \quad w'_{13} = \alpha w'_{23}, \quad w''_{13} = \alpha^5 w''_{23}, \quad w'''_{13} = \alpha^5 w'''_{23} \quad \text{at} \quad x = \eta \tag{23}$$

$$w_{13} = w''_{13} = 0 \quad \text{at} \quad x = 0, \quad w_{23} = w''_{23} = 0 \quad \text{at} \quad x = 1 \tag{24}$$

Eqs. (20) and (24) are the boundary conditions corresponding to Case I. Boundary conditions for other cases can be written similarly.

3.1 Linear problem

The problem at order is linear. We assume a solution of the form

$$w_{11} = [A(T_2)e^{i\omega T_0} + cc]Y_1(x), \quad w_{21} = [A(T_2)e^{i\omega T_0} + cc]Y_2(x) \tag{25, 26}$$

where cc represents the complex conjugate of the preceding terms. Substituting Eqs. (25) and (26) into Eqs. (17)-(20), one will have

$$Y_1^{iv} - \omega^2 Y_1 = 0, \qquad Y_2^{iv} - \frac{1}{\alpha^2}\omega^2 Y_2 = 0 \tag{27, 28}$$

$$Y_1 = \alpha Y_2, \quad Y'_1 = \alpha Y'_2, \quad Y''_1 = \alpha^5 Y''_2, \quad Y'''_1 = \alpha^5 Y'''_2 \quad \text{at} \quad x = \eta \tag{29}$$

Solving Eqs. (27)-(29) exactly for different end conditions yields the mode shapes Y_i and natural frequencies ω. The transcendental equations were numerically solved for the first five modes. For each case, the natural frequencies are listed for different α and $\eta = 0.5$ in Table 1.

Table 1 The first five natural frequencies for different stepped ratio and end conditions ($\eta = 0.5$)

Cases	α	ω_1	ω_1 [10]	ω_2	ω_2 [10]	ω_3	ω_3 [10]	ω_4	ω_4 [10]	ω_5	ω_5 [10]
Pinned-pinned	1	9.8696	9.8696	39.4784	39.4784	88.8264	88.8260	157.9136	157.9140	246.7401	246.7400
	$5^{1/4}$	10.4129	10.4129	50.6566	50.6566	103.7111	103.7110	195.1266	195.1270	295.4998	295.5000
	$20^{1/4}$	9.0747	9.0747	60.1464	60.1464	124.3604	124.3600	213.3760	213.3760	367.8333	367.8330
Clamped-pinned	1	15.4182	15.4182	49.9649	49.9649	104.2476	104.2480	178.2697	178.2700	272.0494	272.0310
	$5^{1/4}$	16.2811	16.2811	63.5852	63.5852	121.7557	121.7560	221.9135	221.9140	322.3577	322.3580
	$20^{1/4}$	14.2538	14.2568	80.1265	80.1265	137.1124	137.1120	251.7603	251.7600	393.9259	393.9260
Clamped-clamped	1	22.3733	22.3733	61.6728	61.6728	120.9575	120.9030	199.9749	199.8590	299.9258	298.5560
	$5^{1/4}$	25.9591	25.9591	78.1518	78.1518	142.0877	142.0880	245.5919	245.5912	359.0943	359.0970
	$20^{1/4}$	30.3213	30.3213	90.2097	90.2097	173.2790	173.2790	266.8390	266.8390	444.5712	444.3510
Clamped-sliding	1	5.5933	5.5933	30.2258	30.2258	74.6389	74.6390	138.7913	138.7910	222.6850	222.6830
	$5^{1/4}$	5.6912	5.6912	34.9709	34.9710	92.0034	92.0030	167.6614	167.6610	267.8909	267.8910
	$20^{1/4}$	5.3573	5.3573	43.0595	43.0595	98.8486	98.8490	211.3065	211.3070	291.9155	291.9160
Sliding-pinned	1	2.4674	2.4674	22.2066	22.2066	61.6850	61.6850	120.9026	120.9030	199.8595	199.8590
	$5^{1/4}$	2.4372	2.4372	26.8677	26.8677	75.8534	75.8530	143.4017	143.4020	247.3280	247.3280
	$20^{1/4}$	2.1841	2.1841	27.5026	27.5026	97.8727	97.8730	154.6877	154.6880	292.9891	292.9890
Sliding-sliding	1	9.8696	9.8696	39.4784	39.4784	88.8264	88.8264	157.9136	157.9140	246.7401	246.7400
	$5^{1/4}$	13.5124	13.5124	45.0027	45.0027	111.3453	111.3450	187.1320	187.1320	301.7942	301.7940
	$20^{1/4}$	18.2949	18.2949	50.3222	50.3222	125.0623	125.0620	231.8222	231.8220	327.4873	327.4870

3.2 Non-linear problem

Solving order ε^3, one obtains the non-linear corrections to the problem. Because the homogeneous Eqs. (15)-(20) have a non-trivial solution, the non-homogeneous problem (21)-(24) will have a solution only if a solvability condition is satisfied. To determine this condition, we first separate the secular and nonsecular terms by assuming a solution of the from

$$w_{13} = \phi_1(x, T_2)e^{i\omega T_0} + W_1(x, T_0, T_2) + cc, \quad w_{23} = \phi_2(x, T_2)e^{i\omega T_0} + W_2(x, T_0, T_2) + cc \quad (30, 31)$$

Substituting this solution into (21)-(24), we eliminate the terms producing secularities. Hence we deal with that part of the equation determining ϕ_l as follows

$$\phi_1^{iv} - \omega^2\phi_1 = -2i\omega(A' + \mu A)Y_1 + \frac{3}{2}\frac{A^2\bar{A}}{[\eta + (1-\eta)/\alpha^2]}\left[\int_0^\eta Y_1'^2 dx + \alpha^2\int_\eta^1 Y_2'^2 dx\right]Y_1'' + \frac{1}{2}F_1 e^{i\sigma T_2} \quad (32)$$

$$\alpha^2\phi_2^{iv} - \omega^2\phi_2 = -2i\omega(A' + \mu A)Y_2 + \frac{3}{2}\frac{A^2\bar{A}}{\alpha^2[\eta + (1-\eta)/\alpha^2]}\left[\int_0^\eta Y_1'^2 dx + \alpha^2\int_\eta^1 Y_2'^2 dx\right]Y_2'' + \frac{1}{2}F_2 e^{i\sigma T_2} \quad (33)$$

$$\phi_1 = \alpha\phi_2, \quad \phi_1' = \alpha\phi_2', \quad \phi_1'' = \alpha^5\phi_2'', \quad \phi_1''' = \alpha^5\phi_2''' \quad \text{at} \quad x = \eta \quad (34)$$

$$\phi_1 = \phi_1'' = 0 \quad \text{at} \quad x = 0 \quad \phi_2 = \phi_2'' = 0 \quad \text{at} \quad x = 1 \quad (35)$$

In obtaining these equations, we substituted the substituted first order solutions (25) and (26) into Eqs. (21)-(24). We also assumed that the external excitation frequency is close to one of the natural frequencies of the system; that is

$$\Omega = \omega + \varepsilon^2\sigma \quad (36)$$

Where σ is a detuning parameter of order 1. After some algebraic manipulations, one obtains the solvability condition for Eqs. (32)-(35) as

$$2i\omega(A' + \mu A) + \frac{3}{2[\eta + (1-\eta)/\alpha^2]}A^2\bar{A}b^2 - \frac{1}{2}fe^{i\sigma T_2} = 0 \quad (37)$$

where the equations are normalized by requiring and the coefficients are defined as follows

$$\int_0^\eta Y_1^2 dx + \alpha^4\int_\eta^1 Y_2^2 dx = 1, \quad b = \int_0^\eta Y_1'^2 dx + \alpha^2\int_\eta^1 Y_2'^2 dx, \quad f = \int_0^\eta F_1 Y_1 dx + \alpha^4\int_\eta^1 F_2 Y_2 dx \quad (38)$$

Note that condition (36) is valid for all Cases I to VI but, of course, the numerical values of b and $Y_1(\eta)$ differ for each case.

Eq. (37) determines the modulations in the complex amplitudes. We use the polar form to calculate real amplitudes and phases

$$A = \frac{1}{2}a(T_2)e^{i\theta T_2} \quad (39)$$

Substituting Eq. (39) into Eq. (37) and separating real and imaginary parts, one finally obtains

$$\omega a \gamma' = \omega a \sigma - \frac{3}{16} b^2 a^3 \Lambda + \frac{1}{2} f \cos\gamma, \quad \omega a' = -\omega \mu a + \frac{1}{2} f \sin\gamma \tag{40, 41}$$

where γ and Λ are defined by

$$\gamma = \sigma T_2 - \theta, \quad \Lambda = \frac{1}{[\eta + (1-\eta)/\alpha^2]} \tag{42}$$

4. Numerical results

In this section numerical examples are presented for different cases. Firstly, the linear natural frequencies for various boundary conditions and different stepped ratio are given in Table 1 and compared with those given by Jang and Bert (1989).

Then, the non-linear frequencies for free, undamped vibrations are calculated. In Eq. (40) and Eq. (41), by taking $\mu = f = \sigma = 0$, one obtains

$$a' = 0 \quad \text{and} \quad a = a_0 \text{ (constant)} \tag{43}$$

Note that a_0 is the steady-state real amplitude of response. Hence the non-linear frequency is

$$\omega_{n1} = \omega + \theta' = \omega + \lambda a_0^2, \qquad \lambda = (3/16)(\Lambda b^2/\omega) \tag{44, 45}$$

Table 2 The non-linear frequency correction coefficients for different stepped ratio, step location and end conditions

		Cases											
α	η	Case-I		Case-II		Case-III		Case-IV		Case-V		Case-VI	
		ω_1	λ	ω_1	λ	ω_1	λ	ω_1	λ	ω_1	λ	ω_1	λ
0.5	0.2	4.7613	14.2781	11.2659	20.1648	16.1990	14.7906	4.1563	4.1617	1.2189	4.1005	5.2086	8.5587
	0.4	4.5198	9.1014	13.5509	12.0201	17.0716	5.9488	6.1189	4.5853	1.1538	3.6157	7.3835	9.5764
	0.6	5.1545	5.8585	10.5991	6.3422	13.6119	3.7483	6.5614	1.4288	1.0940	2.6539	9.4960	5.9104
	0.8	7.7399	2.8663	11.7411	4.3228	16.2339	1.4176	5.1411	0.8739	1.1619	2.0553	8.1674	3.9549
0.8	0.2	7.9133	3.7081	14.4048	3.4214	20.5444	2.5293	5.4331	0.7043	1.9745	0.9600	8.3815	2.8224
	0.4	8.1407	3.2076	14.1931	3.1407	19.8208	2.2641	5.8588	0.6263	1.9833	0.9576	9.0454	2.5652
	0.6	8.8135	2.3870	13.9883	2.6100	19.8115	1.5584	5.6908	0.5788	2.0233	0.9133	9.1719	2.6562
	0.8	9.6399	1.7995	14.9713	1.7187	20.0827	1.2335	5.4273	0.5015	2.1424	0.6440	9.0631	2.5729
2.0	0.2	15.4799	0.3582	21.8931	0.1558	32.4679	0.1772	6.4545	0.0325	4.5486	0.0661	16.3348	0.4943
	0.4	10.3090	0.7323	16.4368	0.3648	27.2238	0.4685	5.9935	0.0313	3.3475	0.0460	18.9920	0.7388
	0.6	9.0397	1.1376	13.9313	0.7190	34.1433	0.7436	4.9838	0.0500	2.5050	0.0695	14.7671	1.1970
	0.8	9.5227	1.7847	14.6087	1.4030	32.3980	1.8488	4.5042	0.0877	2.1685	0.1380	10.4172	1.0698
3.0	0.2	13.8815	0.2882	22.7937	0.0965	33.4141	0.1622	7.9418	0.0742	5.2319	0.0135	24.7660	0.1284
	0.4	7.8317	0.3570	12.7181	0.1878	31.7823	0.1652	5.2539	0.0146	2.7301	0.0136	28.5641	1.4068
	0.6	7.1493	0.5836	10.7547	0.3455	49.6127	1.0978	3.7116	0.0198	1.8566	0.0243	15.7834	1.4410
	0.8	8.7401	1.4009	12.9824	1.0061	34.4018	2.2833	3.3233	0.0347	1.6277	0.0552	9.7674	0.8611
1.0	1.0	9.8696	1.8505	15.4182	1.6117	22.3733	1.2684	5.5933	0.3171	2.4674	0.4626	9.8696	1.8505

To this order of approximation, then, the non-linear frequencies have a parabolic relation with the maximum amplitude of vibration. λ can be defined as the non-linear correction coefficient. The non-linear correction coefficients are listed in Table 2 for different condition values of α and η and first mode corresponding to different case. λ is a measure of the effect of stretching. The non-linearities are of hardening type.

One can from Table 2 that the effects of stretching generally decrease as α increases for all cases. Pakdemirli and Boyacı (2001) compared with case I of this paper by assuming $\eta = 1$ and $\alpha = 1$. It was observed that the results of both cases were 1.8505. The curves that show the relationship between nonlinear frequency and amplitude are displayed in Figs. 2-8 for different α, η values and different boundary conditions. Fig. 2, for Case-I, shows the variation of non-linear frequencies with amplitude. As η increases, so do the effects of stretching. For Case-II, as η increases, the effects of stretching decrease (Fig. 3). For Case-III, as the stepped shifts left to right, the stretching effects increase (Fig. 4). For Case-IV, as η increases, the effects of stretching decrease (Fig. 5). For all cases, the stretching effects decrease as α increases. The results for Case V and Case VI are given in Fig. 6 and Fig. 7 for different α parameters. As α increases, the effects of stretching decrease. The non-linear frequencies for all cases are shown in Fig. 8. We now consider the case in which there is damping and external excitation. In Eqs. (40) and (41), when the system reaches the steady state region, a' and γ' vanish and hence one obtains.

$$\omega\mu a = \frac{1}{2}f\sin\gamma, \qquad -\omega a\sigma + \frac{3}{16}b^2 a^3\Lambda = \frac{1}{2}f\cos\gamma \tag{46, 47}$$

Squaring and adding both equations and solving for the detuning parameter σ yield

$$\sigma = \lambda a^2 \mp \sqrt{\frac{f^2}{4\omega^2 a^2} - \mu^2} \tag{48}$$

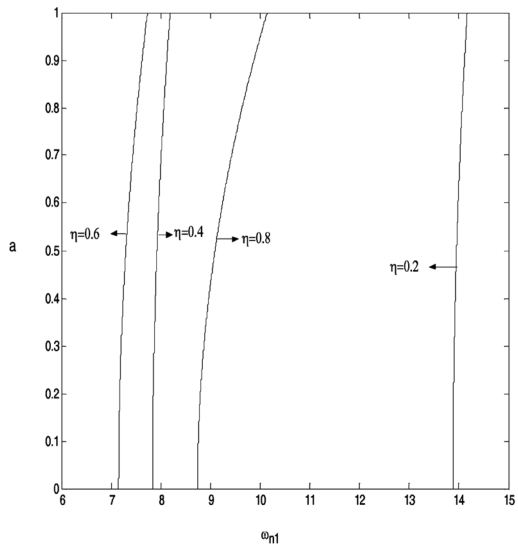

Fig. 2 Non-linear frequency versus amplitude for different stepped location values; first mode, Case-I. $\alpha = 3$

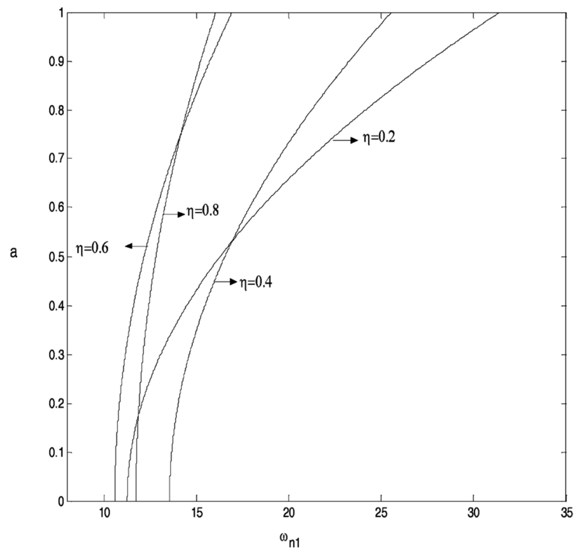

Fig. 3 Non-linear frequency versus amplitude for different stepped location values; first mode, Case-II. $\alpha = 0.5$

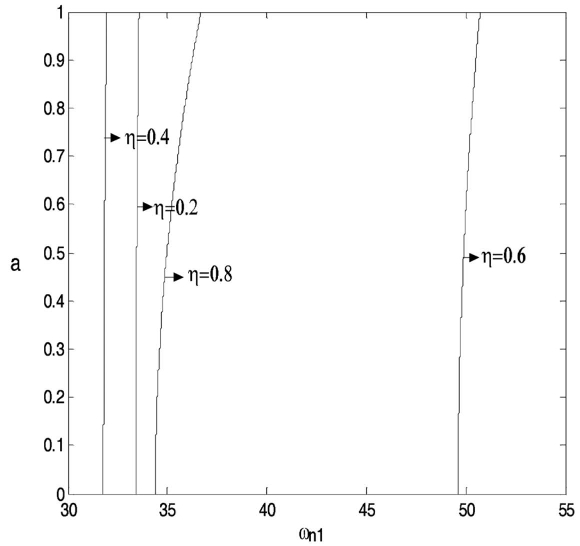

Fig. 4 Non-linear frequency versus amplitude for different stepped location values; first mode, Case-III. $\alpha = 3$

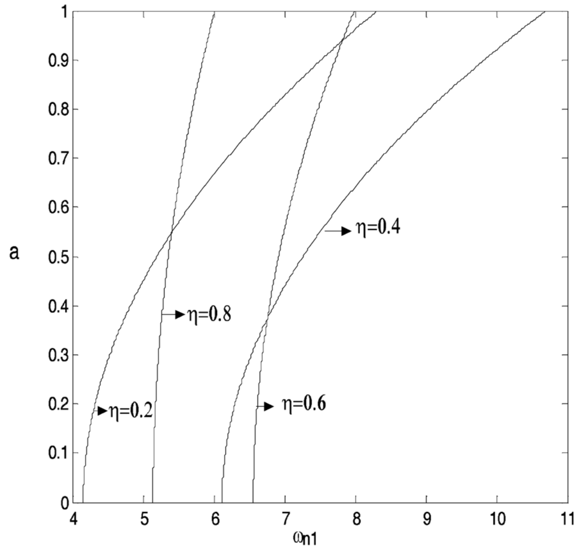

Fig. 5 Non-linear frequency versus amplitude for different stepped location values; first mode, Case-IV. $\alpha = 0.5$

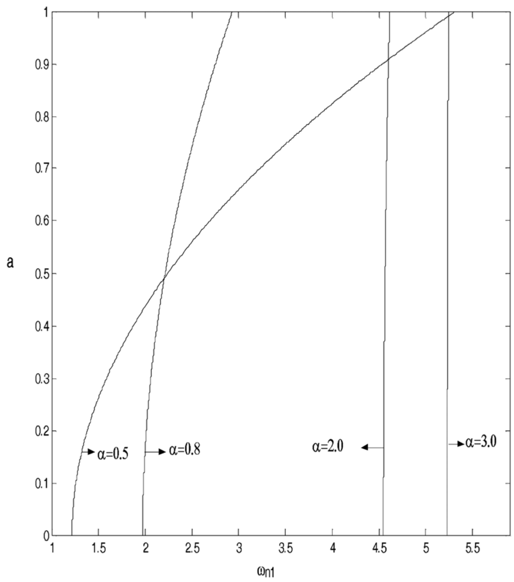

Fig. 6 Non-linear frequency versus amplitude for different stepped ration values; first mode, Case-V. $\eta = 0.2$

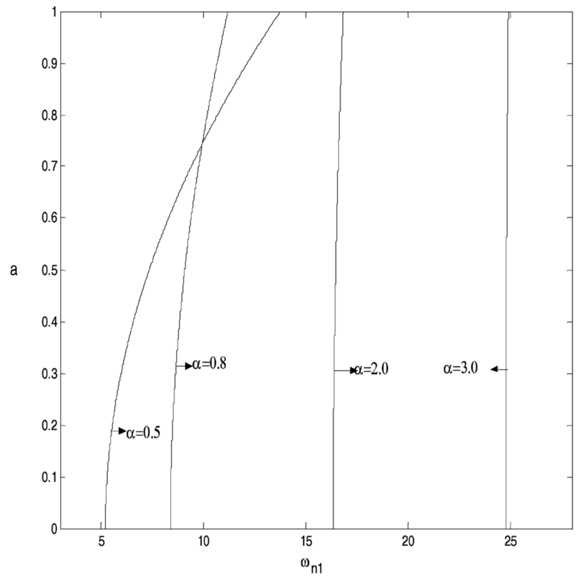

Fig. 7 Non-linear frequency versus amplitude for different stepped ration values; first mode, Case-VI. $\eta = 0.2$

and λ is defined in Eq. (45). The detuning parameter shows the nearness of the external excitation frequency to the natural frequency of system, several figures are drawn using Eq. (48) assuming $f = 1$ and damping coefficient $\mu = 0.2$.

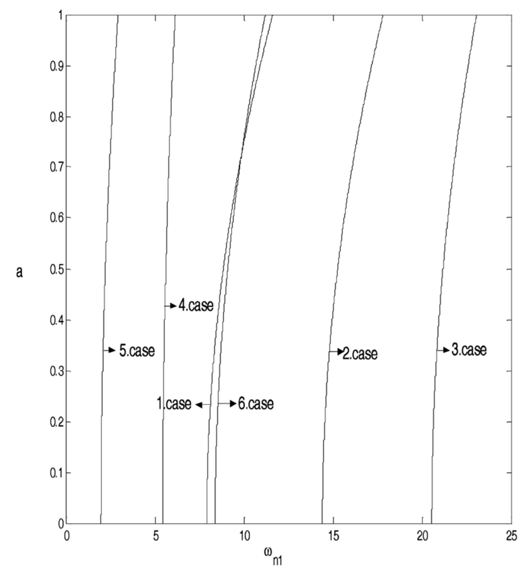

Fig. 8 Non-linear frequency versus amplitude for different boundary conditions; first mode, $\alpha = 0.8$, $\eta = 0.2$

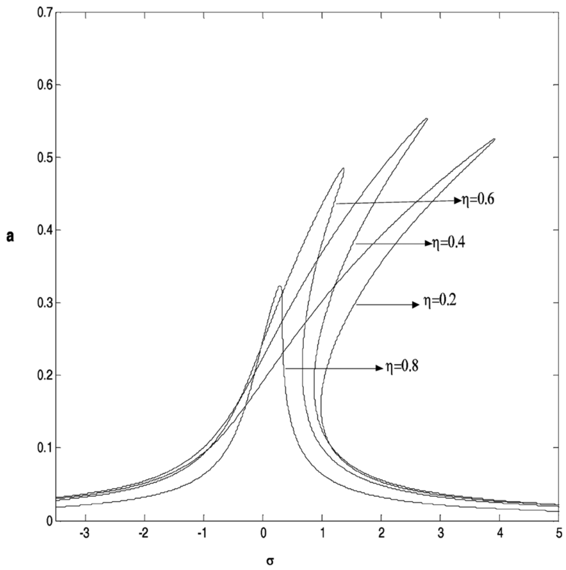

Fig. 9 Frequency-response curves for different stepped locations; first mode, Case-I $\alpha = 0.5$

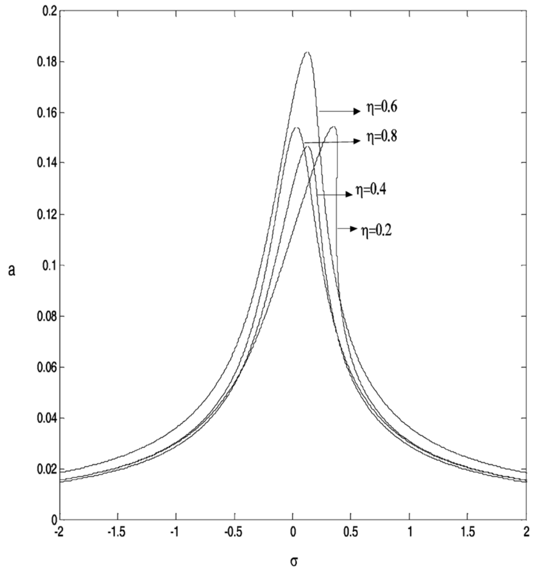

Fig. 10 Frequency-response curves for different stepped locations; first mode, Case-III. $\alpha = 0.5$

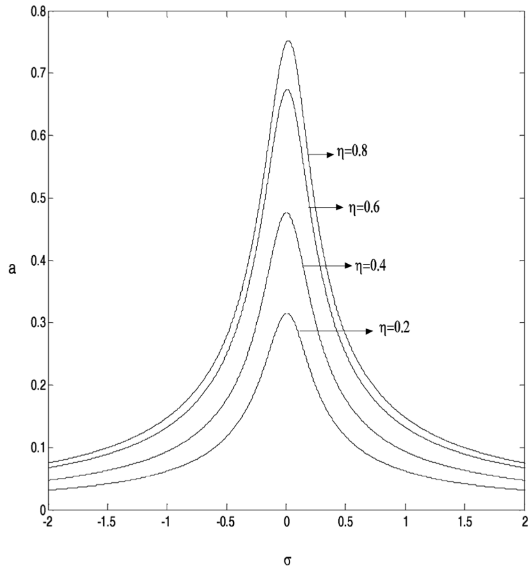

Fig. 11 Frequency-response curves for different stepped locations; first mode, Case-IV $\alpha = 3.0$

Frequency response curves are presented in Figs. 9-13. In Fig. 9, the frequency-response, curves for Case-I are shown for different η values ($\alpha = 0.5$). The effect of stretching bends the curves to the right causing multi-valued regions of solution. This phenomenon is the well-known jump

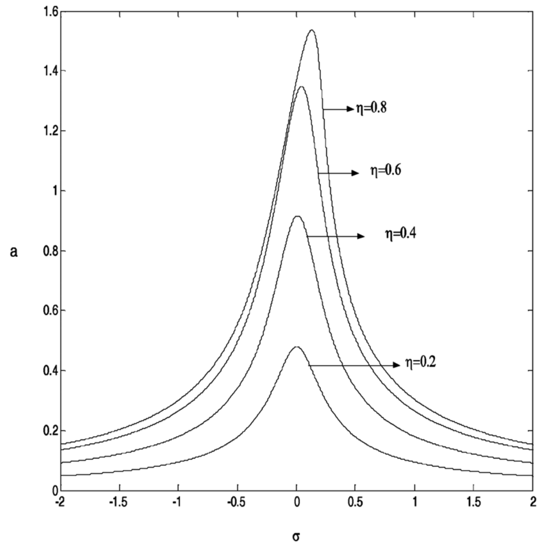

Fig. 12 Frequency-response curves for different stepped locations; first mode, Case-V. $\alpha = 3.0$

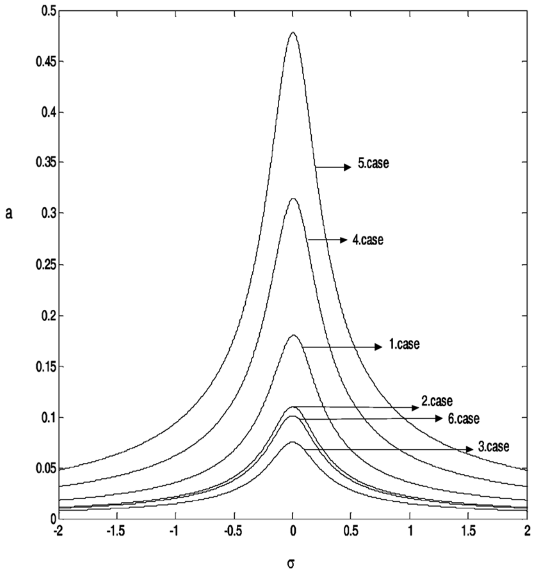

Fig. 13 Frequency-response curves for different end conditions; first mode, $\eta = 0.2$, $\alpha = 3.0$

phenomena. For Case I, when η decreases and provided that other parameters are kept constant, the multi – valued regions increases drastically. Fig. 10 shows frequency-response curves for Case-III ($\eta = 0.2$, 0.4, 0.6, 0.8). The effect of forcing is maximum when $\eta = 0.6$ and, is minimum when $\eta = 0.4$. In Fig. 11, the frequency-response curves for Case-IV are shown for different η values ($\alpha = 3$). When the step position is shifted from left to right, the amplitudes increase. This same result can be seen from the Fig. 12, which was drawn for Case-V. In Fig. 13, for fixed α and η ($\alpha = 3$, $\eta = 0.2$) frequency response curves for Case I-VI are shown on the same plot.

5. Conclusions

The non-linear response of a stepped beam supported by six different end conditions was investigated. The non-linear equations of motion including stretching due to immovable end conditions were derived. Forcing and damping terms were added to the equations. Linear and non-linear analyses were performed. Approximate solutions were searched by applying the method of multiple scales directly to the partial differential equations. The first term lead to the linear problem. Mode shapes and natural frequencies were calculated for different stepped ratios, step location and end conditions. The second terms provide the non-linear corrections to the linear problem. Non-linear frequency-amplitude and forcing frequency-amplitude relations were investigated and plotted.

As the step ratio is increased, the natural frequencies and nonlinear frequencies generally increase, but it decreases some case. One can observe that the stretching caused a non-linearity of the hardening type. When the ratio step is increased (α), the effect of stretching on the non-linear frequencies generally decreases. At the step ratios 0.5 and 0.8, when the step position (η) is increased, the effect of stretching on the non-linear frequencies (λ) generally decreases. At the step ratios 2 and 3, when the step position (η) is increased, the effect of stretching on the non-linear frequencies generally increases (λ).

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