Structural Engineering and Mechanics, Vol. 27, No. 1 (2007) 17-32
DOI: http://dx.doi.org/10.12989/sem.2007.27.1.017

Non-linear vibration and stability analysis of a partially supported conveyor belt by a distributed viscoelastic foundation

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(Received July 13, 2006, Accepted April 10, 2007)

Abstract. The main source of transverse vibration of a conveyor belt is frictional contact between pulley and belt. Also, environmental characteristics such as natural dampers and springs affect natural frequencies, stability and bifurcation points of system. These phenomena can be modeled by a small velocity fluctuation about mean velocity. Also, viscoelastic foundation can be modeled as the dampers and springs with continuous characteristics. In this study, non-linear vibration of a conveyor belt supported partially by a distributed viscoelastic foundation is investigated. Perturbation method is applied to obtain a closed form analytic solutions. Finally, numerical simulations are presented to show stiffness, damping coefficient, foundation length, non-linearity and mean velocity effects on location of bifurcation points, natural frequencies and stability of solutions.

Keywords: non-linear vibration; stability; perturbation method; viscoelastic foundation.

1. Introduction

Because of widespread usage of conveyor belt systems in many engineering devices, vibration of this class of systems has been lately considered by many researchers.

In classical belt models, speed is a constant (Wickert and Mote 1998, Wickert 1992, Pakdemirli and Ozkaya 1998), but recently some researchers have considered time-varying velocity and have investigated the effects of external frequency on system responses (Oz and Pakdemirli 1999, Oz *et al.* 2001, Chen *et al.* 2002, Chen and Yang 2005, Zhang and Chen 2005, Chen and Zhao 2005, Chen *et al.* 2005). The role of principle parametric resonance investigated analytically (Oz and Pakdemirli 1999). Multiple scales method is applied to the Euler-Bernoulli beam theory to solve non-linear equations analytically (Oz *et al.* 2001). Chaos and bifurcation of belt system were investigated (Chen *et al.* 2002). An axially moving, viscoelastic beam with time-dependent velocity is investigated Chen and Yang (2005). Non-linear behavior of viscoelastic moving string was studied (Zhang and Chen 2005).

Belt systems may be modeled as a string or a beam. Some researchers have considered it as a

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string, (Zhang and Chen 2005, Chen *et al*. 2005, Parker 1999, Wickert and Mote 1991, Hou and Zu 2002), and the others as a beam (Wickert 1992, Pakdemirli and Ozkaya 1998, Oz and Pakdemirli 1999, Oz *et al.* 2001, Chen and Yang 2005, Chen and Zhao 2005, Suweken and Van Horssen 2003). Free non-linear vibration of an axially moving beam in both sub and super critical constant speeds was investigated (Wickert 1992). Perturbation method was applied in order to obtain boundary layer solution for a beam under a constant transport speed (Pakdemirli and Ozkaya 1998). Initial-boundary-value problem equations for beam and string models were extracted under low velocity assumption (Suweken and Van Horssen 2003a, 2003b). Kartik and Wickert (2006) considered an axially moving string which is guided by a partial elastic foundation. The traveling speed was considered to be constant and the linear equation of motion was solved to obtain natural frequencies and mode shapes of the transversal vibration of system. Suweken and Van Horssen (2003c) used Hamilton's principle to obtain coupled longitudinal and transversal equations of the motion of a conveyor belt system, then used Kirchhoff's approach to reduce the equations to a single partial differential equation for the transverse vibration. The velocity in this investigation is considered to be small and time-variant. Chen (2005) considered an axially moving string. Particularly, transverse and parametric vibration due to axial speed variation was investigated.

In this study, free non-linear transverse vibration of a conveyor belt supported partially by a distributed viscoelastic foundation, is investigated. The multiple scales method is used in order to examine effects of mean velocity, stiffness, damping coefficient, coefficient of non-linearity and foundation length on natural frequencies, location of bifurcation points and stability of trivial and non-trivial solutions.

2. Model development

According to the real model, the main reason for transverse vibration of a conveyor belt is frictional contact between pulley and belt. This system can be modeled as an axially moving string under time-varying velocity which comprises a mean velocity plus small periodic fluctuations. There are some assumptions as below:

- The string model is considered (Suweken and Van Horssen 2003, Hou and Zu 2002)
- Only the transversal displacement is assumed
- Variations of cross-sectional dimensions are assumed to be negligible
- The axial stiffness is supposed to be large enough to neglect deformation resulting from pretension.

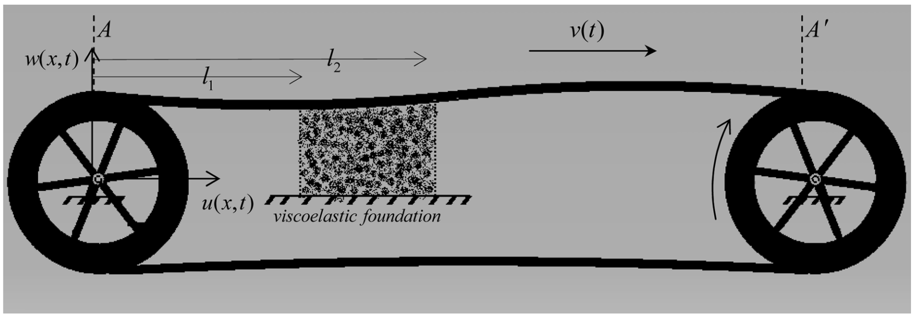


Fig. 1 Pulley-belt system

After all these assumptions energy method is used to obtain partial-differential equation of motion for a conveyor belt which is supported partially by a distributed viscoelastic foundation (Fig. 1).

Let us consider

$$\zeta = \sqrt{\frac{c^2L^2}{\rho A P_0}}, \quad \kappa = \sqrt{\frac{kL^2}{P_0}}, \quad v_1 = \sqrt{\frac{EA}{P_0}} \tag{1}$$

$$x^* = \frac{x}{L}, \quad w^* = \frac{w}{L}, \quad t^* = t\sqrt{\frac{P_0}{\rho A L^2}}, \quad v^* = \frac{v}{\sqrt{\frac{P_0}{\rho A}}} \tag{2}$$

$$H(x-l) = \begin{cases} 1 & x > l \\ 0 & x < l \end{cases} \tag{3}$$

$$v(t) = v_0 + \varepsilon v_1 \sin\omega t \tag{4}$$

Where $\omega, P_0, c, k, EA, \rho A, L, v_0, v$ and ε are frequency of speed, pretension, damping coefficient per unit length, stiffness coefficient per unit length, axial stiffness, mass per unit length, length of model, mean velocity, transport speed and small amount ($\varepsilon << 1$), respectively.

Considering only the transverse vibration, the kinetic and potential energy of the segment $A - A'$ in Fig. 1 is given by Wickert (1992)

$$T = \frac{1}{2}\rho A\int_0^L \left(\frac{\partial w}{\partial t} + v\frac{\partial w}{\partial x}\right)^2 dx \tag{5}$$

$$U = \int_0^L \left(P_0 e_{xx} + \frac{1}{2}EAe_{xx}^2\right)dx \tag{6}$$

in which the e_{xx} is nonlinear strain and can be illustrated by

$$e_{xx} = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{7}$$

The variation of works of the conservative and non-conservative forces can be written as

$$\begin{aligned} \delta w_c + \delta w_{nc} = & -\int_0^L kw[H(x-l_1) - H(x-l_2)]\delta w dx - \\ & \int_0^L c[H(x-l_1) - H(x-l_2)]\left(\frac{\partial w}{\partial t} + v\frac{\partial w}{\partial x}\right)\delta w dx \end{aligned} \tag{8}$$

The Hamilton's principle is used to derive the equation of the motion and boundary conditions

$$\delta I = \int_{t_1}^{t_2} (\delta T - \delta U + \delta w_c + \delta w_{nc})dt = 0 \tag{9}$$

Substituting the Eqs. (5)-(8) into the Eq. (9), the equation of the motion and boundary conditions can be obtained as

$$\rho A\left[\frac{\partial^2 w}{\partial t^2}+\frac{\partial v}{\partial t}\frac{\partial w}{\partial x}+2v\frac{\partial^2 w}{\partial x\partial t}+v^2\frac{\partial^2 w}{\partial x^2}\right]-P_0\frac{\partial^2 w}{\partial x^2}-\frac{3}{2}EA\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2}$$
$$+kw[H(x-l_1)-H(x-l_2)]+c[H(x-l_1)-H(x-l_2)]\left(\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial x}\right) = 0 \tag{10}$$

$$w(0,t) = w(1,t) = 0 \tag{11}$$

Substituting of Eqs. (1) and (2) into Eqs. (10) and (11), with getting rid of the symbol "*", one has

$$\frac{\partial^2 w}{\partial t^2}+\frac{\partial v}{\partial t}\frac{\partial w}{\partial x}+2v\frac{\partial^2 w}{\partial x\partial t}+(v^2-1)\frac{\partial^2 w}{\partial x^2}+\kappa^2 w[H(x-l_1)-H(x-l_2)]$$
$$+\zeta[H(x-l_1)-H(x-l_2)]\left(\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial x}\right) = \frac{3}{2}v_1^2\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2} \tag{12}$$

$$w(0,t) = w(1,t) = 0 \tag{13}$$

Real systems show that transverse vibrations are very small, then it makes the non-linear term of system weak. Under transformation $w = \sqrt{\varepsilon}w^{**}$ ($\varepsilon << 1$) and getting rid of the symbol "**", the Eqs. (12) and (13) will lead to

$$\frac{\partial^2 w}{\partial t^2}+\frac{\partial v}{\partial t}\frac{\partial w}{\partial x}+2v\frac{\partial^2 w}{\partial x\partial t}+(v^2-1)\frac{\partial^2 w}{\partial x^2}+\kappa^2 w[H(x-l_1)-H(x-l_2)]$$
$$+\zeta[H(x-l_1)-H(x-l_2)]\left(\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial x}\right) = \frac{3}{2}\varepsilon v_1^2\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2} \tag{14}$$

$$w(0,t) = w(1,t) = 0 \tag{15}$$

3. Perturbation method, stability and bifurcations

Analytical methods often easily delineate general phenomena, yielding useful results in closed form (Nayfeh 1993). The simple asymptotic expansions often fail to correctly result in appropriate solutions for problems which have secular terms. Using the method of multiple scales and assuming the solution to be a function of multiple independent scales of time, this method leads to a set of equations in different orders. Elimination of secular terms from these equations provides solutions. For more general form of the Multiple Scales Method see (Nayfeh 1981).

In Perturbation Method, $w(x, t, \varepsilon)$ is generally assumed as an asymptotically expansion (Thomsen 2003, Kevorkian and Cole 1981).

$$w(x,t;\varepsilon) = \sum_{j=0}^{1}\varepsilon^j w_j(x, T_0, T_1) + O(\varepsilon^2) \tag{16}$$

Substitution of Eq. (16) into Eq. (14)

$$O(\varepsilon^0): \quad \frac{\partial^2 w_0}{\partial T_0^2} + 2v_0\frac{\partial^2 w_0}{\partial T_0 \partial x} + (v_0^2 - 1)\frac{\partial^2 w_0}{\partial x^2} + \kappa^2[H(x-l_1) - H(x-l_2)]w_0$$
$$+ \zeta[H(x-l_1) - H(x-l_2)]\left[\frac{\partial w_0}{\partial T_0} + v_0\frac{\partial w_0}{\partial x}\right] = 0 \tag{17}$$

$$O(\varepsilon): \quad \frac{\partial^2 w_1}{\partial T_0^2} + 2v_0\frac{\partial^2 w_1}{\partial T_0 \partial x} + (v_0^2 - 1)\frac{\partial^2 w_1}{\partial x^2} + \kappa^2[H(x-l_1) - H(x-l_2)]w_1$$
$$+ \zeta[H(x-l_1) - H(x-l_2)]\left[\frac{\partial w_1}{\partial T_0} + v_0\frac{\partial w_1}{\partial x}\right] = -2\frac{\partial^2 w_0}{\partial T_1 dT_0} - \frac{\partial w_0}{\partial x}v_1\omega\cos(\omega T_0)$$
$$-2v_0\frac{\partial^2 w_0}{\partial T_1 \partial x} - 2\left(\frac{\partial^2 w_0}{\partial T_0 \partial x}\right)v_1\sin(\omega T_0) - 2\frac{\partial^2 w_0}{\partial x^2}v_0 v_1\sin(\omega T_0) \tag{18}$$
$$-\zeta[H(x-l_1) - H(x-l_2)]\left[\frac{\partial w_0}{\partial T_1} + v_0\frac{\partial w_0}{\partial x}\sin(\omega T_0)\right]$$
$$+\frac{3}{2}v_1^2\left(\frac{\partial w_0}{\partial x}\right)^2\frac{\partial^2 w_0}{\partial x^2}$$

in which $T_0 = t$ and $T_1 = \varepsilon t$.

Assuming the solution of Eq. (17) as (Wickert 1992, Oz and Pakdemirli 1999, Oz *et al.* 2001, Chen and Yang 2005, Hou and Zu 2002)

$$w_0(x, T_0, T_1) = \sum_{n=1}^{\infty}[X_n(T_1)e^{i\omega_n T_0}w_n(x) + \bar{X}_n(T_1)e^{-i\omega_n T_0}\bar{w}_n(x)] \tag{19}$$

in which ω_n is the nth natural frequency, $X_n(T_1)$ is the nth amplitude and $w_n(x)$ is the nth complex mode shape.

Substitution of Eq. (19) into Eq. (17) yields

$$-\omega_n^2 w_n(x) + 2iv_0\omega_n\frac{\partial w_n(x)}{\partial x} + (v_0^2 - 1)\frac{\partial^2 w_n(x)}{\partial x^2} + \kappa^2 w_n(x)[H(x-l_1) - H(x-l_2)]$$
$$+\zeta[H(x-l_1) - H(x-l_2)]\left[i\omega_n w_n(x) + v_0\frac{\partial w_n(x)}{\partial x}\right] = 0 \tag{20}$$

In the spans $(0 < x < l_1)$ and $(l_2 < x < l)$, one assumes that mode shapes $w_n(x)$ is of the form $w_n(x) = \exp(i\mu_n x)$, then the Eq. (20) can be rewritten as

$$\mu_n^2(v_0^2 - 1) + 2v_0\omega_n\mu_n + \omega_n^2 = 0 \tag{21}$$

Solving the Eq. (21) for μ_n will lead to

$$\mu_{1n} = \frac{-\omega_n}{v_0 + 1} \tag{22}$$

$$\mu_{2n} = \frac{-\omega_n}{v_0 - 1} \tag{23}$$

Then the nth mode shape in the region $(0 < x < l_1)$ becomes

$$w_n^{(1)}(x) = c_{1n} e^{i\left(\frac{-\omega_n}{v_0+1}\right)x} + c_{2n} e^{i\left(\frac{-\omega_n}{v_0-1}\right)x} \tag{24}$$

and in the region $(l_2 < x < l)$ becomes

$$w_n^{(3)}(x) = c_{1n}^{*} e^{i\left(\frac{-\omega_n}{v_0+1}\right)x} + c_{2n}^{*} e^{i\left(\frac{-\omega_n}{v_0-1}\right)x} \tag{25}$$

where $c_{1n}, c_{2n}, c_{1n}^{*}$ and c_{2n}^{*} are constants.

In the span $(l_1 < x < l_2)$ which is located on the viscoelastic foundation, assuming the nth mode shape $w_n(x)$ of the form $w_n(x) = \exp(i\beta_n x)$, the Eq. (20) will lead to

$$\beta_n^2(v_0^2 - 1) + \beta_n(2v_0\omega_n - i\zeta v_0) + \omega_n^2 - \kappa^2 - i\omega_n\zeta = 0 \tag{26}$$

Solving the Eq. (26) for β_n will lead to

$$\beta_{1n,2n} = \frac{i\zeta v_0 - 2v_0\omega_n \pm \sqrt{4\kappa^2(v_0^2-1) + 4\omega_n^2 - 4i\omega_n\zeta - \zeta^2 v_0^2}}{(v_0^2 - 1)} \tag{27}$$

and the nth mode shape of this region $(l_1 < x < l_2)$ is

$$w_n^{(2)}(x) = c_{1n}' e^{i\beta_{1n}x} + c_{2n}' e^{i\beta_{2n}x} \tag{28}$$

where the c_{1n}' and c_{2n}' are constants.

The boundary and compatibility conditions of the system are

$$\begin{gathered} w_n^{(1)}(0) = 0, \quad w_n^{(1)}(l_1) = w_n^{(2)}(l_1), \quad w_n^{(2)}(l_2) = w_n^{(3)}(l_2), \quad w_n^{(3)}(l) = 0 \\ \left.\frac{\partial w_n^{(1)}}{\partial x}\right|_{x=l_1} = \left.\frac{\partial w_n^{(2)}}{\partial x}\right|_{x=l_1}, \quad \left.\frac{\partial w_n^{(2)}}{\partial x}\right|_{x=l_2} = \left.\frac{\partial w_n^{(3)}}{\partial x}\right|_{x=l_2} \end{gathered} \tag{29}$$

In the Eqs. (24), (25) and (28) the constants $c_{1n}, c_{2n}, c_{1n}^{*}, c_{2n}^{*}, c_{1n}'$ and c_{2n}' can be obtained through boundary and compatibility conditions. Then using the Eqs. (24), (25), (28) and (29) one has

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ e^{i\mu_{1n}l_1} & e^{i\mu_{2n}l_1} & -e^{i\beta_{1n}l_1} & -e^{i\beta_{2n}l_1} & 0 & 0 \\ i\mu_{1n}e^{i\mu_{1n}l_1} & i\mu_{2n}e^{i\mu_{2n}l_1} & -i\beta_{1n}e^{i\beta_{1n}l_1} & -i\beta_{2n}e^{i\beta_{2n}l_1} & 0 & 0 \\ 0 & 0 & e^{i\beta_{1n}l_2} & e^{i\beta_{2n}l_2} & -e^{i\mu_{1n}l_2} & -e^{i\mu_{2n}l_2} \\ 0 & 0 & i\beta_{1n}e^{i\beta_{1n}l_2} & i\beta_{2n}e^{i\beta_{2n}l_2} & -i\mu_{1n}e^{i\mu_{1n}l_2} & -i\mu_{2n}e^{i\mu_{2n}l_2} \\ 0 & 0 & 0 & 0 & e^{i\mu_{1n}l} & e^{i\mu_{2n}l} \end{bmatrix} \begin{bmatrix} c_{1n} \\ c_{2n} \\ c_{1n}' \\ c_{2n}' \\ c_{1n}^{*} \\ c_{2n}^{*} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{30}$$

To have the non-trivial solution, the determinant of the coefficient matrix of the Eq. (30) should be equal to zero (Oz and Pakdemirli 1999), so the *n*th natural frequency can be easily obtained through numerical simulations.

Elimination process in the Eq. (30) would lead to (the c_{2n}^* is assumed to be unity)

$$c_{1n}^* = -e^{i(\mu_{2n}-\mu_{1n})l} \tag{31}$$

$$c'_{1n} = \frac{(\mu_{1n}-\beta_{2n})e^{i(\mu_{1n}l_2+\mu_{2n}l-\mu_{1n}l+\beta_2 l_2)} + (\beta_{2n}-\mu_{2n})e^{i(\beta_{2n}+\mu_{2n})l_2}}{(\beta_{2n}-\beta_{1n})e^{i(\beta_{1n}+\beta_{2n})l_2}} \tag{32}$$

$$c'_{2n} = \frac{(\beta_{1n}-\mu_{1n})e^{i(\mu_{1n}l_2+\mu_{2n}l-\mu_{1n}l+\beta_1 l_2)} + (\mu_{2n}-\beta_{1n})e^{i(\beta_{1n}+\mu_{2n})l_2}}{(\beta_{2n}-\beta_{1n})e^{i(\beta_{1n}+\beta_{2n})l_2}} \tag{33}$$

$$c_{1n} = \frac{c'_{1n}(\mu_{2n}-\beta_{1n})e^{i(\beta_{1n}+\mu_{2n})l_1} + c'_{2n}(\mu_{2n}-\beta_{2n})e^{i(\beta_{2n}+\mu_{2n})l_1}}{(\mu_{2n}-\mu_{1n})e^{i(\mu_{1n}+\mu_{2n})l_1}} \tag{34}$$

$$c_{2n} = \frac{c'_{1n}(\beta_{1n}-\mu_{1n})e^{i(\beta_{1n}+\mu_{1n})l_1} + c'_{2n}(\beta_{2n}-\mu_{1n})e^{i(\beta_{2n}+\mu_{1n})l_1}}{(\mu_{2n}-\mu_{1n})e^{i(\mu_{1n}+\mu_{2n})l_1}} \tag{35}$$

In the case in which the speed frequency is near to the twice of any natural frequency of the system, principal parametric resonance would arise (Oz and Pakdemirli 1999, Oz *et al.* 2001, Chen and Yang 2005). To investigate the *n*th principal resonance, assuming there is no internal resonance, the generality for w_0 will not be lost if only *n*th mode of the vibration is considered (Oz and Pakdemirli 1999, Oz *et al.* 2001, Chen and Yang 2005, Hou and Zu 2002)

$$w_0(x, T_0, T_1) = X_n(T_1)e^{i\omega_n T_0}w_n(x) + \overline{X}_n(T_1)e^{-i\omega_n T_0}\overline{w}_n(x) \tag{36}$$

Nearness of the speed frequency to the twice of the *n*th natural frequency can be illustrated by

$$\omega = 2\omega_n + \varepsilon\sigma \tag{37}$$

in which σ is detuning parameter.

Substitution of the Eq. (36) into the Eq. (18) yields

$$O(\varepsilon): \quad \frac{\partial^2 w_1}{\partial T_0^2} + 2v_0\frac{\partial^2 w_1}{\partial T_0 \partial x} + (v_0^2-1)\frac{\partial^2 w_1}{\partial x^2} + \kappa^2[H(x-l_1)-H(x-l_2)]w_1$$
$$+\zeta[H(x-l_1)-H(x-l_2)]\left[\frac{\partial w_1}{\partial T_0} + v_0\frac{\partial w_1}{\partial x}\right] = e^{i\omega_n T_0}\left[-2i\omega_n\frac{dX}{dT_1}w_n\right.$$
$$-2v_0\frac{dX_n}{dT_1}\frac{dw_n}{dx} - \zeta[H(x-l_1)-H(x-l_2)]\frac{dX_n}{dT_1}w_n$$
$$\left.+\frac{3}{2}v_1^2X_n^2\overline{X}_n\left(\frac{dw_n}{dx}\right)^2\frac{d^2\overline{w}_n}{dx^2} + 3v_1^2X_n^2\overline{X}_n\frac{d^2w_n}{dx^2}\frac{dw_n}{dx}\frac{d\overline{w}_n}{dx}\right] +$$

$$+e^{i(\omega+\omega_n)T_0}\begin{bmatrix} -\frac{1}{2}v_1\omega X_n\frac{dw_n}{dx} - v_1\omega_n X_n\frac{dw_n}{dx} + iv_0v_1X_n\frac{d^2w_n}{dx^2} \\ +\frac{1}{2}\zeta[H(x-l_1)-H(x-l_2)]iv_1X_n\frac{dw_n}{dx} \end{bmatrix}$$

$$+e^{i(\omega-\omega_n)T_0}\begin{bmatrix} -\frac{1}{2}v_1\omega\overline{X}_n\frac{d\overline{w}_n}{dx} + v_1\omega_n\overline{X}_n\frac{d\overline{w}_n}{dx} + iv_0v_1\overline{X}_n\frac{d^2\overline{w}_n}{dx^2} \\ +\frac{1}{2}\zeta[H(x-l_1)-H(x-l_2)]iv_1\overline{X}_n\frac{d\overline{w}_n}{dx} \end{bmatrix}$$

$$+cc+NST \tag{38}$$

where "*cc*" and *NST* stand for complex conjugate and non-secular terms.

Using the Eqs. (37) and (38), the solvability condition will lead to Nayfeh (1981)

$$\frac{dX_n}{dT_1}[-2i\omega_n C_1 - 2v_0C_2 - \zeta C_3] + 3X_n^2\overline{X}_n v_1^2\left[\frac{1}{2}C_4 + C_5\right]$$
$$+\overline{X}_n e^{i\sigma T_1}v_1\left[\left(\omega_n - \frac{1}{2}\omega\right)C_6 + iv_0C_7 + \frac{1}{2}i\zeta C_8\right] = 0 \tag{39}$$

in which

$$\left\{\begin{aligned}
C_1 &= \int_0^{l_1} w_n^{(1)}\overline{w}_n^{(1)}dx + \int_{l_1}^{l_2} w_n^{(2)}\overline{w}_n^{(2)}dx + \int_{l_2}^{l} w_n^{(3)}\overline{w}_n^{(3)}dx \\
C_2 &= \int_0^{l_1}\frac{\partial w_n^{(1)}}{\partial x}\overline{w}_n^{(1)}dx + \int_{l_1}^{l_2}\frac{\partial w_n^{(2)}}{\partial x}\overline{w}_n^{(2)}dx + \int_{l_2}^{l}\frac{\partial w_n^{(3)}}{\partial x}\overline{w}_n^{(3)}dx \\
C_3 &= \int_{l_1}^{l_2} w_n^{(2)}\overline{w}_n^{(2)}dx \\
C_4 &= \int_0^{l_1}\left(\frac{\partial w_n^{(1)}}{\partial x}\right)^2\frac{\partial^2\overline{w}_n^{(1)}}{\partial x^2}\overline{w}_n^{(1)}dx + \int_{l_1}^{l_2}\left(\frac{\partial w_n^{(2)}}{\partial x}\right)^2\frac{\partial^2\overline{w}_n^{(2)}}{\partial x^2}\overline{w}_n^{(2)}dx + \int_{l_2}^{l}\left(\frac{\partial w_n^{(3)}}{\partial x}\right)^2\frac{\partial^2\overline{w}_n^{(3)}}{\partial x^2}\overline{w}_n^{(3)}dx \\
C_5 &= \int_0^{l_1}\frac{\partial^2 w_n^{(1)}}{\partial x^2}\frac{\partial w_n^{(1)}}{\partial x}\frac{\partial\overline{w}_n^{(1)}}{\partial x}\overline{w}_n^{(1)}dx + \int_{l_1}^{l_2}\frac{\partial^2 w_n^{(2)}}{\partial x^2}\frac{\partial w_n^{(2)}}{\partial x}\frac{\partial\overline{w}_n^{(2)}}{\partial x}\overline{w}_n^{(2)}dx + \int_{l_2}^{l}\frac{\partial^2 w_n^{(3)}}{\partial x^2}\frac{\partial w_n^{(3)}}{\partial x}\frac{\partial\overline{w}_n^{(3)}}{\partial x}\overline{w}_n^{(3)}dx \\
C_6 &= \int_0^{l_1}\frac{\partial\overline{w}_n^{(1)}}{\partial x}\overline{w}_n^{(1)}dx + \int_{l_1}^{l_2}\frac{\partial\overline{w}_n^{(2)}}{\partial x}\overline{w}_n^{(2)}dx + \int_{l_2}^{l}\frac{\partial\overline{w}_n^{(3)}}{\partial x}\overline{w}_n^{(3)}dx \\
C_7 &= \int_0^{l_1}\frac{\partial^2\overline{w}_n^{(1)}}{\partial x^2}\overline{w}_n^{(1)}dx + \int_{l_1}^{l_2}\frac{\partial^2\overline{w}_n^{(2)}}{\partial x^2}\overline{w}_n^{(2)}dx + \int_{l_2}^{l}\frac{\partial^2\overline{w}_n^{(3)}}{\partial x^2}\overline{w}_n^{(3)}dx \\
C_8 &= \int_{l_1}^{l_2}\frac{\partial\overline{w}_n^{(2)}}{\partial x}\overline{w}_n^{(2)}dx
\end{aligned}\right\} \tag{40}$$

Assuming the $X_n(T_1)$ of the polar form

$$X_n(T_1) = \frac{1}{2}x_n e^{i\alpha_n} \tag{41}$$

and substituting the Eq. (41) into the Eq. (39) results in

$$\frac{dx_n}{dT_1} = x_n \times \mathrm{Im}\left\{\frac{(0.5v_1\omega - v_1\omega_n)C_6 - iv_0v_1C_7 - 0.5i\zeta v_1C_8}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right\}\sin(\eta_n)$$

$$-x_n \times \mathrm{Re}\left\{\frac{(0.5v_1\omega - v_1\omega_n)C_6 - iv_0v_1C_7 - 0.5i\zeta v_1C_8}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right\}\cos(\eta_n) \quad (42)$$

$$+ 0.25x_n^3 \times \mathrm{Re}\left\{\frac{\frac{3}{2}v_1^2C_4 + 3v_1^2C_5}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right\}$$

$$\frac{d\eta_n}{dT_1} = 2\mathrm{Im}\left\{\frac{(0.5v_1\omega - v_1\omega_n)C_6 - iv_0v_1C_7 - 0.5i\zeta v_1C_8}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right\}\cos(\eta_n)$$

$$+ 2\mathrm{Re}\left\{\frac{(0.5v_1\omega - v_1\omega_n)C_6 - iv_0v_1C_7 - 0.5i\zeta v_1C_8}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right\}\sin(\eta_n) \quad (43)$$

$$- 0.5x_n^2 \times \mathrm{Im}\left\{\frac{\frac{3}{2}v_1^2C_4 + 3v_1^2C_5}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right\} + \sigma$$

where $\eta_n = \sigma T_1 - 2\alpha_n$.

At stationary response ($t \to \infty$), eliminating the η_n between Eqs. (42) and (43) results in

$$\sigma_{1,2} = \mp\left[\left|\frac{(0.5v_1\omega - v_1\omega_n)C_6 - iv_0v_1C_7 - 0.5i\zeta v_1C_8}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right|^2 - \left(0.25x_n^2\mathrm{Re}\left\{\frac{\frac{3}{2}v_1^2C_4 + 3v_1^2C_5}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right\}\right)^2\right]^{0.5}$$

$$+ 0.5x_n^2\mathrm{Im}\left\{\frac{\frac{3}{2}v_1^2C_4 + 3v_1^2C_5}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right\} \quad (44)$$

Constructing the Jacobian matrix from Eqs. (42) and (43) and evaluating the eigenvalues will lead to

$$\left[0.5x_n^2\mathrm{Re}\left\{\frac{\frac{3}{2}v_1^2C_4 + 3v_1^2C_5}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right\} - \lambda\right]^2 + 0.5x_n^2\mathrm{Im}\left\{\frac{\frac{3}{2}v_1^2C_4 + 3v_1^2C_5}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right\} \times$$

$$\left[0.5x_n^2\mathrm{Im}\left\{\frac{\frac{3}{2}v_1^2C_4 + 3v_1^2C_5}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right\} - \sigma\right] = 0 \quad (45)$$

From Eqs. (44) and (45), through Routh-Hurwitz criterion, the stability conditions for the first detuning parameter, σ_1, become

$$\left|\frac{(0.5v_1\omega - v_1\omega_n)C_6 - iv_0v_1C_7 - 0.5i\zeta v_1C_8}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right|^2 > \left(0.25x_n^2\mathrm{Re}\left\{\frac{\frac{3}{2}v_1^2C_4 + 3v_1^2C_5}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right\}\right)^2$$

$$\mathrm{Re}\left\{\frac{\frac{3}{2}v_1^2C_4 + 3v_1^2C_5}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right\} < 0, \quad \mathrm{Im}\left\{\frac{\frac{3}{2}v_1^2C_4 + 3v_1^2C_5}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right\} > 0 \tag{46}$$

and for the second detuning parameter (σ_2), the stability conditions will lead to

$$\left|\frac{(0.5v_1\omega - v_1\omega_n)C_6 - iv_0v_1C_7 - 0.5i\zeta v_1C_8}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right|^2 > \left(0.25x_n^2\mathrm{Re}\left\{\frac{\frac{3}{2}v_1^2C_4 + 3v_1^2C_5}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right\}\right)^2$$

$$\mathrm{Re}\left\{\frac{\frac{3}{2}v_1^2C_4 + 3v_1^2C_5}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right\} < 0, \quad \mathrm{Im}\left\{\frac{\frac{3}{2}v_1^2C_4 + 3v_1^2C_5}{2i\omega_nC_1 + 2v_0C_2 + \zeta C_3}\right\} < 0 \tag{47}$$

Using the Eqs. (22), (23) and (27) and setting determinant of the coefficient matrix of the Eq. (30) equal to zero, the natural frequencies of the system can be obtained. Then, one may investigate the effects of system parameters on natural frequencies. Fig. 2(a) shows that increasing not only the mean velocity but also the damping coefficient factor would lead to a reduction in real part of natural frequencies. Also, Fig. 2(b) shows that the increasing mean velocity would lead to a reduction in imaginary part of natural frequencies, while the increasing damping coefficient increase the imaginary part of the natural frequencies.

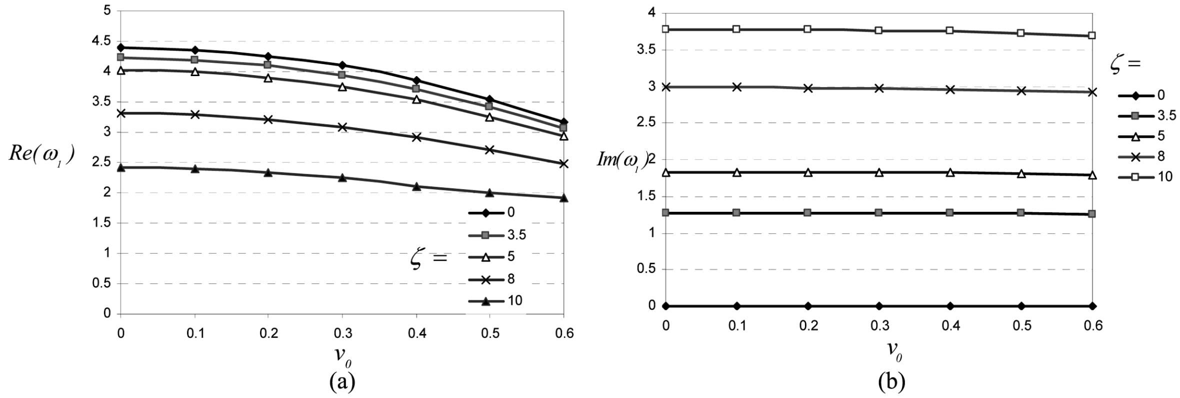
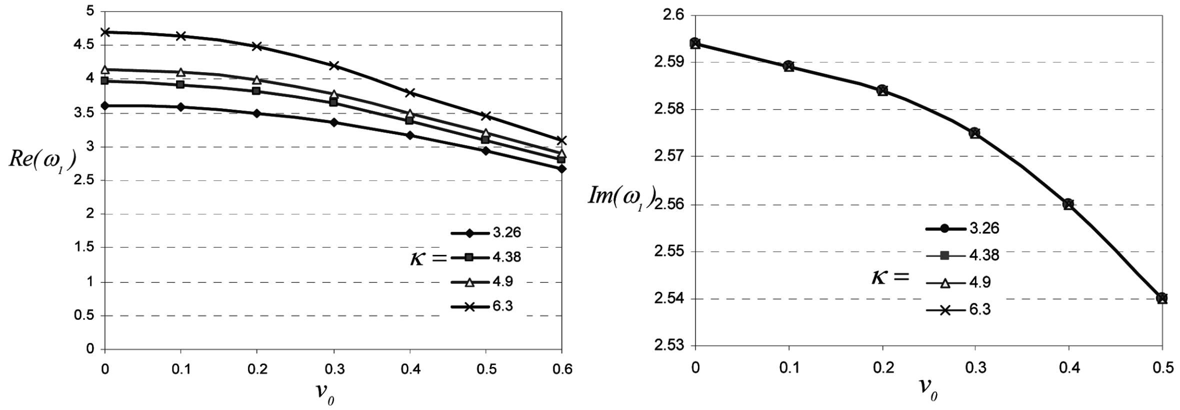
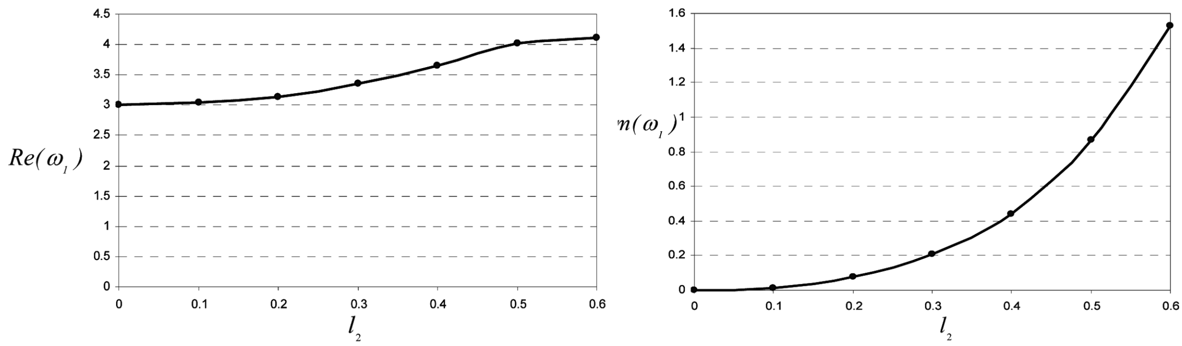


Fig. 2 (a) Real part of the first natural frequency versus the mean velocity and damping coefficient factor, (b) Imaginary part of the first natural frequency versus the mean velocity and damping coefficient factor ($l_1 = 0.2$, $l_2 = 0.8$, $l = 1$, $\kappa = 3.26$)



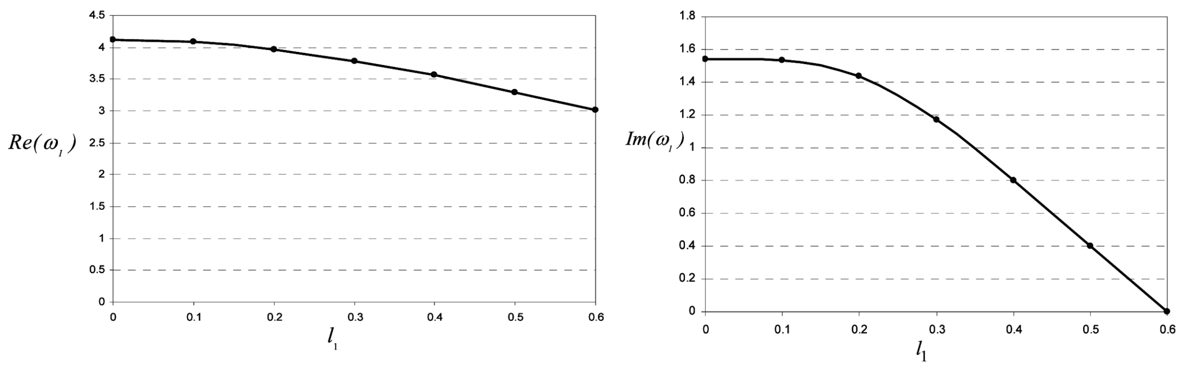
(a) (b)

Fig. 3 (a) Real part of the first natural frequency versus the mean velocity and stiffness factor, (b) Imaginary part of the first natural frequency versus the mean velocity and stiffness factor ($l_1 = 0.2$, $l_2 = 0.8$, $l = 1$, $\zeta = 7$)



(a) (b)

Fig. 4 (a) Real part of the first natural frequency versus the l_2, (b) Imaginary part of the first natural frequency versus the l_2. ($l_1 = 0$, $l = 1$, $\zeta = 5.5$, $\kappa = 3.26$, $v_0 = 0.2$)



(a) (b)

Fig. 5 (a) Real part of the first natural frequency versus the l_1, (b) Imaginary part of the first natural frequency versus the l_1 ($l_2 = 0.6$, $l = 1$, $\zeta = 5.5$, $\kappa = 3.26$, $v_0 = 0.2$)

Fig. 3(a) shows that increasing the mean velocity will reduce the real part of natural frequencies but increasing stiffness factor will increase the real part of natural frequencies. Fig. 3(b) shows that the increasing mean velocity would lead to a reduction in imaginary part of natural frequencies, while the increasing stiffness factor makes the imaginary part of natural frequencies be a constant.

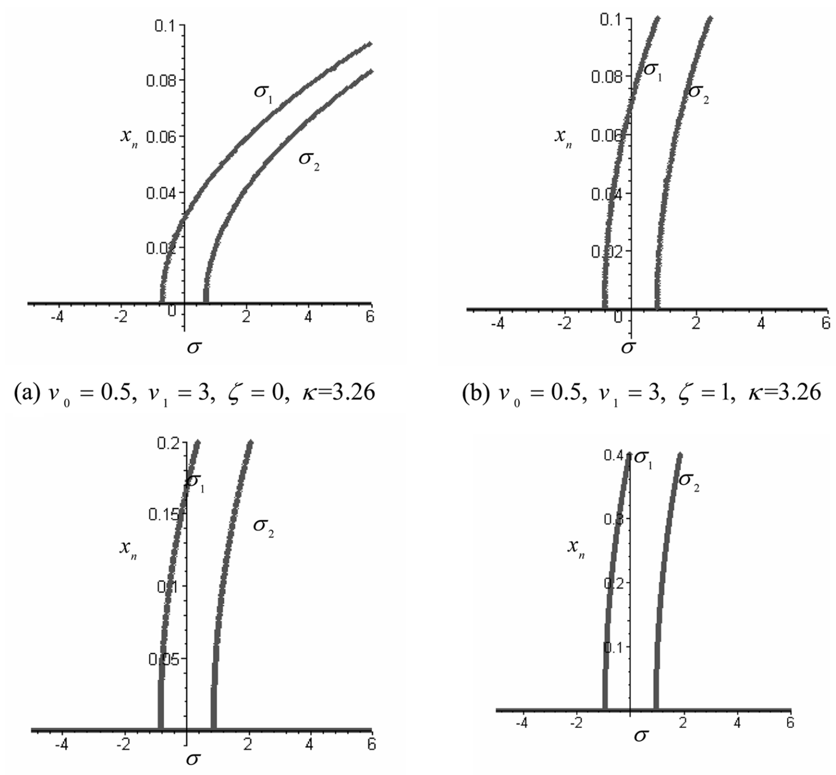
Fig. 4 shows that the increasing l_2 makes both real and imaginary part of natural frequencies increase. Fig. 5 shows that the decreasing viscoelastic foundation length will decrease the natural frequencies.

Next, the numerical simulations are presented to illustrate how stiffness and damping coefficient variations, viscoelastic foundation length, and coefficient of the non-linearity will affect the location of bifurcation points and stability of solutions.

In Fig. 6, when $\sigma < \sigma_1$, there is a stable trivial solution. When $\sigma = \sigma_1$, the trivial solution starts to be unstable and a stable nontrivial solution bifurcates. When $\sigma = \sigma_2$, the trivial solution starts to be stable again, and then, an unstable nontrivial solution appears. Increasing damping coefficient will lead to a larger instability interval for trivial solution (It is not obvious in Fig. 6). It means that the bifurcation point will appear earlier.

Also numerical simulations (Fig. 7) show that, increasing stiffness factor leads to a smaller instability interval for trivial solution, which means that bifurcation point will appear later.

Fig. 8 shows the effect of increasing viscoelastic foundation length on stability and location of



(a) $v_0 = 0.5,\ v_1 = 3,\ \zeta = 0,\ \kappa$=3.26 (b) $v_0 = 0.5,\ v_1 = 3,\ \zeta = 1,\ \kappa$=3.26

(c) $v_0 = 0.5,\ v_1 = 3,\ \zeta = 2,\ \kappa$=3.26 (d) $v_0 = 0.5,\ v_1 = 3,\ \zeta = 3,\ \kappa$=3.26

Fig. 6 Stability and location of bifurcation point variation under the damping factor variation for the first mode (σ_1: stable, σ_2: unstable, $l_2 = l$, and $l_1 = 0$)

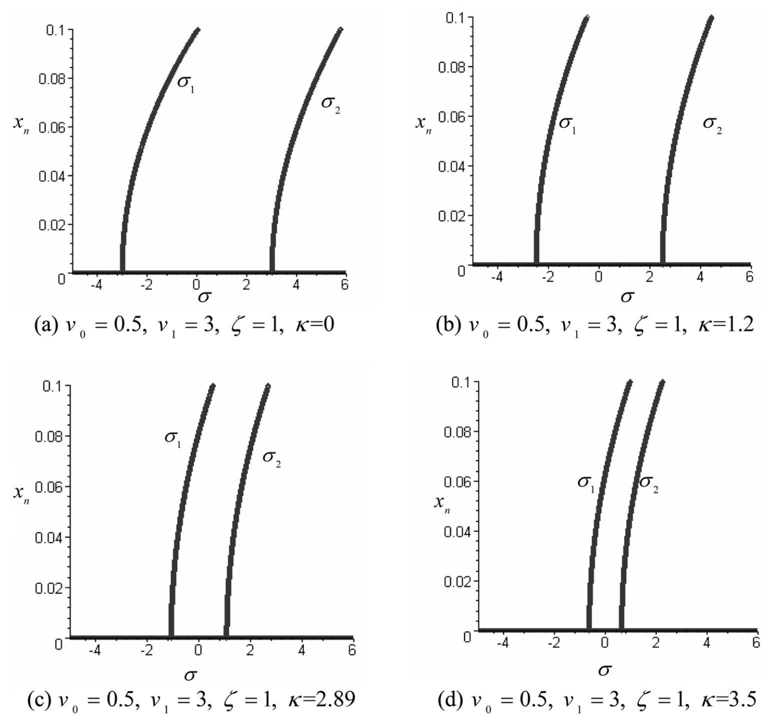


Fig. 7 Stability and location of bifurcation point variation under the stiffness factor variation for the first mode (σ_1: stable, σ_2: unstable, $l_2 = l$, and $l_1 = 0$)

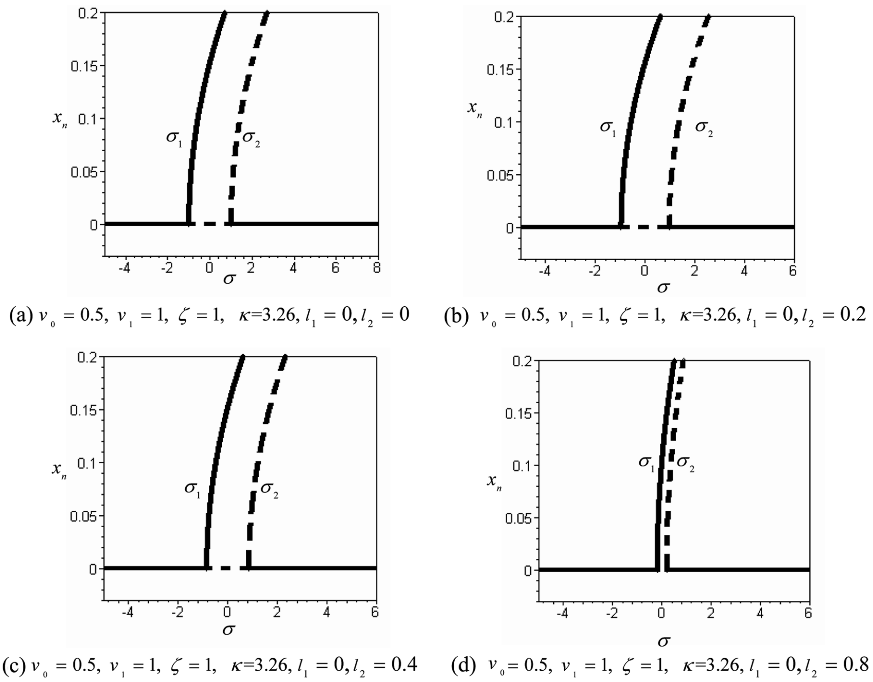
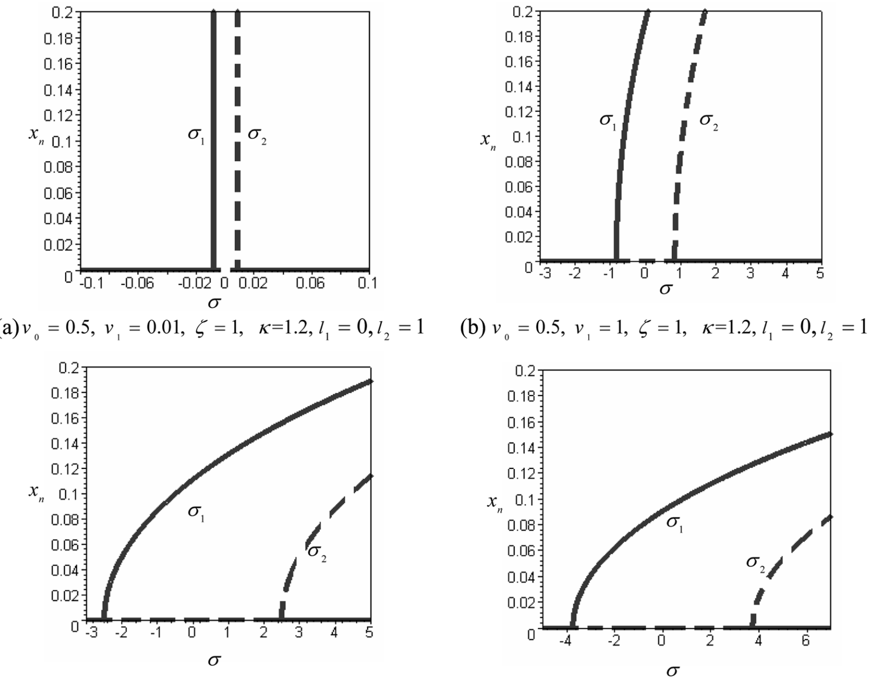


Fig. 8 Stability and bifurcation point variation under l_2 variation for the first mode (dashed line: unstable, and solid line: stable)



(a) $v_0 = 0.5,\ v_1 = 0.01,\ \zeta = 1,\ \kappa=1.2, l_1 = 0, l_2 = 1$ (b) $v_0 = 0.5,\ v_1 = 1,\ \zeta = 1,\ \kappa=1.2, l_1 = 0, l_2 = 1$

(c) $v_0 = 0.5,\ v_1 = 3,\ \zeta = 1,\ \kappa=1.2, l_1 = 0, l_2 = 1$ (d) $v_0 = 0.5,\ v_1 = 4.5,\ \zeta = 1,\ \kappa=1.2, l_1 = 0, l_2 = 1$

Fig. 9 Stability and bifurcation point variation under coefficient of non-linearity, v_1, variation for the first mode (dashed line: unstable, and solid line: stable)

bifurcation point. In Fig. 8, when $\sigma < \sigma_1$, only stable trivial solution exists. At $\sigma = \sigma_1$, the trivial solution starts to be unstable and a stable nontrivial solution bifurcates. At $\sigma = \sigma_2$, the trivial solution starts to be stable again, and then, an unstable nontrivial solution happens. Also, instability interval of width of the resonant region will decrease when l_2 increases.

Fig. 9 shows the effect of the coefficient of non-linearity on the frequency- response curve using Eq. (44). For the case in which $v_1 \approx 0$, the system is linear and the peak of the curve is located in the line of $\sigma = 0$. Increasing v_1 will increase the slope of the curve to the right side, also increases the width of the resonant region.

4. Conclusions

Non-linear vibrations of a conveyor belt which is supported partially by a distributed viscoelastic foundation was investigated. The multiple scales method was applied. It was realized that when frequency of speed fluctuation, is close to twice of the natural frequency, the principal parametric resonance would occur. At the stationary response the stability conditions were investigated through Ruth-Hurwitz criterion. Numerical simulations helped us to show the effects of mean velocity, viscoelastic foundation length, stiffness and damping factors, on natural frequencies. Increasing mean velocity will reduce both real and imaginary parts of natural frequencies, while the increasing damping factor makes the real part of natural frequencies decrease and it makes imaginary part of natural frequencies increase. Increasing the stiffness factor does not affect imaginary part of natural frequencies but will increase the real part of natural frequencies. Stability and bifurcation of non-trivial and trivial steady state responses were analyzed and the effects of damping and stiffness factors, viscoelastic foundation length, and coefficient of non-linearity on location of bifurcation points and stability of trivial and non-trivial solutions were also investigated, and the frequency-response curves were drawn. It was realized that the first detuning parameter is stable and the second one is unstable. Also the instability area of trivial solution is located between the first and the second detuning parameters. Increasing stiffness factor leads to a smaller instability area for trivial solutions but increasing damping factor will lead to a larger instability area for trivial solutions. Instability interval of width of the resonant region will decrease when l_2 increases. Increasing the coefficient of non-linearity will increase the slope of the curve to the right side, also increases the width of the resonant region.

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