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Stability and vibration analysis of composite plates using spline finite strips with higher-order shear deformation

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Abstract. In the present study, a spline finite strip with higher-order shear deformation is formulated for the stability and free vibration analysis of composite plates. The analysis is conducted based on Reddy's third-order shear deformation theory, Touratier's "Sine" model, Afaq's exponential model and Cho's higher-order zigzag laminate theory. Consequently, the shear correction coefficients are not required in the analysis, and an improved accuracy for thick laminates is achieved. The numerical results, based on different shear deformation theories, are presented in comparison with the three-dimensional elasticity solutions. The effects of length-to-thickness ratio, fibre orientation, and boundary conditions on the critical buckling loads and natural frequencies are investigated through numerical examples.

Keywords: stability; free vibration; composite plate; spline finite strip method; Reddy's third-order shear deformation theory; Touratier's "Sine" model; Afaq's Exponential model; Cho's higher-order zigzag laminate theory.

1. Introduction

In recent decades, extensive use of fibre-reinforced composite materials in all types of engineering structures has drawn increased attention from many researchers to develop more accurate and more efficient analysis methods for predicting the stability and free vibration behaviours of composite structures.

In composite plate structures, the elastic modulus to shear modulus ratio is relatively high (e.g., of the order of 25 to 40 for graphite-epoxy and boron-epoxy composites, instead of 2.6 for typical isotropic materials). Therefore, the classical thin plate theory often overestimates critical buckling loads and natural frequencies of composite structures with unacceptable errors, since this theory totally neglects transverse shear deformation.

The transverse shear deformation in the composite plates can be taken into consideration using the first-order shear deformation theory according to the Mindlin assumptions, which assumes that any straight line originally normal to the plate middle surface remains straight, but not generally normal to the middle surface after deformation. This theory accounts for the transverse shear deformation, but results in a constant distribution of the transverse shear strains in the thickness direction of the plate.

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Although this theory yields satisfactory results for the critical buckling loads and natural frequencies of moderately thick composite plates, it fails to provide accurate results for very thick plates with a length-to-thickness ratio lower than 10. Moreover, even for a moderately thick plate, the wavelength of a higher-order vibration mode may be only a fraction of the plate side length, so that the plate may dynamically behave like a very thick plate under the higher-order vibrations. In addition, analysis based on that theory requires a shear correction to the transverse shear stiffness in order to achieve acceptable accuracy, and evaluation of the correction coefficients is often tedious and inaccurate.

Analysis can be improved by means of higher-order shear deformation theories. In 1984, J. N. Reddy published a third-order shear deformation theory (Reddy 1984). That theory assumes a parabolic variation of the transverse shear strains through the plate thickness, and accounts for zero transverse shear stresses on the surfaces of the plates. Consequently, there is no need to use shear correction coefficients in computing the shear stiffness. Based on that theory, analytical solutions have been obtained for simply supported cross-ply laminates and anti-symmetrical angle-ply laminates. The results are satisfactory when compared with the three-dimensional elasticity solutions for thick laminates with the length-to-thickness ratio as low as 4, while the number of degrees of freedom required in the analysis is the same as that in the analysis based on the first-order shear deformation theory. As the length-to-thickness ratio increases, the solution automatically converges to that based on the first-order shear deformation theory or thin plate theory without any numerical difficulties. In later years, Touratier (1991) proposed a "Sine" model, whereas Afaq *et al*. (2003) introduced an exponential model, by respectively assuming sinusoidal and exponential variations of inplane displacement in the thickness direction, yielding further improved accuracy. In 1993, Cho developed a higher-order zigzag laminate theory by superimposing a zigzag linearly varying inplane displacement on a cubic varying displacement field (Cho and Parmerter 1993). The cubic variation proposed by Reddy accounts for the overall parabolic distribution of transverse shear strains while the zigzag approach accounts for the strain discontinuities required for stress continuity conditions. The unknowns of different planes are expressed in terms of those of midplane after imposing the interlaminar shear stress continuity conditions and ensuring the plate surfaces free of transverse shear stresses. Thus, the number of involved unknowns remains the same, but a continuous distribution of shear stresses is achieved, so that the accuracy of static analysis is significantly improved with only a little extra computational cost. However, all the above-mentioned higher-order theories experience difficulties in their finite element implementation, because the required shape functions must guarantee the inter-element continuity not only for the deflection but also for its first derivatives. Construction of such elements often requires extra degrees of freedom and lengthy calculation (Phan and Reddy 1985, Putcha and Reddy 1986), resulting in extra computational cost.

The difficulties can be readily overcome by the finite strip simulation, in which the plate structure is modeled by a number of longitudinal strips. Within each strip, a series of beam eigenfunctions are used to express the displacement variations in the longitudinal direction, while the Hermitian cubic polynomials are employed to interpolate the deflection variation in the lateral inplane direction (Cheung *et al*. 1996). Thus, the deflection and its first derivatives are all continuous across the nodal lines between the finite strips. The finite strip analysis of composite laminates based on Reddy's third-order shear deformation theory was published by Akhras *et al*. in 1994 and 1995. Although excellent efficiency and satisfactory accuracy have been achieved, the analysis can only be carried out efficiently for the plates with two opposite ends simply supported because of some limitations inherent in the semi-analytical solution procedures.

The efficiency of the analysis for the plates with other types of boundary conditions can be significantly enhanced by introducing the B-spline function, a piecewise cubic polynomial, as the longitudinal displacement function (Cheung *et al*. 1996). This measure makes the finite strip method as flexible as the finite element method in dealing with different boundary and loading conditions, while maintaining other features of the finite strip method. This alternative method is referred to as the spline finite strip method and has been extensively used in the analysis of composite plates based on the classical thin plate theory and the first-order shear deformation theory (Dawe 2002). In parallel to the above-mentioned development, the spline finite strip method has been successfully generalized to incorporate Reddy's third-order shear deformation for linear static and stability analysis of thick isotropic and laminated plates (Kong and Cheung 1993, Cheung and Kong 1993). The strip has two nodal lines. In the transverse in-plane direction of the strip, the Hermitian cubic polynomials are used for the interpolation of deflection, while the linear functions are used for in-plane displacements and normal rotations. The application was further extended to the free vibration and geometrically nonlinear analysis of laminated plates (Kong and Cheung 1995). In that analysis, the normal rotations were replaced by transverse shear strains as nodal degrees of freedom, so that the convergence for thin plates was improved. Recently, the approach has been applied to the analysis of piezolaminated plates by Ramos Loya *et al*. (2001), and improved flexibility has been achieved by adopting the spline functions with uneven lengths of section. However, in all these works, only the results on simply supported cross-ply laminates were presented in the application of this method to thick composite plates.

Recently, a spline finite strip with higher-order shear deformation has been developed for the static and free vibration analysis of composite plates using Reddy's third-order shear deformation theory (Akhras and Li 2005) and for the static analysis based on Cho's zigzag theory (Akhras and Li 2007). The strip has three nodal lines, each of which has an identical number of knots (Fig. 1). The knot on a side nodal line has six degrees of freedom, while the knot on the middle nodal line has only four degrees of freedom. This means that the present method employs approximately the same number of degrees of freedom as the spline finite strip method based on the first-order shear deformation theory, but will yield improved results for thick composite plates. In the lateral inplane direction, the quadratic interpolation is adopted for the inplane displacements, while the Hermitian interpolation is employed for the deflection. This interpolation combination can accurately simulate a linear variation of the transverse bending moment in the lateral inplane direction for the laminates with bending and inplane coupling, and consequently the convergence of the analysis is enhanced.

In the present study, this spline finite strip is further formulated for the stability and free vibration analysis of composite plates using either Reddy's third-order plate theory, Touratiers "Sine" model, Afaq's exponential model and Cho's zigzag laminate theory. Details are described in the following sections. The numerical results based on different shear deformation theories are presented in comparison with the three-dimensional elasticity solutions. The effects of length-to-thickness ratio, fibre orientation, number of plies and boundary conditions on the critical buckling load and natural frequencies are investigated through numerical examples.

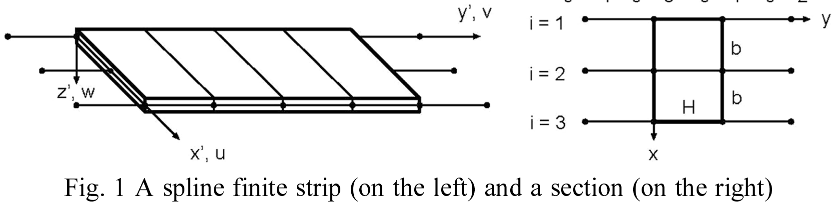

Fig. 1 A spline finite strip (on the left) and a section (on the right)

2. Spline finite strip formulation

In this study, the proposed spline finite strip has three equally spaced nodal lines, labelled by i = 1, 2, 3 respectively (Fig. 1). In the longitudinal direction, the strip is divided into a number of sections of identical length H. Within a section, the displacement components at any point can be interpolated from the displacement parameters at 12 knots located on the three nodal lines at local coordinates $y = j\,H$ with j equal to −1, 0, 1 and 2, respectively. For the knot identified by i and j, the following displacement parameters are adopted

$$\{\delta\}_{ij} = \begin{cases} \left[u,\, v,\, w,\, \dfrac{\partial w}{\partial x},\, \gamma_x,\, \gamma_y\right]_{ij}^T & \text{for} \quad i = 1, 3 \\ [u,\, v,\, \gamma_x,\, \gamma_y]_{ij}^T & \text{for} \quad i = 2 \end{cases} \tag{1}$$

where u, v, w denote the displacement components as shown in Fig. 1, while γ_x and γ_y represent the transverse shear deformation measured at midplane.

The midplane displacements u_o, v_o, w_o and shear deformation γ_x and γ_y at any point $(x, y, 0)$ within a section can be expressed in terms of the above displacement parameters as

$$\begin{aligned} u_o &= \sum_{i=1}^{3}\sum_{j=-1}^{2} N_i(x)\Phi_j(y)u_{ij} \\ v_o &= \sum_{i=1}^{3}\sum_{j=-1}^{2} N_i(x)\Phi_j(y)v_{ij} \\ w_o &= \sum_{i=1,3}\sum_{j=-1}^{2}\left[W_i(x)\Phi_j(y)w_{ij} + R_i(x)\Phi_j(y)\left(\frac{\partial w}{\partial x}\right)_{ij}\right] \\ \gamma_x &= \sum_{i=1}^{3}\sum_{j=-1}^{2} N_i(x)\Phi_j(y)\gamma_{xij} \\ \gamma_y &= \sum_{i=1}^{3}\sum_{j=-1}^{2} N_i(x)\Phi_j(y)\gamma_{yij} \end{aligned} \tag{2}$$

where $N_i(x)$ for i = 1, 2, 3 are the quadratic interpolation functions defined as

$$\begin{aligned} N_1(x) &= (b-x)(2b-x)/(2b^2) \\ N_2(x) &= x(2b-x)/b^2 \\ N_3(x) &= x(x-b)/(2b^2) \end{aligned} \tag{3}$$

and b is the spacing between two adjacent nodal lines.

$W_i(x)$ and $R_i(x)$ for i = 1, 3 are the following Hermitian cubic polynomials

$$\begin{aligned} W_1(x) &= 1 - 3(\bar{x})^2 + 2(\bar{x})^3 \\ W_3(x) &= 3(\bar{x})^2 - 2(\bar{x})^3 \\ R_1(x) &= x[1 - 2(\bar{x}) + (\bar{x})^2] \end{aligned} \tag{4}$$

$$R_3(x) = x[(\bar{x})^2 - (\bar{x})]$$

and $\bar{x} = \dfrac{x}{2b}$

while $\Phi_j(y)$ for $j = -1, 0, 1, 2$ denote B-spline functions, which have the following expressions within the section with $y \in [0, H]$

$$\begin{aligned}
\Phi_{-1}(y) &= \frac{1}{6H^3}(H-y)^3 \\
\Phi_0(y) &= \frac{1}{6H^3}[H^3 + 3H^2(H-y) + 3H(H-y)^2 - 3(H-y)^3] \\
\Phi_1(y) &= \frac{1}{6H^3}[H^3 + 3H^2 y + 3Hy^2 - 3y^3] \\
\Phi_2(y) &= \frac{1}{6H^3}y^3
\end{aligned} \tag{5}$$

The required spline functions for unequal lengths of section are provided in Cheung *et al.* (1996).

The analysis can be conducted based on Reddy's third-order shear deformation theory (Reddy 1984), Touratier's "Sine" model (Touratier 1991), Afaq's exponential model (Afaq *et al.* 2003) and Cho's higher-order zigzag laminate theory (Cho and Parmerter 1993). Based on these theories, displacements u, v and w at any point (x, y, z) of the laminate have the following relationships with the midplane displacements

$$\begin{aligned}
\{u\} &= \{u_o\} - z\{w'\} + [F(z)]\{\gamma\} \\
w &= w_o(x, y)
\end{aligned} \tag{6}$$

where $\{u\} = [u, v]^T$, $\{w'\} = [w'_x, w'_y]^T$, $\{\gamma\} = [\gamma_x, \gamma_y]^T$, while $[F(z)]$ is a 2 by 2 matrix related to assumed variation of transverse shear deformation through plate thickness.

Reddy's theory assumes a parabolic variation of transverse shear strains through plate thickness. Therefore, $[F(z)]$ has the following form

$$F(z)_{11} = F(z)_{22} = z\left(1 - \frac{4z^2}{3h^2}\right) \quad \text{and} \quad F(z)_{12} = F(z)_{21} = 0 \tag{7}$$

in which h is the plate thickness.

Touratier's model assumes a cosine variation of transverse shear strains. Thus, $[F(z)]$ is written as

$$F(z)_{11} = F(z)_{22} = \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right) \quad \text{and} \quad F(z)_{12} = F(z)_{21} = 0 \tag{8}$$

Afaq's model introduces an exponential variation of inplane displacement in the thickness direction, and $[F(z)]$ has the following form

$$F(z)_{11} = F(z)_{22} = ze^{-2(z/h)^2} \quad \text{and} \quad F(z)_{12} = F(z)_{21} = 0 \tag{9}$$

Cho's theory superimposes a zigzag linearly varying inplane displacement on a cubic varying displacement field, yielding continuous distribution of transverse shear stresses in the thickness direction of plate. $[F(z)]$ takes a more complicated form as follows (Shu and Sun 1994)

$$[F(z)]_k = [F_1]_k + z[F_2]_k + z^2[F_3] + z^3[F_4] \tag{10}$$

in which, the subscript k represents the ordinal number of a layer, while $[F_1]_k, [F_2]_k, [F_3]$ and $[F_4]$ are determined from the material properties and laminate thicknesses as below

$$[F_3] = 4[h[C_1]_N + 4[C_2]_N]^{-1}[C_2]_N/h$$
$$[F_4] = -4[h[C_1]_N + 4[C_2]_N]^{-1}[C_1]_N/3h$$
$$[F_2]_k = 2([Q_{sk}]^{-1}[C_1]_k - z_k[I])[F_3] + 3(2[Q_{sk}]^{-1}[C_2]_k - z_k^2[I])[F_4]$$
$$[F_1]_k = \sum_{l=2}^{k} z_{l-1}([F_2]_{l-1} - [F_2]_l) - \sum_{l=2}^{m} z_{l-1}([F_2]_{l-1} - [F_2]_l)$$

and

$$[C_1]_k = \sum_{l=1}^{k} [Q_{sl}](z_l - z_{l-1})$$
$$[C_2]_k = \sum_{l=1}^{k} [Q_{sl}](z_l^2 - z_{l-1}^2)/2$$

where, N is the number of layers in the laminate, z_k is the z coordinate of the interface between layer k and layer $k+1$, $[I]$ is the unit matrix, m is the ordinal number of the layer across or next to the plate midplane ($z_m = 0$ or $z_m z_{m+1} < 0$), $[Q_{sk}]$ is the transverse shear stiffness matrix of layer k defined by $[\tau_{xz}, \tau_{yz}]^T = [Q_{sk}][\gamma_{xz}, \gamma_{yz}]^T$.

Eq. (6) is also applicable to the analysis based on the classical plate theory by assuming

$$F(z)_{11} = F(z)_{12} = F(z)_{21} = F(z)_{22} = 0 \tag{11}$$

as well as for the analysis using the first-order shear deformation theory by taking

$$F(z)_{11} = F(z)_{22} = z \quad \text{and} \quad F(z)_{12} = F(z)_{21} = 0 \tag{12}$$

By substituting Eq. (2) into Eq. (6), the displacement field within a section can be expressed in terms of the displacement parameters as

$$\{f\} = [u, v, w]^T = \sum_{i=1}^{3} \sum_{j=-1}^{2} [N]_{ij}\{\delta\}_{ij} \tag{13}$$

where $[N]_{ij}$ is the displacement matrix. For $i = 1$ or 3, it is written as

$$[N]_{ij} = \begin{bmatrix} N_i\Phi_j & 0 & -zW_i'\Phi_j & -zR_i'\Phi_j & F(z)_{11}N_i\Phi_j & F(z)_{12}N_i\Phi_j \\ 0 & N_i\Phi_j & -zW_i\Phi_j' & -zR_i\Phi_j' & F(z)_{21}N_i\Phi_j & F(z)_{22}N_i\Phi_j \\ 0 & 0 & W_i\Phi_j & R_i\Phi_j & 0 & 0 \end{bmatrix} \tag{14}$$

where $W_i' = dW_i/dx$, $R_i' = dR_i/dx$ and $\Phi_j' = d\Phi_j/dy$. For $i = 2$, the above expression is applicable without the third and the fourth columns.

The following relationships between displacements and linear strains are used in the analysis

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{15}$$

By substituting Eq. (13) into Eq. (15), the strain vector can be expressed in terms of the displacement parameters as

$$\{\varepsilon\} = [\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}]^T = \sum_{i=1}^{3}\sum_{j=-1}^{2}[B]_{ij}\{\delta\}_{ij} \tag{16}$$

in which $[B]_{ij}$ is the strain matrix. For i = 1 or 3, it takes the following form

$$[B]_{ij} = \begin{bmatrix} N_i'\Phi_j & 0 & -zW_i''\Phi_j & -zR_i''\Phi_j & F_{11}N_i'\Phi_j & F_{12}N_i'\Phi_j \\ 0 & N_i\Phi_j' & -zW_i\Phi_j'' & -zR_i\Phi_j'' & F_{21}N_i\Phi_j & F_{22}N_i\Phi_j' \\ N_i\Phi_j' & N_i'\Phi_j & -2zW_i'\Phi_j' & -2zR_i'\Phi_j' & F_{11}N_i\Phi_j' + F_{21}N_i'\Phi_j & F_{12}N_i\Phi_j' + F_{22}N_i'\Phi_j \\ 0 & 0 & 0 & 0 & F_{21}'N_i\Phi_j & F_{22}'N_i\Phi_j \\ 0 & 0 & 0 & 0 & F_{11}'N_i\Phi_j & F_{12}'N_i\Phi_j \end{bmatrix} \tag{17}$$

where $N_i' = dN_i/dx$, $W_i'' = d^2W_i/dx^2$, $R_i'' = d^2R_i/dx^2$, $\Phi_j'' = d^2\Phi_j/dy^2$, while $F_{mn}' = dF_{mn}/dz$ with m = 1, 2 and n = 1, 2. For i = 2, the above expression is valid without the third and fourth columns.

From Eqs. (16) and (17) it can be observed that at any given coordinate z, ε_x varies linearly in the x direction, so that the current formulation can accurately simulate a linear variation of the transverse bending moment in the lateral inplane direction. Consequently, the convergence of the analysis can be improved significantly over the strips using linear interpolations in the x direction for inplane displacements (Kong and Cheung 1993, 1995). This is especially important for the analysis of the composite laminates with the coupling between bending and inplane deformation.

It is assumed that the laminates are manufactured from orthotropic layers (or plies) of preimpregnated unidirectional fibrous composite materials. Neglecting σ_z, for each layer, the stress-strain relationships in the x-y-z coordinate system can be stated as

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{26} & 0 & 0 \\ Q_{16} & Q_{26} & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{45} & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = [Q]\{\varepsilon\} \tag{18}$$

Then, the stiffness matrix, the mass matrix and the geometrical stiffness matrix of the section can be formed by following standard procedures in the spline finite strip analysis (Cheung *et al.* 1996, Dawe 2002). The submatrices corresponding to knot ij and knot mn are obtained as

$$[K]_{ij,mn} = \iiint_V [B]_{ij}^T[Q][B]_{mn}dV \tag{19}$$

$$[M]_{ij,mn} = \iiint_V [N]_{ij}^T\rho[N]_{mn}dV \tag{20}$$

$$[L]_{ij,mn} = \iiint_V \{[G_u]_{ij}^T[\sigma^o][G_u]_{mn} + [G_v]_{ij}^T[\sigma^o][G_v]_{mn} + [G_w]_{ij}^T[\sigma^o][G_w]_{mn}\}dV \tag{21}$$

where ρ is the mass density of the laminate, V denotes the volume of the section, $[G_u]_{ij}$, $[G_v]_{ij}$ and $[G_w]_{ij}$ are defined as

$$\begin{aligned}\left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right]^T &= \sum_{i=1}^{3}\sum_{j=-1}^{2}[G_u]_{ij}\{\delta\}_{ij}\\ \left[\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right]^T &= \sum_{i=1}^{3}\sum_{j=-1}^{2}[G_v]_{ij}\{\delta\}_{ij}\\ \left[\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right]^T &= \sum_{i=1}^{3}\sum_{j=-1}^{2}[G_w]_{ij}\{\delta\}_{ij}\end{aligned} \tag{22}$$

$[\sigma^o]$ is the initial inplane stress matrix expressed as

$$[\sigma^o] = \begin{bmatrix}\sigma_x^o & \tau_{xy}^o\\ \tau_{xy}^o & \sigma_y^o\end{bmatrix} \tag{23}$$

which are calculated from the initial edge loads by assuming that the laminate is initially in a state of uniform strain (Srinivas *et al.* 1970). In some works on the stability analysis, $[G_u]_{ij}$ and $[G_v]_{ij}$ in Eq. (21) were neglected based on the classical plate theory. However, these two items compose the so-called "curvature" terms, which gain importance as the plate becomes relatively thicker (Dawe and Roufaeil 1982). Therefore, $[G_u]_{ij}$ and $[G_v]_{ij}$ are included in the current analysis, particularly for the thick and moderately thick plates.

In the present study, the boundary conditions are imposed by means of the penalty function method (Cheung *et al.* 1996). For instance, deflection w at knot ij can be fixed by adding an imaginary elastic support to this knot in the z direction. The stiffness matrix of this support is the square matrix combined with a factor C in the following equilibrium matrix equation

$$\begin{Bmatrix}Z_{i,j-1}\\ Z_{i,j}\\ Z_{i,j+1}\end{Bmatrix} = C\begin{bmatrix}1 & 4 & 1\\ 4 & 16 & 4\\ 1 & 4 & 1\end{bmatrix}\begin{Bmatrix}w_{i,j-1}\\ w_{i,j}\\ w_{i,j+1}\end{Bmatrix} \tag{24}$$

in which $Z_{i,j}$ represents the general force applied by the support to the degree of freedom $w_{i,j}$ of knot ij, while C is the stiffness coefficient of the support. It should be much higher (in the order of 100 times) than the maximum value of the diagonal items in the structural stiffness matrix.

After assembling the stiffness matrices, mass matrices and geometrical stiffness matrices over all the sections in the structure, followed by imposing boundary conditions, the natural frequencies ω and the critical load factors λ of the laminate can be determined by solving the following structural matrix equations using standard computer subroutines

$$[K]\{\delta\} = \omega^2[M]\{\delta\} \tag{25}$$

$$[K]\{\delta\} = \lambda[L]\{\delta\} \tag{26}$$

3. Numerical examples

Several numerical examples are presented to verify the convergence of the proposed method, and to show the effects of length-to-thickness ratio, fibre orientation, number of plies and boundary conditions on the critical buckling load and natural frequencies of composite laminates. In all the examples, the lamina properties are assumed to be $E_1 = 40.0E_2$, $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$ and $\nu_{12} = 0.25$, where the subscripts 1 and 2 respectively denote the fibre and transverse to fibre inplane directions, while 3 refers to the direction normal to the plate midplane. In addition, all the layers in a plate have equal thickness.

3.1 Free vibration of simply supported square (0°/90°/90°/0°) and (0°/90°/0°/90°) laminates

Free vibration of square (0°/90°/90°/0°) and (0°/90°/0°/90°) laminates with side length, a, and thickness, h, are analyzed using the present method. Each plate is simply supported on all four edges with permissible displacement in the normal inplane direction. By virtue of the symmetry,

Table 1 Dimensionless fundamental frequencies $\bar{\omega} = \omega a^2(\rho/E_2h^2)^{0.5}$ of simply supported square (0°/90°/90°/0°) laminates

a/h	Mesh & Solution	Theory in the present method			
		Reddy	Touratier	Afaq	Cho
5	1 × 1 Section	10.799	10.804	10.821	10.840
	2 × 2 Sections	10.788	10.794	10.812	10.831
	3 × 3 Sections	10.787	10.793	10.811	10.830
	Theory	10.787[1]	10.793[2]	10.810[2]	10.830[2]
	Elasticity[3]	10.752			
	FSDT[1]	10.854			
	CLPT[1]	18.299			
10	1 × 1 Section	15.140	15.143	15.158	15.155
	2 × 2 Sections	15.109	15.115	15.130	15.127
	3 × 3 Sections	15.108	15.113	15.128	15.125
	Theory[2]	15.107	15.113	15.128	15.125
	Elasticity[4]	15.069			
	FSDT[2]	15.143			
	CLPT[1]	18.738			
100	1 × 1 Section	19.076	18.905	18.905	18.905
	2 × 2 Sections	18.848	18.840	18.841	18.840
	3 × 3 Sections	18.837	18.837	18.837	18.837
	Elasticity[4]	18.835			
	FSDT[2]	18.836			
	CLPT[2]	18.890			

References: **1**. Reddy 2004, **2**. Rayleigh-Ritz solution using exact displacement functions, **3**. Noor 1973, **4**. Finite layer solution (Cheung *et al.* 1996).

Table 2 Dimensionless fundamental frequencies $\overline{\omega} = \omega a^2(\rho/E_2h^2)^{0.5}$ of simply supported square (0º/90º/0º/90º) laminates

a/h	Mesh & Solution	Theory in the present method			
		Reddy	Touratier	Afaq	Cho
5	1 × 1 Section	11.183	11.176	11.176	11.054
	2 × 2 Sections	11.172	11.165	11.165	11.043
	3 × 3 Sections	11.172	11.163	11.163	11.042
	Theory[1]	11.172	11.163	11.163	11.041
	Elasticity[2]	10.680			
	FSDT[1]	11.271			
	CLPT[1]	16.673			
10	1 × 1 Section	14.873	14.863	14.859	14.792
	2 × 2 Sections	14.848	14.841	14.837	14.770
	3 × 3 Sections	14.847	14.838	14.835	14.768
	Theory[1]	14.846	14.838	14.834	14.767
	Elasticity[3]	14.495			
	FSDT[1]	14.921			
	CLPT[1]	17.145			
100	1 × 1 Section	17.500	17.339	17.339	17.338
	2 × 2 Sections	17.290	17.283	17.283	17.282
	3 × 3 Sections	17.280	17.279	17.279	17.278
	Elasticity[3]	17.273			
	FSDT[1]	17.280			
	CLPT[1]	17.308			

References: **1**. Rayleigh-Ritz solution using exact displacement functions, **2**. Noor 1973, **3**. Finite layer solution (Cheung *et al*. 1996).

only a quarter of the laminate is modeled. In each case, one, two or three proposed spline strips with identical number of sections are employed. The resulting dimensionless fundamental frequencies $\overline{\omega} = \omega a^2(\rho/E_2h^2)^{0.5}$ are listed in Table 1 and Table 2.

Results show that the present solutions based on all the included higher-order shear deformation theories converge quickly to the respective theoretical solutions (Theory), which are taken from references or obtained by means of Rayleigh-Ritz method using exact displacement functions. In comparison with three-dimensional elasticity solutions (Elasticity), the errors of the present results are less than 1% for the symmetrical cross-ply laminates and less than 5% for the anti-symmetrical ones. In most cases, the proposed method yields improved accuracy over the first-order shear deformation theory (FSDT with shear correction coefficients $k_1^2 = k_2^2 = 5/6$) and the classical plate theory (CLPT). Among all the included higher-order shear deformation theories, Reddy's theory yields the best results for the symmetrical cross-ply laminates, whereas Cho's zigzag theory yields the best ones for antisymmetric laminates. However, the differences between the results based on different higher-order shear deformation theories are not as significant as in the static analysis (Reddy 1984, Touratier 1991, Afaq *et al*. 2003, Cho and Parmerter 1993).

Table 3 Dimensionless critical buckling loads $\overline{N} = N_x^o a^2/E_2 h^3$ of simply supported square (0°/90°/90°/0°) and (0°/90°/0°/90°) laminates

a/h	Method	(0°/90°/90°/0°)	(0°/90°/0°/90°)
10	Present (Reddy)	22.967	22.332
	Present (Touratier)	22.984	22.307
	Present (Afaq)	23.028	22.295
	Present (Cho)	23.020	22.094
	Elasticity[1]	22.881	21.280
	FSDT[2]	23.077	22.558
	CLPT[2]	35.168	29.782
20	Present (Reddy)	31.474	27.831
	Present (Touratier)	31.484	27.820
	Present (Afaq)	31.507	27.815
	Present (Cho)	31.494	27.737
	Elasticity[3]	31.499	27.396
	FSDT[2]	31.521	27.923
	CLPT[2]	35.907	30.213
100	Present[3] (Reddy)	35.944	30.250
	Present (Touratier)	35.944	30.249
	Present (Afaq)	35.945	30.249
	Present (Cho)	35.945	30.245
	Elasticity[3]	35.944	30.228
	FSDT[2]	35.945	30.253
	CLPT[2]	36.150	30.353

References: **1**. Noor 1975, **2**. Rayleigh-Ritz solution using exact displacement functions, **3**. Finite layer solution (Cheung *et al.* 1996)

3.2 Buckling of simply supported square (0°/90°/90°/0°) and (0°/90°/0°/90°) laminates

The square cross-ply laminates in the previous example are subjected to a uniform compressive force N_x. The stability of the laminates is evaluated using the present method with a quarter of each plate being modeled by four proposed strips and the same number of spline sections. A further refined model yields little improvement. The resulting critical buckling loads are given in Table 3. The findings from the previous example can also be observed in this example.

3.3 Free vibration and buckling of clamped square cross-ply laminates

The square cross-ply laminates in Example 1 are clamped on all the edges. The stability under a uniform compressive force N_x and free vibration of the plates are analyzed using the present method. All the degrees of freedom on the plate edges are fixed in the analysis (Reddy 2004). In addition, $w_y' = 0$ is also imposed on the two edges $y = 0$ and $y = a$.

A quarter of each laminate is modeled by four strips of four spline sections. Upgraded models yield negligible changes. The results of dimensionless fundamental frequencies $\overline{\omega} = \omega a^2(\rho/E_2 h^2)^{0.5}$ and uniaxial buckling loads $\overline{N} = N_x^o a^2/E_2 h^3$ are given in Table 4 as the function of the length-to-

Table 4 Dimensionless fundamental frequencies $\bar{\omega} = \omega a^2(\rho/E_2h^2)^{0.5}$ and uniaxial buckling loads $\bar{N} = N_x^o a^2/E_2h^3$ of square cross-ply laminates with all the edges clamped

a/h	Method	(0°/90°/90°/0°)		(0°/90°/0°/90°)	
		$\bar{N}$	$\bar{\omega}$	$\bar{N}$	$\bar{\omega}$
10	Present (Reddy)	41.850	23.033	37.560	24.015
	Present (Touratier)	42.182	23.107	37.715	24.094
	Present (Afaq)	42.580	23.169	37.875	24.112
	Present (Cho)	42.386	23.206	36.600	23.803
	FSDT	-	22.593	37.076	23.636
	CLPT	127.79	41.408	82.192	37.733
20	Present (Reddy)	87.146	32.622	76.301	32.356
	Present (Touratier)	87.272	32.684	76.303	32.366
	Present (Afaq)	87.437	32.728	76.284	32.360
	Present (Cho)	87.379	32.717	75.334	32.188
	FSDT	87.196	32.341	76.221	32.243
	CLPT	134.78	41.722	108.93	38.070
100	Present (Reddy)	134.10	41.265	108.32	37.884
	Present (Touratier)	134.10	41.274	108.30	37.880
	Present (Afaq)	134.11	41.277	108.29	37.879
	Present (Cho)	134.10	41.275	108.36	37.869
	FSDT	134.18	41.257	110.10	37.884
	CLPT	137.28	41.825	110.15	38.179

thickness ratio *a/h*. The converged finite strip solutions based on the first-order shear deformation theory (FSDT) and the classical thin plate theory (CLPT) obtained by the authors are also listed in the table.

Comparison of the results in Tables 1 to 4 shows that changing boundary conditions from simple supported edges to clamped edges raises natural frequencies and buckling loads of thin plates substantially. However, the increment is not as obvious for thick plates.

3.4 Square (45°/–45°/…) anti-symmetrical angle-ply laminate: buckling and free vibration

A square (45°/–45°/...) anti-symmetrical angle-ply laminate is either clamped or simply supported on its four edges. The boundary conditions for the clamped edges are basically imposed in the same manner as in the previous example. However, because of the coupling between bending and inplane deformations, the inplane displacements are included in the analysis. In addition, the edge of the plate is assumed to be free in the tangential inplane direction, but immovable in the normal inplane direction for both the simply-supported and clamped boundaries. The critical buckling loads under uniform in-plane load N_x and fundamental frequencies are determined using the present method based on Reddy's theory. The entire laminate is modeled by eight strips and eight spline sections. Results are summarized in Table 5. The analytical solutions obtained by Reddy and Phan (1985) are also listed in this table for comparison. The agreement is excellent.

It can be observed from the table that the fundamental frequency and critical buckling load of the

Table 5 Dimensionless fundamental frequencies $\overline{\omega} = \omega a^2(\rho/E_2h^2)^{0.5}$ and uniaxial buckling loads $\overline{N} = N_x^o a^2/E_2h^3$ of square anti-symmetrical angle-ply (45°/–45°/...) laminates

a/h	Number of layers	Four edges clamped		Four edges simply-supported		
		$\overline{N}$	$\overline{\omega}$	$\overline{N}$	$\overline{\omega}$	$\overline{\omega}$ (Reddy)
10	2	25.659	18.780	17.823	13.263	13.263
	4	34.812	23.012	32.705	18.321	-
	8	37.311	23.938	35.063	19.264	19.266
20	2	36.502	21.796	20.564	14.246	14.246
	4	70.571	30.706	48.180	21.805	-
	8	78.225	32.410	54.718	23.238	23.239
100	2	42.703	23.370	21.661	14.622	14.621
	4	104.52	36.500	55.722	23.451	-
	8	119.84	39.078	64.214	25.175	25.174

laminates increase with the number of layers. This effect is particularly significant for the thin plates (*a/h* = 100) when the number of layers is doubled from two to four. However, the effect is not as noteworthy when the number of layers is further increased or when the plates become relatively thicker.

3.5 Square symmetrical angle-ply laminate: buckling and free vibration

The stability and free vibration of a square (45°/–45°/–45°/45°) laminate are analyzed using the present method based on Reddy's third order shear deformation theory. The four edges of the plate are either all simply supported or all clamped. In both cases, the entire laminate is modeled by eight strips and eight spline sections. The results are listed in Table 6, which shows a close agreement with the solutions obtained by Chen *et al.* (1997) using the *p*-Ritz method.

Similarly to the findings from the previous examples, clamping plate edges raises the fundamental frequency and critical buckling load substantially for the thin plate. However, the effect becomes not as significant for the thick plate.

The effects of fibre-orientation can be found by comparing the data in all the tables. For a visual comparison, the results for four layer laminates based on Reddy's theory are illustrated in Figs. 2 to 5. For simply supported square thin plates, the anti-symmetrical angle-ply yields the highest

Table 6 Dimensionless fundamental frequency $\overline{\omega} = \omega a^2(\rho/E_2h^2)^{0.5}$ and uniaxial buckling loads $\overline{N} = N_x^o a^2/E_2h^3$ of (45°/–45°/–45°/45°) laminate

a/h	Method	Four edges clamped		Four edges simply-supported	
		$\overline{N}$	$\overline{\omega}$	$\overline{N}$	$\overline{\omega}$
10	Present	29.811	21.883	27.678	17.433
	p-Ritz	-	21.672	-	17.381
20	Present	59.201	29.936	41.106	20.809
100	Present	89.198	36.575	48.451	22.534

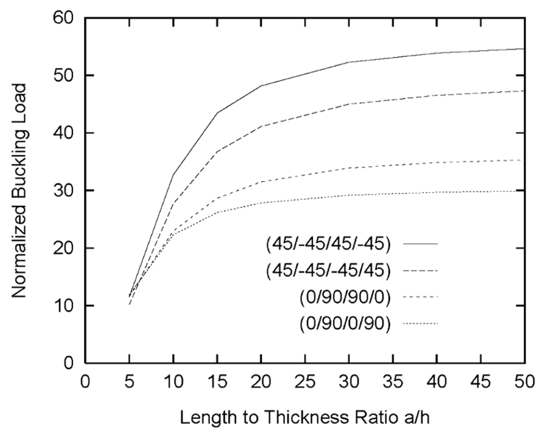

Fig. 2 Critical buckling loads of simply supported laminates

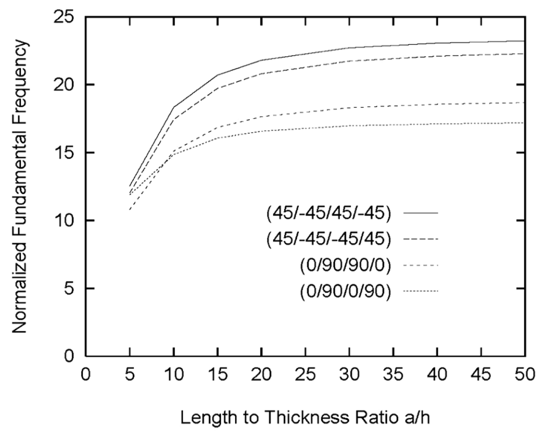

Fig. 3 Fundamental frequencies of simply supported laminates

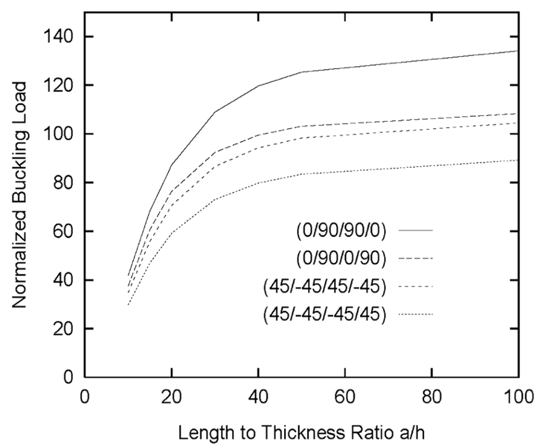

Fig. 4 Critical buckling loads of clamped laminates

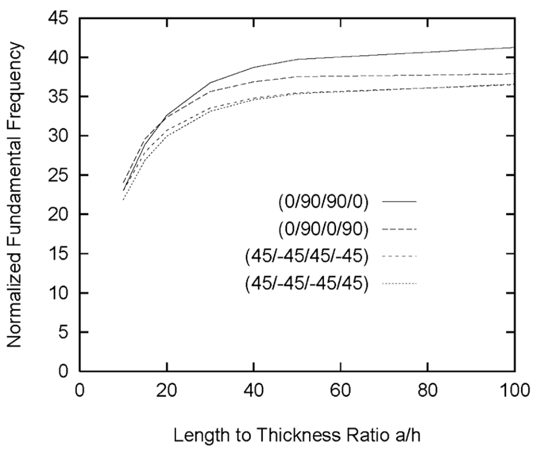

Fig. 5 Fundamental frequencies of clamped laminates

fundamental frequency and critical buckling load, whereas the cross-ply laminates yield higher fundamental frequency and critical buckling load than the angle-ply laminates for clamped laminates. However, the results may be different for thick plates. Therefore, a case to case study is recommended.

4. Conclusions

In this study, a spline finite strip method has been formulated for stability and free vibration analysis of composite plates respectively based on Reddy's third-order shear deformation theory, Touratier's "Sine" model, Afaq's exponential model and Cho's zigzag laminate theory. This method does not require shear correction coefficients but yields improved accuracy for thick laminates. In addition, the selected shape functions can accurately simulate a linear variation of transverse bending moment in the transverse inplane direction, and avoid shear locking for thin plates.

The effects of boundary conditions, length-to-thickness ratios, fibre-orientations and number of layers on the critical buckling loads and fundamental frequencies of composite plates are investigated. In general, these effects are found to be more significant for thin plates than for thick plates.

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References

Afaq, K., Karama, M. and Mistou S. (2003), "A new refined model for laminated structures", *Comptes Rendus des JNC13*, Strasbourg, France, 12-14 Mars, 283-292.

Akhras, G. and Li, W. (2005), "Static and free vibration analysis of composite plates using spline finite strips with higher-order shear deformation", *Composites: Part B,* **36**, 496-503.

Akhras, G. and Li, W. (2007), "Spline finite strip analysis of composite plates based on higher-order zigzag composite plate theory", *Compos. Struct.,* **78**(1), 112-118.

Akhras, G., Cheung, M.S. and Li, W. (1995), "Vibration and stability analysis of thick anisotropic composite plates by finite strip method", *Struct. Eng. Mech.*, **3**(1), 49-60.

Akhras, G., Cheung, M.S. and Li, W. (1994), "Finite strip analysis of anisotropic laminated composite plates using higher-order shear deformation theory", *Comput. Struct.*, **52**(3), 471-477.

Chen, C.C., Liew, K.M., Lim, C.W. and Kitipornchai, K. (1997), "Vibration analysis of symmetrically laminated thick rectangular plates using the higher-order theory and *p*-Ritz method", *J. Acoustical Soc. Am.*, **102**(3), 1600-1611.

Cheung, M.S., Li, W. and Chidiac, S.E. (1996), *Finite Strip Analysis of Bridges*, E & FN SPON, London.

Cheung, Y.K. and Kong, J. (1993), "Linear elastic stability analysis of shear deformable plates using a modified spline finite strip method", *Comput. Struct.,* **47**(2), 189-192.

Cho, M. and Parmerter, R.R. (1993), "Efficient higher order composite plate theory for general lamination configurations", *AIAA J.*, **31**(7), 1299-1306.

Dawe, D.J. (2002), "Use of the finite strip method in predicting the behaviors of composite laminated structures", *Compos. Struct.*, **57**(1), 11-36.

Dawe, D.J. and Roufaeil, O.L. (1982), "Buckling of rectangular Mindlin plates", *Comput. Struct.,* **15**(4), 461-471.

Kong, J. and Cheung, Y.K. (1995), "A generalized spline finite strip for the analysis of plates", *Thin Wall. Struct.*, **22**(3), 181-202.

Kong, J. and Cheung, Y.K. (1993), "Application of the spline finite strip to the analysis of shear deformable plates", *Comput. Struct.*, **46**(6), 985-988.

Noor, A.K. (1975), "Stability of multilayered composite plates", *Fibre Sci. Technol.*, **8**, 81-89.

Noor, A.K. (1973), "Free vibrations of multilayered composite plates", *AIAA J.*, **11**(7), 1038-1039.

Phan, N.D. and Reddy, J.N. (1985), "Analysis of laminated composite plates using a higher-order deformation theory", *Int. J. Numer. Meth. Eng.*, **21**, 2201-2219.

Putcha, N.S. and Reddy, J.N. (1986), "A refined mixed shear flexible finite element for the nonlinear analysis of laminated plates", *Comput. Struct.*, **22**(4), 529-538.

Ramos Loja, M.A., Mota Soares, C.M. and Mota Soares, C.A. (2001), "Higher-order B-spline finite strip model for laminated adaptive structures", *Compos. Struct.*, **52**, 419-427.

Reddy, J.N. (2004), *Mechanics of Laminated Composite Plates, Theory and Analysis, Second Edition,* CRC

Press, New York.
Reddy, J.N. (1984), "A simple higher-order theory for laminated composite plates", *J. Appl. Mech.*, **51**, 745-752.
Reddy, J.N. and Phan N.D. (1985), "Stability and vibration of isotropic, orthotropic and laminated plates according to a higher-order shear deformation theory", *J. Sound Vib.*, **98**(2), 157-170.
Srinivas, S., Joga Rao, C.V. and Rao, A.K. (1970), "Some results from an exact analysis of thick laminates in vibration and buckling", *J. Appl. Mech.,* **37**, 868-870.
Shu, X. and Sun, L. (1994), "An improved simple higher-order theory for laminated composite plates", *Comput. Struct.,* **50**(2), 231-236.
Touratier, M. (1991), "An efficient standard plate theory", *Int. J. Eng. Sci.*, **29**(8), 901-916.