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Effective buckling length of steel column members based on elastic/inelastic system buckling analyses

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Abstract. This study presents an improved method that uses the elastic and inelastic system buckling analyses for determining the *K*-factors of steel column members. The inelastic system buckling analysis is based on the tangent modulus theory for a single column and the application is extended to the frame structural system. The tangent modulus of an inelastic column is first derived as a function of nominal compressive stress from the column strength curve given in the design codes. The tangential stiffness matrix of a beam-column element is then formulated by using the so-called stability function or Hermitian interpolation functions. Two inelastic system buckling analysis procedures are newly proposed by utilizing nonlinear eigenvalue analysis algorithms. Finally, a practical method for determining the *K*-factors of individual members in a steel frame structure is proposed based on the inelastic and/or elastic system buckling analyses. The *K*-factors according to the proposed procedure are calculated for numerical examples and compared with other results in available references.

Keywords: inelastic system buckling; effective buckling length; column; plane frame; tangent modulus theory.

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1. Introduction

Generally the effective buckling length can be defined as the distance between two inflection points at the buckled shape of framed structures. The effective buckling length is one of the important parameters in the design of column and/or beam-column members in frame structures, and the stability design method based on the effective buckling length is quite a common practice to engineers for many years.

To present, various methods have been reported for the calculation of the effective buckling length and among them, the most widely accepted method for designing frame members is the alignment chart method (ACM) proposed by Julian and Lawrence (1959). And now, various design codes such as AISC (2005), ACI (2005), and AASHTO (1998) adopt this method. As an alternative method, the isolated subassembly approach also has been applied extensively following the early works of Galambos (1968) and adopted widely in specifications such as DIN 18800 (1990) and Eurocode 2 (2002).

However, the major drawback of both ACM and methods based on the isolated subassembly approach is that they do not properly reflect the interaction effects of neighboring members except for the very closely adjacent columns and beams. Since these methods are based on several fundamental assumptions, which basically limit the use of those methods only for the idealized cases, they may lead to inadequate estimation of effective buckling length when these assumptions are not satisfied. To overcome this drawback, many researches (Bridge and Fraster 1987, Duan and Chen 1989, 1988, Mahini and Seyyedian 2006) have been carried out. Particularly Mahini and Seyyedian (2006) presented a new iterative approach with a high convergence rate to improve the effectiveness of the G-factor method (Bridge and Fraster 1987). In their study, using the curve-fitting principles, the explicit forms of the stability functions were replaced by more simple rational forms, and also members with the non-hinged far-end condition were converted to equivalent hinged far-end ones, considering the rotational stiffness in each case.

Another improved approach to calculate effective lengths without system buckling analysis is the so called "storey-based buckling approach", which accounts for the horizontal interaction between columns in a storey (Yura 1971, LeMessurier 1977, Xu *et al.* 2001, Liu and Xu 2005). Two methods of determining the storey-based effective length factor K, namely the storey stiffness method (LeMessurier 1977) and the storey buckling method (Yura 1971), were presented in the Commentary of the LRFD Specification (AISC 2005). Also, the elastic in-plane buckling characteristics of the unbraced frames under non-proportional loading were investigated by Xu *et al.* (2001) adopting the storey-based buckling concept. In their study, to overcome the difficulty associated with non-proportional loading, the problem of the lateral buckling of unbraced frames was expressed as a minimization problem subjected to stability constraints. Recently, Liu and Xu (2005) presented a practical method for evaluating the K-factors for columns in multi-storey unbraced frames based on the storey-based elastic buckling concept, which decomposes a multi-storey frame into a series of single-storey partially-restrained frames. The lateral stiffness of a multi-storey frame is derived and expressed as the product of the lateral stiffness of each individual storey.

On the other hand, various studies (Yura 1971, White and Hajjar 1997, Essa 1998, AISC 2005) had been carried out to account for the effects of inelastic column behavior on evaluation of K-factor. Yura (1971) proposed an iterative procedure to determine the K-factor in the inelastic range of column behavior. White and Hajjar (1997) addressed the advantages, proper usage, and limitations of ACM, story and system buckling models and discussed certain anomalies that can occur in calculating effective buckling lengths with relation to elastic/inelastic buckling models.

Also Essa (1998) developed a design approach to predict the *K*-factor for columns in unbraced frames, considering the inelastic behavior, semi-rigid connections, far-end conditions, and differentiated stiffness parameters of the connected columns. He used a slope-deflection equation with the column stiffness reduction factors, which were evaluated from the column-strength curve proposed by the Structural Stability Research Council (SSRC). In AISC (2005), inelastic *K*-factors are determined from ACM with the stiffness reduction factor applied elastic modulus in the equation for *G*-factor. Depending on how the stiffness reduction factors are calculated, they might account for both the reduction in the stiffness of columns due to geometric imperfections and the spread of plasticity due to residual stresses under high compression loading. However, those of studies were based on the ACM, which adopted several simplifying assumptions.

The stability of a column in a framed structure, although often expressed as a stability problem restricted to an individual column, can be regarded as a problem in related with the whole stability of the structures by considering interactions between all members. Therefore, the column design is a system-related problem, not an individual member-restricted problem. AISC (1999) states that "…the effective length factor *K* of compression members shall be determined by structural analysis". Shanmugam and Chen (1995) presented a brief review and assessment of four different methods in determination of *K*-factor including the alignment chart, LeMessurier's formula, Lui's formula, and the system buckling method. Roddis *et al.* (1998) also pointed out the practical limitations of ACM using the elastic system buckling analysis. Geschwindner (2002) reviewed a wide range of analytical approaches including the elastic system buckling analysis. Özmen and Girgin (2005) developed a simplified procedure for determining approximate values for the buckling length of columns in multi-storey frames. This procedure utilized lateral load analysis of frames and yielded errors in the order of 10%, which might be considered suitable for design purpose. Also Girgin *et al.* (2006) applied their previous practical method (Özmen and Girgin 2005) to irregular frames and obtained good results with errors less than 5% for all examples considered.

Even though significant amounts of research have been reported on *K*-factor evaluation for the stability design of steel frames, several studies are still on going as mentioned. The purpose of this study is also to propose a more reliable procedure to determine *K*-factors of steel columns in braced and unbraced frames based on the elastic/inelastic system buckling analyses using tangent modulus-nominal compressive stress curves of steel columns. The important points considered are summarized as follows:

1. Firstly, based on the tangent modulus theory for inelastic buckling of columns, nominal stress-strain, tangent modulus-slenderness ratio, and *tangent modulus-nominal stress relationships* are consistently derived from column strength curves proposed in the AISC (2005) and SSRC (Galambos 1998) for columns with a wide range of slenderness ratio.
2. Next, tangential stiffness matrices of a beam-column member using tangent modulus, stability functions, and Hermitian interpolation polynomials are described, and two inelastic system buckling analysis procedures based on tangent modulus-compressive stress relationship are presented using nonlinear eigenvalue analysis algorithms.
3. Then, two effective buckling lengths for each column are evaluated from the elastic and inelastic system buckling analyses and a practical method to take the smaller one of two lengths as the final effective buckling length is proposed.
4. Through numerical examples, *K*-factors of columns in braced and unbraced frames are evaluated and compared with those from available references. In addition, the characteristics of effective lengths of relatively weak members are investigated from system buckling analysis results.

2. Alignment chart method

As mentioned, the concept of the effective length has been introduced to consider the structural interaction between the structure and its individual members. The alignment chart is widely used because it offers a straightforward method of obtaining the effective length of a column. This Section briefly summarizes ACM presented in the AISC-LRFD Specification (2005).

Basically, the alignment charts for the design of braced and unbraced frames, respectively, are graphical representations of the transcendental equation of a buckling solution of a subassemblage

$$\frac{G_A G_B}{4}\left(\frac{\pi}{K}\right)^2 + \left(\frac{G_A + G_B}{2}\right)\left(1 - \frac{\pi/K}{\tan(\pi/K)}\right) + \frac{2\tan(\pi/2K)}{\pi/K} - 1 = 0 \tag{1a}$$

$$\frac{G_A G_B(\pi/K)^2 - 36}{6(G_A + G_B)} - \frac{\pi/K}{\tan(\pi/K)} = 0 \tag{1b}$$

where K is the effective length factor, G_A and G_B are the stiffness ratios of columns and girders at joints A and B, respectively. Note that Eqs. (1a, b) may be analytically derived under several severe fundamental assumptions that the structure should follow in order to guarantee a reasonable result. One of them is the frames should be regular.

To improve the accuracy of the effective length factors based on ACM, the commentary of the AISC Specification recommends Equation C-C2-8 based on the storey buckling approach (LeMessurier 1977) as

$$K_i' = \sqrt{\frac{\pi^2 E_i I_i / L_i^2}{P_{r,i}}\left(\frac{\sum P_{r,i}}{\sum \frac{\pi^2 E_i I_i}{(K_{n,i} L_i)^2}}\right)} \geq \sqrt{\frac{5}{8}} K_{n,i} \tag{2}$$

where $E_i I_i$ and L_i are the flexural rigidity and the actual length of individual columns, respectively. $K_{n,i}$ denotes K_i value determined directly from the alignment chart. $P_{r,i}$ is the required axial compressive strength for the ith rigid column. ΣP_r is the total axial compressive strength of columns in the story.

To consider the inelastic stiffness reduction of a column, the elastic modulus can be lowered with the inelasticity adjustment factor τ_a as

$$E^* = \tau_a E \tag{3}$$

with

$$\tau_a = 1.0 \quad \text{for} \quad P_u/P_y \leq 0.39 \tag{4a}$$

$$\tau_a = -2.724(P_u/P_y)\ln(P_u/P_y) \quad \text{for} \quad P_u/P_y > 0.39 \tag{4b}$$

where E is the elastic modulus, P_u and P_y are the acting and yielding axial forces for the column, respectively.

3. Inelastic buckling of a column member

In this Section, the tangent modulus theory for inelastic buckling analysis is briefly described for a simply-supported column. Then, the tangent modulus-nominal stress relationship is derived from the column strength curves to consider the inelastic buckling behavior of a column. This relationship is used in the development of the inelastic system buckling analysis procedure in the next Section.

3.1 Tangent modulus theory for a single steel column

Historically, several inelastic buckling theories have been developed as such a form of the tangent modulus theory, the double modulus theory, and the Shanley's theory to consider the load-carrying capacity in inelastic range. Among these available theories, the tangent modulus theory is widely adopted in practice because of its simplicity in application but relatively high accuracy compared to the double modulus theory, which has been verified by experiments.

In the tangent modulus theory, the equilibrium equation of a simply-supported column subjected to axial load P is expressed as

$$E_t I y'''' + P y'' = 0 \tag{5}$$

where E_t is the tangent modulus corresponding to the stress level at inelastic buckling, I is the second moment of area. Note that the axial rigidity EA is tacitly assumed to remain unchanged in the tangent modulus theory.

Then, by a similar process to the elastic buckling problem, the inelastic buckling stress and the corresponding buckling mode of a simply supported column are obtained as

$$f_{cr} = \frac{\pi^2 E_t}{\lambda^2}, \quad \phi(x) = \sin\frac{\pi x}{l} \tag{6a,b}$$

where f_{cr} and λ are the inelastic buckling stress and the slenderness ratio of column, respectively.

Here, it is important to note that similarly to an elastic buckling mode, the inelastic buckling mode is composed of rigid body modes and deformable mode of sine function. The inelastic buckling mode for a simply-supported column is, therefore, expressed as Eq. (6b). In addition, it should be emphasized that the inelastic buckling stress (6a) is clearly proportional to the tangent modulus for a given slenderness ratio, which is utilized in developing an inelastic system buckling analysis procedure.

Provided that the nonlinear relationship between the compressive stress and strain is given for a column, the inelastic buckling load (i.e., ultimate load-carrying capacity) can be evaluated by using the tangent modulus E_t even when the compressive stress exceeds a proportional limit. The following illustrates the evaluation procedure of the ultimate load-carrying capacity for a simply-supported column as a function of the slenderness ratio when a stress-strain curve is given.

i) Assume inelastic buckling stress f_{cr}.
ii) From a given stress-strain curve, obtain the tangent modulus E_t corresponding to the assumed f_{cr}.
iii) Calculate the slenderness ratio $\lambda = \pi\sqrt{E_t/f_{cr}}$ corresponding to the inelastic buckling stress.
iv) Repeat i)~iii) by altering f_{cr}.

With this procedure, a load-carrying capacity curve can be drawn as a function of the slenderness ratio λ_t. Or inversely, when a load-carrying capacity curve is given in design codes such as a form of the column strength curve, the nominal stress-strain curve can be derived with the help of Eq. (6a).

3.2 Derivation of nonlinear compressive stress-strain and tangent modulus-stress relationships

As external loads increase on a frame structure, the inelastic buckling of columns can lead to system collapse prior to the formation of a plastic hinge on the beam members. The load-carrying capacity of a column member is codified in several design specifications based on such information as theoretical buckling analysis results, various experimental results that considered initial imperfections due to manufacturing or constructional errors, residual stress due to welding, dependency on cross-sectional shape and etc.

Two kinds of load-carrying capacity curves are considered for further development in inelastic buckling formulation, one from the AISC-LRFD (2005) and the other from the SSRC (Galambos 1998).

In the development of the LRFD specification, the AISC (2005) specification committee decided to continue using only one column strength curve as

$$f^* = 0.658^{\lambda^{*2}} \quad \text{for} \quad 0 \le \lambda^* \le 1.5 \tag{7a}$$

$$f^* = \frac{0.877}{\lambda^{*2}} \quad \text{for} \quad \lambda^* > 1.5 \tag{7b}$$

with

$$f^* = \frac{f_{cr}}{f_y}, \qquad \lambda^* = \frac{L_e}{\pi r}\sqrt{\frac{f_y}{E}} \tag{8a,b}$$

where, f^* and f_y are the dimensionless strength and the yield stress of the column, respectively; λ^* is the slenderness parameter; $L_e(=KL)$ and r are the effective buckling length and the radius of gyration, respectively. This equation was calibrated to fit closely the SSRC (Galambos 1998) curve 2, which was modified to reflect an initial out-of-straightness of about 1/1500.

On the other hand, SSRC proposed a column strength curve in terms of the slenderness parameter as

$$f^* = 1 - \frac{\lambda^{*2}}{4} \quad \text{for} \quad 0 \le \lambda^* \le \sqrt{2} \tag{9a}$$

$$f^* = \frac{1}{\lambda^{*2}} \quad \text{for} \quad \lambda^* \ge \sqrt{2} \tag{9b}$$

The parabola curve in Eq. (9a) and the Euler hyperbola curve in Eq. (9b) become tangent to each other when $\lambda^* = \sqrt{2}$. This curve has been developed based on the average critical stress for small and medium sized hot-rolled wide-flange sections of mild structural steel with symmetrical residual stress distribution typical of such members. These two column strength curves are depicted in Fig. 1, in terms of f^* and λ^*.

For the mathematical expression of the tangent modulus at the moment of buckling, the stress-strain relationship and the tangent modulus-nominal stress relationship need to be inversely derived

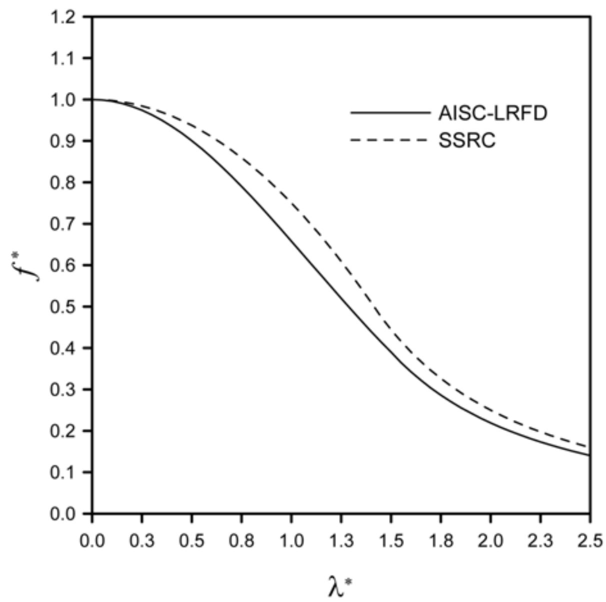

Fig. 1 Load-carrying capacity curves in AISC and SSRC

from the column strength curve. As the first step, dimensionless parameters are defined as

$$\varepsilon^* = \frac{\varepsilon}{\varepsilon_y}, \qquad E_t^* = \frac{E_t}{E} = \frac{df^*}{d\varepsilon^*} \tag{10}$$

where ε_y is the yield strain corresponding to the yield stress.

Using Eq. (10), Eq. (6a) can be expressed as the dimensionless form of the inelastic buckling stress.

$$f^* = \frac{E_t^*}{\lambda^{*2}} \tag{11}$$

The mathematical formulations of required relationships are illustrated here with the column strength curve in AISC-LRFD. For the column of $\lambda^* \le 1.5$, using Eqs. (7a) and (11), the dimensionless tangent modulus is expressed as

$$E_t^* = \lambda^{*2} f^* = \frac{\ln f^*}{\ln 0.658} f^* = -2.389 f^* \ln f^* = \frac{df^*}{d\varepsilon^*} \tag{12}$$

Separating variables of Eq. (12) gives

$$\frac{df^*}{f^* \ln f^*} = -2.389 d\varepsilon^* \tag{13}$$

Then, the general solution of Eq. (13) for the initial condition of $f^* = 0.390$ at $\varepsilon^* = 0.444$ for λ^* of 1.5 is obtained as follows

$$f^* = \exp[-2.725\{\exp(-2.389\varepsilon^*)\}] \quad \text{for} \quad \varepsilon^* \ge 0.444 \tag{14}$$

Following the similar procedure for a column of $\lambda^* > 1.5$, the differential equation can also be expressed as

$$\frac{df^*}{d\varepsilon^*} = \lambda^{*2} f^* = 0.877 \tag{15}$$

In Eq. (15), integration of f^* with respect to ε^* gives

$$f^* = 0.877\varepsilon^* \qquad \text{for} \qquad 0 \le \varepsilon^* < 0.444 \tag{16}$$

Eqs. (14) and (16) are the stress-strain relationship and presented in Fig. 2(a) as a graphical form with solid line.

The tangent modulus-nominal stress relationship is also basically required for the inelastic system buckling analysis. The relationship for tangent modulus versus compressive stress can be obtained,

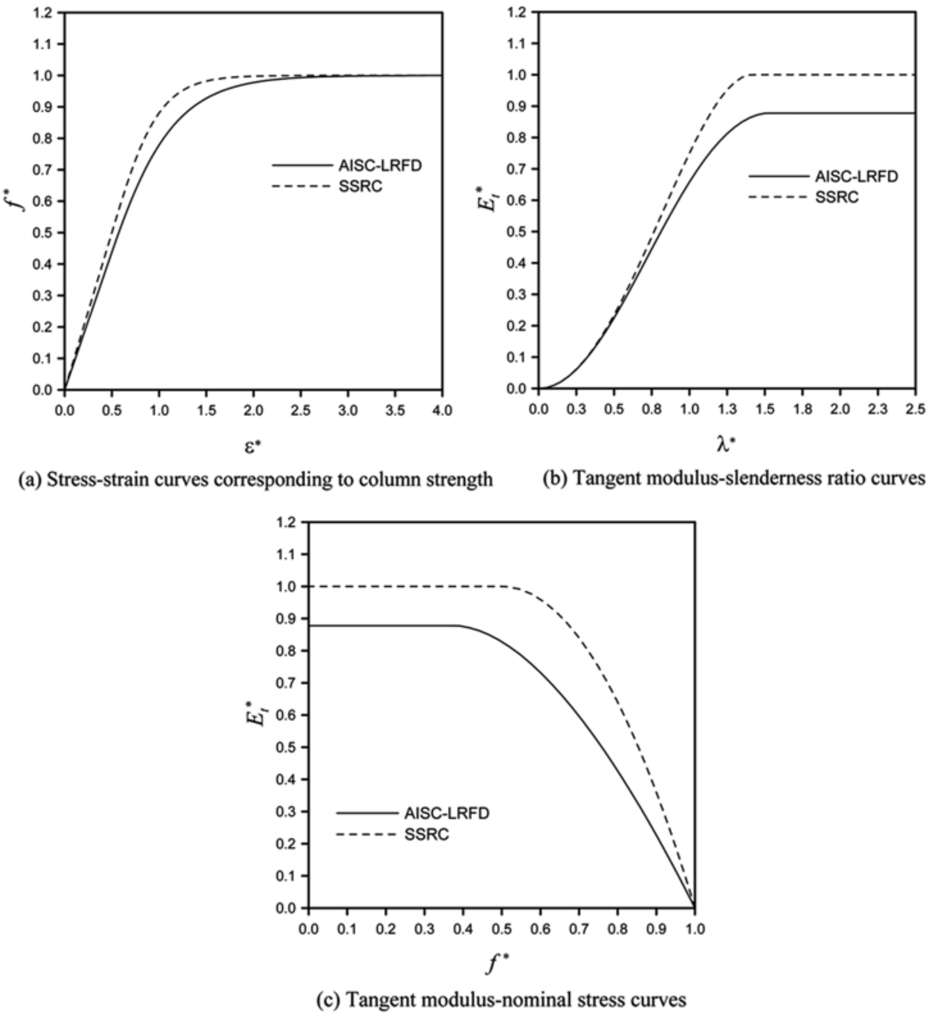

Fig. 2 Formulated curves according to the load-carrying capacity curves in AISC and SSRC

by eliminating slenderness ratio in Eqs. (7) and (11) as

$$E_t^* = 0.877 \qquad \text{for} \qquad 0 \le f^* < 0.39 \tag{17a}$$

$$E_t^* = -2.389 f^* \ln f^* \qquad \text{for} \qquad f^* \ge 0.39 \tag{17b}$$

Finally, the tangent modulus-slenderness ratio relationship is obtained by inserting Eq. (7) into Eq. (17) as follows

$$E_t^* = -2.389 \ln 0.658 \lambda^{*2} 0.658 \lambda^{*2} \qquad \text{for} \qquad 0 \le \lambda^* \le 1.5 \tag{18a}$$

$$E_t^* = 0.877 \qquad \text{for} \qquad \lambda^* > 1.5 \tag{18b}$$

For the case of a load-carrying capacity curve in SSRC, the similar procedure can also be applied and details are presented in APPENDIX. Fig. 2 shows the graphical form of the stress-strain, tangent modulus-slenderness ratio, and tangent modulus-stress relationships derived from the column strength curves proposed by AISC-LRFD and SSRC together.

4. Elastic and inelastic system buckling analyses

This Section describes a procedure for elastic and inelastic system buckling analyses based on the tangent modulus theory and tangent modulus-stress relationships discussed in the Section 3. The procedure of evaluating the effective buckling length for each frame member is then proposed using the solutions of the system buckling analysis.

4.1 Tangential stiffness matrix of a steel beam-column element

Consider the inelastic tangential stiffness matrix of the conventional 2-node plane frame element for the system buckling analysis of frame structures. Here, $\mathbf{d}_i$ $(= \langle u^p, v^p, \omega^p, u^q, v^q, \omega^q \rangle^T)$ represents the member or element displacement vector and $\mathbf{f}_i$ the corresponding force vector of the ith frame element in the local coordinate system. Then, the incremental force-displacement relationship of the element subjected to an initial axial force P_i is given as

$$\mathbf{f}_i = \mathbf{k}_t^i(E_t^i, P_i)\mathbf{d}_i \tag{19}$$

with

$$\mathbf{k}_t^i = \begin{bmatrix} a & \cdot & \cdot & -a & \cdot & \cdot \\ \cdot & b_1 & -b_2 & \cdot & -b_1 & -b_2 \\ \cdot & -b_2 & b_3 & \cdot & b_2 & b_4 \\ -a & \cdot & \cdot & a & \cdot & \cdot \\ \cdot & -b_1 & b_2 & \cdot & b_1 & -b_2 \\ \cdot & -b_2 & b_4 & \cdot & -b_2 & b_3 \end{bmatrix} \tag{20}$$

where

$$a = \frac{EA}{L_i}, \quad b_1 = \frac{12E_t^i I}{L_i^3}\phi_1^i, \quad b_2 = \frac{6E_t^i I}{L_i^2}\phi_2^i, \quad b_3 = \frac{4E_t^i I}{L_i}\phi_3^i, \quad b_4 = \frac{2E_t^i I}{L_i}\phi_4^i \tag{21a-e}$$

where ϕ_k is the so-called stability functions that depends on the axial force. The details on the stability function can be referred to Chen and Lui (1987).

If the axial force is small or a member is divided into several finite elements, the stability functions may be approximated using only constant and linear terms with respect to axial forces by the Taylor series expansion. In case, the tangential stiffness matrix can be decomposed into the inelastic stiffness matrix $\mathbf{k}_{\mathbf{te}}^{\mathbf{i}}$ and the geometric stiffness matrix $\mathbf{k}_{\mathbf{tg}}^{\mathbf{i}}$ as follows

$$\mathbf{k}_{\mathbf{te}}^{\mathbf{i}} + \mathbf{k}_{\mathbf{tg}}^{\mathbf{i}} = \begin{bmatrix} a & \cdot & \cdot & -a & \cdot & \cdot \\ \cdot & b_5 & -b_6 & \cdot & -b_5 & -b_6 \\ \cdot & -b_6 & b_7 & \cdot & b_6 & b_8 \\ -a & \cdot & \cdot & a & \cdot & \cdot \\ \cdot & -b_5 & b_6 & \cdot & b_5 & -b_6 \\ \cdot & -b_6 & b_8 & \cdot & -b_6 & b_7 \end{bmatrix} \tag{22}$$

where

$$b_5 = \frac{12E_t^i I}{L_i^3} + \frac{6P_i}{5L_i}, \quad b_6 = \frac{6E_t^i I}{L_i^2} + \frac{P_i}{10}, \quad b_7 = \frac{4E_t^i I}{L_i} + \frac{2P_i L_i}{15}, \quad b_8 = \frac{2E_t^i I}{L_i} - \frac{P_i L_i}{30} \tag{23a-d}$$

and stiffness matrices (22) may be derived by substituting Hermitian interpolation polynomials into total potential energy of beam-column elements and integrating it.

In Eqs. (19), (20), and (22), note that the tangent modulus is replaced with the elastic modulus only for the stiffness component related with flexural rigidity. At the moment of elastic or inelastic system buckling being occurred in the system, the incremental equilibrium equation for the whole structural system can be formulated by the use of the direct stiffness assembly process as follows

$$\mathbf{K_t}(\xi)\Delta\mathbf{U} = 0 \tag{24a}$$

$$(\mathbf{K_{te}} + \xi\mathbf{K_{tg}})\Delta\mathbf{U} = 0 \tag{24b}$$

where

$$\mathbf{K_t}(\xi) = \sum_i \mathbf{k}_{\mathbf{t}}^{\mathbf{i}}(E_t^i, \xi P_i), \quad \mathbf{K_{te}} = \sum_i \mathbf{k}_{\mathbf{te}}^{\mathbf{i}}(E_t^i), \quad \mathbf{K_{tg}}(\xi) = \xi\sum_i \mathbf{k}_{\mathbf{tg}}^{\mathbf{i}}(P_i) \tag{25a-c}$$

and ξ is a buckling parameter multiplied for the axial force of each frame element.

4.2 Elastic and inelastic system buckling analyses

Fig. 3 shows the system buckling mode of a frame as a whole structure together with an enlarged individual column member subjected to axial force P. The support effect of the surrounding frame structure on the buckling of a focused member can be simulated by virtual elastic springs at both

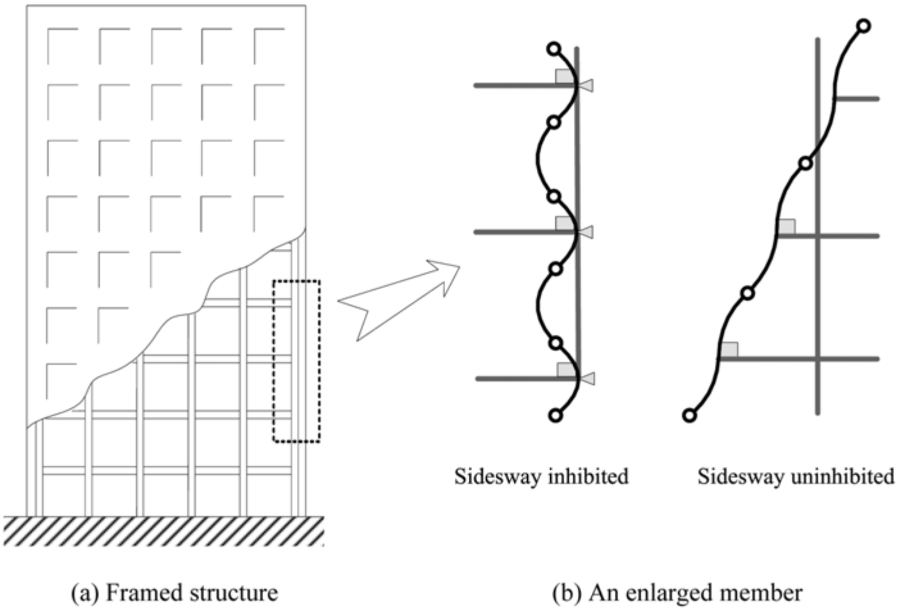

Fig. 3 System buckling of a frame structure

ends of that member. Note that elastic boundary springs are only introduced conceptually, and spring coefficients do not need to be determined in an actual application.

According to Chen and Lui (1987), the best way to evaluate the exact K-factor for each individual member consisting of a frame structural system should be the stability analysis of the entire structure as a whole. Hence, the buckling load P_{cr} of the column can be obtained as multiplication of the initial axial force P acting on a column and the buckling parameter ξ_{cr} obtained from elastic system buckling analysis. After then, the effective buckling length can be calculated as the distance between two inflection points as

$$P_{cr} = P\xi_{cr} = \frac{\pi^2 EI}{(KL)^2} \tag{26}$$

In case of the *inelastic* system buckling analysis, the inelastic behavior of column due to geometric imperfections and the gradual spread of plasticity in the cross-section due to residual stresses should be considered. This study accounts for these inelastic effects with the tangent modulus E_t, which is derived as a function of the compressive stress in subsection 3.2.

In case of the elastic system buckling analysis using Eq. (24b), the critical parameter ξ_{cr} can be determined by solving the standard eigen-problem. However, for the element tangential stiffness matrix formulated using the stability function such as Eqs. (20) and (24a), the elastic stiffness matrix and the geometric stiffness matrix can not be separated. Therefore, a standard eigenvalue procedure is no longer applicable and a nonlinear eigenvalue procedure should be adopted. For this purpose, two system buckling analysis procedures are developed with relation to two nonlinear eigen-problems (24a) and (24b) in this section.

4.2.1 System buckling analysis procedure I

As the load parameter ξ is given, the tangent modulus E_t for the individual column corresponding to the compressive force ξP_i can be evaluated with the help of Eq. (17). Then, the system tangential stiffness matrix $\mathbf{K_t}$ is assembled from the element tangential stiffness $\mathbf{k_t^i}(E_t^i, \xi P_i)$ and finally, the eigen-solution technique is applied to find the ξ_{cr} that satisfies $\det|\mathbf{K}_t(\xi_{cr})| = 0$, iteratively.

The followings describe the elastic and inelastic system buckling analysis procedure adopted.

Step 1) The first order elastic analysis:

For a given load condition, the following linear elastic analysis is performed to evaluate the axial force P_i for each member.

$$\mathbf{K_e U} = \mathbf{F} \tag{27}$$

where $\mathbf{K_e}$ is the system elastic stiffness matrix, $\mathbf{F}$ is the external force vector, and $\mathbf{U}$ is the nodal displacement vector.

Step 2) Elastic system buckling analysis procedure I:

Eq. (24a) denoting a nonlinear eigen-problem may be rewritten as follows:

$$\mathbf{K_t}(\xi_{cr})\Delta\mathbf{U} = \sum_{i=1}^{n} \mathbf{k_t^i}(\xi_{cr}P_i \;\; E)\Delta\mathbf{U} = 0 \tag{28}$$

where the eigenvalue ξ_{cr} should satisfy the condition in which the determinant of $\mathbf{K_t}(\xi_{cr})$ becomes zero. In this study, the ξ_{cr} is firstly calculated utilizing a mathematical property that one of the pivot element of the stiffness matrix transformed to the upper triangular matrix changes its sign when ξ approaches ξ_{cr}. The corresponding buckling mode can be then evaluated by using the so called *penalty method* (Bathe 1996). That is, a constant of relatively large magnitude κ is added to the mth diagonal element of $\mathbf{K_t}(\xi_{cr})$ in Eq. (28) and a corresponding force vector component, respectively, so that the required mth displacement component ΔU_m is approximately equal to one as follows

$$(\mathbf{K_t}(\xi_{cr}) + \kappa \mathbf{e}_m \mathbf{e}_m^T)\Delta\mathbf{U} = \kappa \mathbf{e}_m \tag{29}$$

where $\mathbf{e}_m$ is a vector with all entries equal to zero except its mth entry, which is equal to one.

Now utilizing the Gauss elimination method, we can obtain the frame buckling mode from Eq. (29).

Step 3) Inelastic system buckling analysis procedure I:

Consider the nonlinear eigenvalue analysis problem taking into account tangent modulus E_t^i depending on the compressive load ξP_i instead of the elastic modulus in Eq. (28).

$$\mathbf{K_t}(\xi_{cr})\Delta\mathbf{U} = \sum_{i=1}^{n} \mathbf{k_t^i}(\xi_{cr}P_i, E_t^i)\Delta\mathbf{U} = 0 \tag{30}$$

To get the ξ_{cr} considering inelastic buckling effect, the compressive stress is first calculated for a given compressive load ξP_i for ith member. The corresponding tangent modulus E_t^i is then obtained from Eq. (17) for the evaluation of $\mathbf{k_t^i}$. Now construct the system tangential stiffness matrix

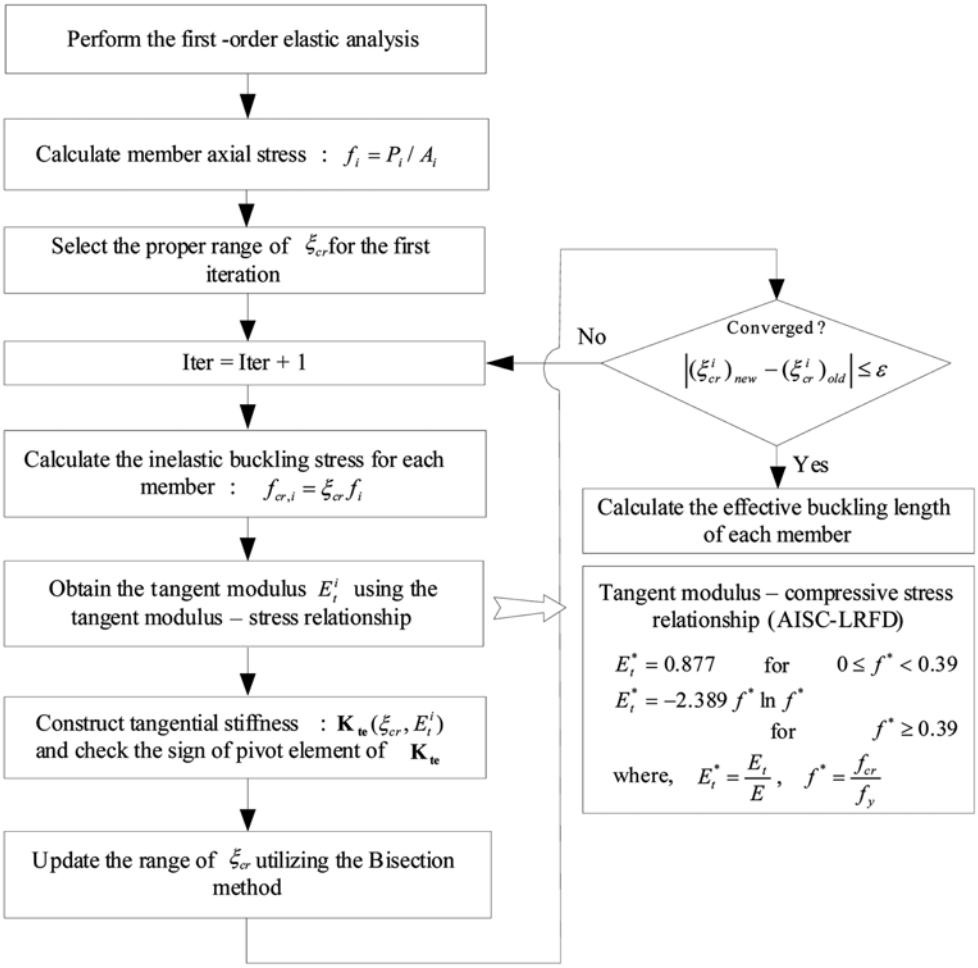

Fig. 4 Flowchart of inelastic system buckling analysis procedure I

increasing the load parameter ξ and $\det|\mathbf{K_t}(\xi_{cr})|$ is repeatedly evaluated to find ξ_{cr} until the determinant of the tangential stiffness matrix vanishes. Finally the inelastic buckling mode can be easily obtained using the *penalty method*.

In addition, Fig. 4 shows the flow chart for elastic/inelastic system buckling analysis procedure I.

Step 4) Calculation of effective buckling lengths based on both elastic and inelastic buckling analyses:

After the two eigenvalues ξ_{cr}^e and ξ_{cr}^{in} are determined from the elastic and inelastic system buckling analyses, respectively, with the axial force P_i evaluated from the first order elastic analysis, the effective buckling length KL can be determined for each element as

$$P_i \xi_{cr}^e = \frac{\pi^2 E I_i}{(K_i^e L_i)^2}, \qquad P_i \xi_{cr}^{in} = \frac{\pi^2 E_t^i I_i}{(K_i^{in} L_i)^2} \tag{31a,b}$$

Step 5) Final determination of the effective buckling length:

The effective lengths obtained from system buckling analysis show that the K-factors for the relatively weak columns (e.g., subjected to relatively large compression or having a relatively small stiffness) tend to be small. Inversely, the members subjected to small compression or having a large stiffness have relatively large K-factors. With respect to the system buckling mode, it means that the surrounding strong members systematically take over the burden of weak ones. Furthermore, this trend is much more prominent for the inelastic buckling analysis.

On the other hand, according to the LRFD Specifications, the K-factors of a column, highly loaded up to the inelastic range, are always smaller than those loaded within the elastic range because inelastic K-factors are also determined from the classical alignment charts by applying the stiffness reduction factors.

Based on these two observations, the following scheme determining the K-factors is proposed in this study: *the smaller one of two effective buckling lengths based on elastic/inelastic system buckling analyses is taken as the final effective buckling length.*

4.2.2 System buckling analysis procedure II

In this subsection, an elastic/inelastic system buckling analysis algorithm for tangential stiffness (24b) based on the Hermitian beam-column element is presented.

Step 1): The same as that of 4.2.1

Step 2) Elastic system buckling analysis procedure II:

The *elastic* system buckling analysis corresponding to Eq. (24b) may be easily solved by adopting a standard eigenvalue algorithm.

$$(\mathbf{K_{te}} + \xi_{cr}\mathbf{K_{tg}})\Delta\mathbf{U} = \sum_{i=1}^{n}(\mathbf{k^i_{te}}(E) + \xi_{cr}\mathbf{k^i_{tg}}(P_i))\Delta\mathbf{U} = 0 \tag{32}$$

Step 3) Inelastic system buckling analysis procedure II:

Now the *inelastic* system buckling analysis corresponding to Eq. (24b) may be written as follows

$$(\mathbf{K_{te}} + \xi_{cr}\mathbf{K_{tg}})\Delta\mathbf{U} = \sum_{i=1}^{n}(\mathbf{k^i_{te}}(E^i_t) + \xi_{cr}\mathbf{k^i_{tg}}(P_i))\Delta\mathbf{U} = 0 \tag{33}$$

where it should be again noticed that tangent modulus E^i_t depends on the compressive load ξP_i.

In this case, the first iteration procedure becomes the elastic system buckling analysis of step 2). The compressive stresses are then calculated from buckling loads $\xi^{(1)}_{cr}P_i$ for each member. After that, the corresponding tangent modulus E^i_t is obtained from Eq. (17) and the inelastic stiffness matrix $\mathbf{k^i_{te}}$ is evaluated. Now the updated inelastic and geometric stiffness matrices using the new tangent modulus E^i_t are constructed and the standard eigen-problem is again solved. The evaluation loop is repeated until the convergence criteria are satisfied.

Step 4) and Step 5): The same as those of 4.2.1

Finally the flow chart for elastic/inelastic system buckling analysis procedure II is displayed in Fig. 5.

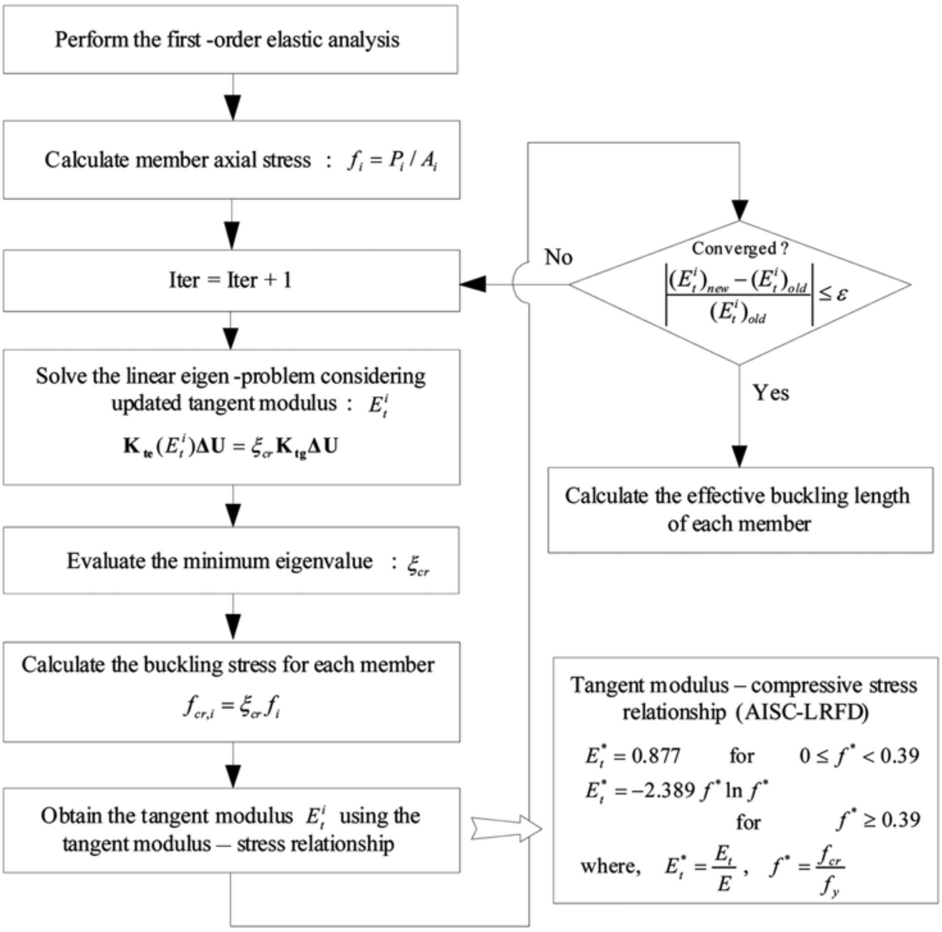

Fig. 5 Flowchart of inelastic system buckling analysis procedure II

5. Numerical examples

Generally two elastic/inelastic system buckling analysis algorithms provide identical solutions if a *single* element based on stability functions and *several* Hermitian elements are used in FE modeling for each prismatic member subject to constant axial force. Therefore, in subsequent numerical examples, only one result for K-factors of columns in braced or unbraced frames is presented using the proposed elastic and/or inelastic system buckling analyses and compared with those from available references. In the following Tables, *ESBA* and *ISBA* denote the elastic and inelastic system bucking analyses, respectively, and the shaded values denote the smaller one of two K-factors.

5.1 Continuous braced column

Five span continuous columns, as shown in Fig. 6, studied by Mahini and Seyyedian (2006) and Bridge and Fraser (1987), are considered. This continuous braced column is a common in building

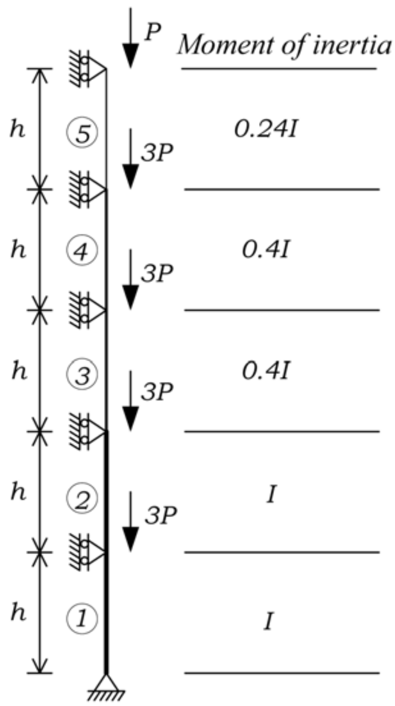

Fig. 6 Tied continuous column

Table 1 K-factors for the braced continuous column

Column No.	This study			Mahini and Seyyedian (2006)	Bridge and Fraser (1987)	ACM
	ISBA		ESBA			
	AISC	SSRC				
①	**0.83**	**0.81**	0.96	0.97	1.02	1.00
②	1.29	1.39	**1.10**	1.11	1.17	1.00
③	1.11	1.15	**0.83**	0.84	0.88	1.00
④	1.48	1.52	**1.10**	1.11	1.17	1.00
⑤	2.29	2.36	**1.70**	1.72	1.73	1.00

structures where the beams are simply connected to the columns and the structure is erected in simple framing.

Table 1 shows K-factors obtained in this study based on *ESBA* and *ISBA* and compared with other's calculations such as Mahini and Seyyedian (2006), Bridge and Fraser (1987), and ACM. Mahini and Seyyedian (2006) used a new iterative approach with high convergence rate and Bridge and Fraser (1987) proposed the improved G-factor method.

The results by *ESBA* are in excellent agreement with those of Mahini and Seyyedian (2006) and in good agreement with those by the improved G-factor method. Also the K-factors by inelastic analysis are larger than those based on elastic analysis except for the column ①.

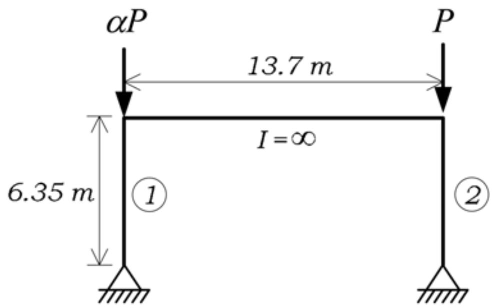

Fig. 7 Portal frame

Table 2 K-factors for the portal frame

α	Column No.	This study			Aristizabal-Ochoa (1994)	Cheong-Siat-Moy (1986)	AISC (C-C2-8)	ACM
		ISBA		ESBA				
		AISC	SSRC					
1.00	①	2.00	2.00	**2.00**	2.00	2.00	2.00	2.00
	②	2.00	2.00	**2.00**	2.00	2.00	2.00	2.00
0.25	①	3.34	3.33	**3.17**	3.15	3.14	3.16	2.00
	②	**1.52**	**1.52**	1.59	1.58	1.57	1.58	2.00
0.00	①	∞	∞	∞	∞	∞	∞	2.00
	②	**1.21**	**1.28**	1.43	1.41	1.41	1.41	2.00

5.2 Portal frame

Fig. 7 shows the typical portal frame consists of columns with W8 × 31's rolled section. The frame is 6.35 m in height and 13.7 m in width. As shown in Fig. 7, proportional loads αP and P act on columns ① and ②, respectively.

Table 2 shows K-factors calculated in this study for the given proportional constant α of 1.00, 0.25 or 0.00, and compared with those by Aristizabal-Ochoa (1994), Cheong-Siat-Moy (1986), and ACM. Since the commentary of the AISC Specification (AISC 2005, C-C2-8) recommends the use of a modified equation for determining the K-factor based on the storey based buckling method, the results are also compared in Table 2. K-factors by *ESBA* and *ISBA* in this study are in good agreement with Aristizabal-Ochoa (1994), Cheong-Siat-Moy (1986), and AISC (C-C2-8) results.

When the inelastic analysis is applied, K-factors become smaller for the column ②, which is subjected to higher compressive force than the column ①, compared with the elastic analysis results. On the contrary, for the column ①, K-factors become larger when the inelastic analysis applied. Consequently this shows K-factors of a frame in inelastic analysis is much more sensitive to the compressive force distribution in each column.

5.3 One-bay three-storey frame

In this example, analyses are carried out on larger frames to investigate the validity of the present

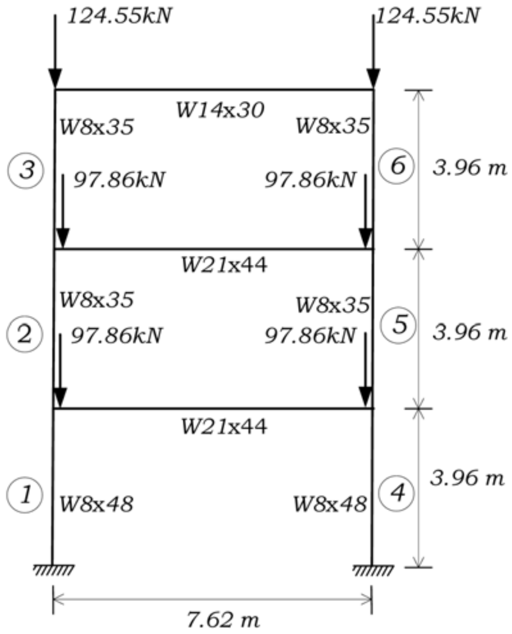

Fig. 8 Simple one-bay three-storey frame

Table 3 *K*-factors for the simple one-bay three-storey frame

Column No.	This study			Lui and Xu (2005)	AISC (C-C2-8)	ACM
	ISBA		ESBA			
	AISC	SSRC				
①, ④	**1.11**	**1.02**	1.15	1.15	1.11	1.11
②, ⑤	1.15	1.23	**1.14**	1.15	1.21	1.21
③, ⑥	2.69	2.89	**1.53**	1.53	1.23	1.23

method to predict *K*-factors of columns in multi-storey frames. One single-bay, three-storey frame, as shown in Fig. 8, has the area and the moment of inertia associated with W-shape sections as follows: W8 × 35, A = 66.45 cm^2, I = 5286 cm^4; W8 × 48, A = 90.97 cm^2, I = 7659 cm^4; W14 × 30, A = 57.1 cm^2, I = 12112 cm^4; W21 × 44, A = 83.87 cm^2, I = 35088 cm^4.

The proposed *K*-factors are compared with other investigations in Table 3. The *ESBA* results showed fairly good agreement with those by Lui and Xu (2005), who used the geometrical stiffness distribution approach in the system buckling method. Whesn the inelastic behavior is considered in system bucking analysis, *K*-factors for columns ① and ④, which are subjected to the highest axial force, became smaller. On the other hand, inelastic analysis introduced higher *K*-factors for columns ③ and ⑥ subjected to a relatively small axial force.

5.4 Un-symmetric frame with three-bay three-storey

Fig. 9 shows an un-symmetric frame consisting of three-bay three-storey. The sectional properties

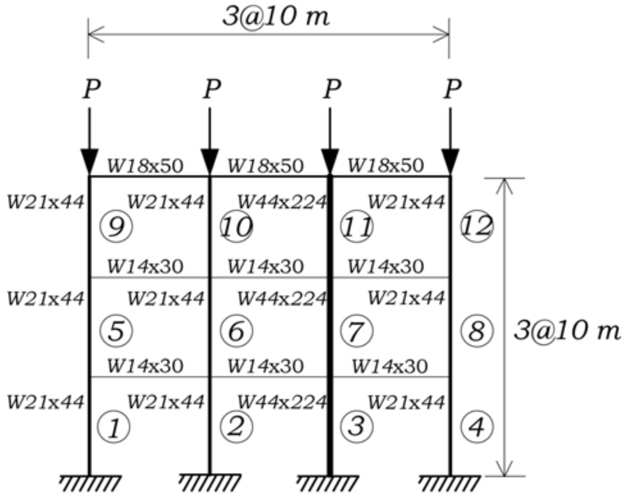

Fig. 9 Un-symmetric frame with three-bay three-storey

Table 4 *K*-Factors of the un-symmetric frame with three-bay three-storey

Column No.	This study			AISC (C-C2-8)	ACM
	ISBA		ESBA		
	AISC	SSRC			
①	**0.62**	**0.62**	1.29	1.70	1.54
②	**0.65**	**0.64**	1.30	1.70	1.36
③	8.37	8.87	**6.10**	8.12	1.93
④	**0.64**	**0.64**	1.30	1.70	1.54
⑤	**0.62**	**0.62**	1.29	2.45	2.37
⑥	**0.64**	**0.64**	1.30	2.45	1.80
⑦	8.38	8.87	**6.11**	11.71	7.42
⑧	**0.64**	**0.64**	1.30	2.45	2.37
⑨	**0.62**	**0.62**	1.29	1.89	1.76
⑩	**0.64**	**0.63**	1.30	1.89	1.46
⑪	8.40	8.89	**6.12**	9.03	4.32
⑫	**0.64**	**0.63**	1.27	1.89	1.76

of the W-shape sections are as follows: W21 × 44, A = 83.87 cm^2, I = 35100 cm^4; W14 × 30, A = 57.10 cm^2, I = 12100 cm^4; W18 × 50, A = 94.84 cm^2, I = 33300 cm^4; W44 × 224, A = 424.52 cm^2, I = 799200 cm^4. Since the distribution of structural member is un-symmetric, this frame can be classified as one of the structure that the alignment chart is not readily applicable to.

Table 4 shows *K*-factors using *ESBA* and *ISBA* in this study, AISC (C-C2-8), and ACM results together. In case of the proposed system buckling analysis, almost equal *K*-factors were evaluated for the columns of same cross-section, length, and initial axial force. Also AISC (C-C2-8) results displayed the similar trend to those by *ESBA* and *ISBA*. However, ACM gave different results even for the same columns according to the stiffness of adjacent beams and columns.

Particularly it is worth noticing that relatively large deviations in estimated *K*-factors between two

ESBA and *ISBA* were observed for strong columns ③, ⑦ and ⑪ and the remaining weak columns. Since the stiffness of these strong columns is much larger than those of other columns, it is judged that this un-symmetric disposition of columns introduced conspicuous discrepancy in the estimation of the *K*-factor. Consequently *K*-factors reduced when the plastic effect was considered for the relatively weak columns of the structure, while they increased for the relatively strong columns of ③, ⑦ and ⑪.

6. Conclusions

The effective buckling length of a steel frame has been studied extensively. The LRFD-AISC (2005) commentary currently recommends the so-called alignment chart in evaluating the *K*-factor of a steel beam-column member for the design of frame structures. Since this ACM is, however, based on many basic assumptions, it may lead to unreasonable or uneconomical *K*-factors if a frame structure does not satisfy these assumptions.

This study proposed newly two inelastic system buckling analysis procedures of steel framed structures by generalizing the iterative inelastic buckling procedure of Yura (1971) using ACM for the evaluation of *K*-factor. The nonlinear tangent modulus-nominal stress relationship was inversely formulated from the column strength curves specified in AISC-LRFD and SSRC. This relationship was utilized to calculate the inelastic tangential stiffness matrix when the compressive load of a column element was given. Consequently *K*-factors obtained by the proposed inelastic procedure exactly conform to the column design capacity specified in AISC-LRFD and SSRC. Comparative numerical examples and parametric studies were conducted and the following conclusions were drawn.

1. If all columns are subjected to almost equal axial force, the system buckling analysis, compared with ACM, produced larger *K*-factors for stiff columns, while on the other, smaller *K*-factors for weak columns. If the column stiffness is almost equal, then the system buckling analysis, compared with ACM, produced larger *K*-factors for the column subjected to smaller axial force and smaller *K*-factors for the column subjected to larger axial force.
2. In calculating the *K*-factor based on the elastic buckling analysis of a frame system, the relatively stiffer column shows larger *K*-factor than weaker one if all columns are subjected to almost equal axial force. If the column stiffness is almost equal, then the column subjected to smaller axial force shows larger *K*-factor.
3. Particularly, the above tendency was more prominent when the inelastic system buckling analysis was applied.
4. Inelastic system buckling analyses can reduce the *K*-factor of a relatively weak or highly stressed column that controls the practical load-carrying capacity of the whole structural system of a frame structure. Consequently a more reasonable and economical design is expected by considering inelastic buckling analyses.

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Appendix. Stress-strain and tangent modulus-stress relationships of column by SSRC

For column of $0 \le \lambda^* \le \sqrt{2}$, the dimensionless inelastic buckling stress is expressed as

$$f^* = \frac{E_t^*}{\lambda^{*2}} = 1 - \frac{\lambda^{*2}}{4} \tag{A-1}$$

The dimensionless tangent modulus is

$$E_t^* = \frac{df^*}{d\varepsilon^*} = \lambda^{*2} f^* = 4f^*(1 - f^*) \tag{A-2}$$

From Eq. (A-2), the following equation can be considered as an integration form.

$$\int \frac{1}{f^*(1 - f^*)} df^* = \int 4\, d\varepsilon^* \tag{A-3}$$

The general solution of Eq. (A-3) is

$$f^* = \frac{1}{C_1 e^{-4\varepsilon^*} + 1} \tag{A-4}$$

The integration constant C_1 is evaluated from the condition of $f^* = 0.5$ at $\varepsilon^* = 0.5$ for λ^* of $\sqrt{2}$. Resultantly

$$f^* = \frac{1}{7.389 e^{-4\varepsilon^*} + 1} \quad \text{at} \quad \varepsilon^* \geq 0.5 \tag{A-5}$$

Similarly, for column of $\lambda^* \geq \sqrt{2}$, the dimensionless inelastic buckling stress and tangent modulus are

$$f^* = \frac{E_t^*}{\lambda^{*2}} = \frac{1}{\lambda^{*2}} \tag{A-6}$$

and

$$E_t^* = \frac{df^*}{d\varepsilon^*} = \lambda^{*2} f^* = 1.0 \tag{A-7}$$

The general solution for buckling stress is obtained from Eq. (A-7).

$$f_* = \varepsilon_* + C_2 \tag{A-8}$$

From the initial condition of $f^* = 0.0$ at $\varepsilon^* = 0.0$, C_2 is zero and as a result

$$f^* = \varepsilon^* \quad \text{at} \quad 0 \leq \varepsilon^* < 0.5 \tag{A-9}$$

Similarly, a tangent modulus-nominal stress relationship is obtained from Eqs. (A-5, 9)

$$E_t^* = 1.0 \qquad \text{for} \qquad 0 \leq f^* < 0.5 \tag{A-10a}$$

$$E_t^* = 4f^*(1 - f^*) \qquad \text{for} \qquad f^* \geq 0.5 \tag{A-10b}$$

In addition, a tangent modulus-slenderness ratio relationship is derived as follows

$$E_t^* = \frac{16}{\lambda^{*2}}\left(1 - \frac{4}{\lambda^{*2}}\right) \qquad \text{for} \qquad 0 \leq \lambda^* \leq \sqrt{2} \tag{A-11a}$$

$$E_t^* = 1.0 \qquad \text{for} \qquad \lambda^* \geq \sqrt{2} \tag{A-11b}$$