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Interval finite element method for complex eigenvalues of closed-loop systems with uncertain parameters

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Abstract. In practical engineering, the uncertain concept plays an important role in the control problems of the vibration structures. In this paper, based on matrix perturbation theory and interval finite element method, the closed-loop vibration control system with uncertain parameters is discussed. A new method is presented to develop an algorithm to estimate the upper and lower bounds of the real parts and imaginary parts of the complex eigenvalues of vibration control systems. The results are derived in terms of physical parameters. The present method is implemented for a vibration control system of the frame structure. To show the validity and effectiveness, we compare the numerical results obtained by the present method with those obtained by the classical random perturbation.

Keywords: matrix perturbation method; interval finite element; stability robustness; random perturbation; uncertain closed-loop system.

1. Introduction

Analyzing uncertain systems has been a major issue of research for a few decades within many practical engineering communities. In the past, problems with uncertainties have been studied to provide an insight into the statistical response variations. The methods used in these studied were based on probabilistic approaches includes simulation (involving sampling and estimation). But the probabilistic modeling is not the only way to describe the uncertainties. And the probabilistic approach is not able to deliver reliable results at the required precision without sufficient experimental data. Uncertainties in parameters can also be modeled on the basis of alterative, non-probabilistic conceptual frameworks. One such approach is an unknown-but-bounded convex model, in which parameters may take any value within a prescribed "ellipsoid" centered at the nominal values and of given radius. Such set models of uncertainties in parameters have drawn interest both from the system control robustness analysis field and from the structural failure measures field. For example, the convex model was introduced by Ben-Haim and Elishakoff (1990). Venini (1998) used the convex model to discuss the robust control of uncertain structures. The convex model has also been applied to the optimal design of structures with uncertain parameters (Ganzerli and Pantelides 1999, 2000, Pantelides and Booth 2000, Pantelides and Gantelides 1998). A new method called the

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interval analysis has appeared since the mid-1960s. Moor (1979) and his co-workers, Alefeld and Herzberger (1983) have done the pioneering work. Recently, the interval models have been applied to the study of the responses of structures with interval parameters (Chen and Lian 2002, Chen and Wu 2004). In structural analysis, Dimarogonas (1995) discussed the interval of vibrating systems. Using the interval method, References (Chen and Ward 1997, Shin and Grandhi 2001) discussed the optimal structural design.

In recent years, the vibration control problem of structures with uncertain parameters has attracted a great deal of interest. For example, the reference discussed the control problem of uncertain system for helicopter rotor blades (Krodkiewski 2000). The control problems for a wide class of mechanical system with uncertainties was presented (Ferrara 2000). A systematic approach is proposed for determining the probability of instability for a control structure with real parameter uncertainties which were modeled as random variables with prescribed probability distributions (Spencer 2000). The robust vibration control of uncertain systems using variable parameter feedback and model-based fuzzy strategies were proposed (Li and Yarn 2001).

However, few papers can be found discussing the complex eigenvalues of closed-loop vibration control systems with uncertain parameters using the interval finite element method.

In this paper, the interval analysis is used to deal with the control systems with uncertain parameters. The uncertainties of the structural parameters are described by interval variables. The state feedback gain matrix of the systems with deterministic parameters can be obtained by using the method of pole allocation, and then it is applied to the actual uncertain system. Using the interval finite element method and matrix perturbation theory, a new method for estimating the upper and lower bounds of the real parts and imaginary parts of the complex eigenvalues of closed-loop systems with interval parameters will be discussed. The lower bounds of eigenvalues can be used to estimate the robustness of the stability of the uncertain controlled system. The method presented in this paper will not require the distribution function of the uncertain parameters of the systems other than their upper and lower bounds. The numerical example is given to illustrate the applications and effectiveness of the approach presented in this study.

2. Perturbation of linear control systems

In general, vibration control problems can be expressed in the second-order system using finite element method

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{q}(t) = \bar{\mathbf{B}}\mathbf{u}(t) \tag{1}$$

where $\mathbf{M} \in \mathbf{R}^{n \times n}$ is the mass matrix, $\mathbf{C} \in \mathbf{R}^{n \times n}$ is the damping matrix, $\mathbf{K} \in \mathbf{R}^{n \times n}$ is the stiffness matrix, $\mathbf{q}(t)$ is the node displacement vector, $\mathbf{u}(t) \in \mathbf{R}^{m_1 \times 1}$ is the feedback force, and $\bar{\mathbf{B}} \in \mathbf{R}^{n \times m_1}$ is the actuator location matrix, respectively.

Now, we transfer second-order system Eq. (1) into state-space system. Suppose the state vector is $\mathbf{x}(t) = [\dot{\mathbf{q}}^T(t), \mathbf{q}^T(t)]^T$, then we can obtain

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{2}$$

where

$$\mathbf{A} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \ \mathbf{B} = \begin{Bmatrix} \mathbf{M}^{-1}\overline{\mathbf{B}} \\ \mathbf{0} \end{Bmatrix} \tag{3}$$

However, uncertainties in structural parameters that are derived from modeling, manufacturing, installation or measurement errors are inevitable. If the design variables have some perturbations $\Delta\mathbf{b}$, i.e., $\mathbf{b} = \mathbf{b}_0 + \Delta\mathbf{b}$, then the mass matrix, stiffness matrix, and damping matrix of the uncertain system can be reformulated as

$$\mathbf{M}(\mathbf{b}) = \mathbf{M}_0(\mathbf{b}_0) + \Delta\mathbf{M}_1, \ \ \mathbf{K}(\mathbf{b}) = \mathbf{K}_0(\mathbf{b}_0) + \Delta\mathbf{K}_1, \ \ \mathbf{C}(\mathbf{b}) = \mathbf{C}_0(\mathbf{b}_0) + \Delta\mathbf{C}_1 \tag{4}$$

in which $\mathbf{M}_0$, $\mathbf{K}_0$ and $\mathbf{C}_0$ are the deterministic parts of corresponding matrices; $\Delta\mathbf{M}_1$, $\Delta\mathbf{K}_1$ and $\Delta\mathbf{C}_1$ are the uncertain parts of corresponding matrices, respectively.

Now expand $(\mathbf{M}_0 + \Delta\mathbf{M}_1)^{-1}$ to become (Alefeld 1991)

$$(\mathbf{M}_0 + \Delta\mathbf{M}_1)^{-1} = \mathbf{M}_0^{-1} - \mathbf{M}_0^{-1}\Delta\mathbf{M}_1\mathbf{M}_0^{-1} + \mathbf{M}_0^{-1}\Delta\mathbf{M}_1\mathbf{M}_0^{-1}\Delta\mathbf{M}_1\mathbf{M}_0^{-1} - \ldots \tag{5}$$

If the norm $\|\Delta\mathbf{M}_1\mathbf{M}_0^{-1}\|$ is less than unity, or in a more rigorously way, if and only if the spectral radius of $\Delta\mathbf{M}_1\mathbf{M}_0^{-1}$ is less than unity, we can obtain from Eq. (5)

$$(\mathbf{M}_0 + \Delta\mathbf{M}_1)^{-1} = \mathbf{M}_0^{-1} + \mathbf{M}_0^{-1}\sum_{i=1}^{\infty}(-\Delta\mathbf{M}_1\mathbf{M}_0^{-1})^i \tag{6}$$

Now neglecting high-order terms, we can arrive at the expressions for the matrices in state-space coordinates

$$\mathbf{A} = \mathbf{A}_0 + \Delta\mathbf{A}_1, \ \ \mathbf{B} = \mathbf{B}_0 + \Delta\mathbf{B}_1 \tag{7}$$

where

$$\mathbf{A}_0 = \begin{bmatrix} -\mathbf{M}_0^{-1}\mathbf{C}_0 & -\mathbf{M}_0^{-1}\mathbf{K}_0 \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \ \mathbf{B}_0 = \begin{Bmatrix} \mathbf{M}_0^{-1}\overline{\mathbf{B}} \\ \mathbf{0} \end{Bmatrix}$$

$$\Delta\mathbf{A}_1 = \begin{bmatrix} \mathbf{M}_0^{-1}(\Delta\mathbf{M}_1\mathbf{M}_0^{-1}\mathbf{C}_0 - \Delta\mathbf{C}_1) & \mathbf{M}_0^{-1}(\Delta\mathbf{M}_1\mathbf{M}_0^{-1}\mathbf{K}_0 - \Delta\mathbf{K}_1) \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{8}$$

$$\Delta\mathbf{B}_1 = \begin{Bmatrix} -\mathbf{M}_0^{-1}\Delta\mathbf{M}_1\mathbf{M}_0^{-1}\overline{\mathbf{B}} \\ \mathbf{0} \end{Bmatrix}$$

where $\mathbf{A}_0$ and $\mathbf{B}_0$ are the deterministic matrices; $\Delta\mathbf{A}_1$ and $\Delta\mathbf{B}_1$ are the uncertain parts, respectively.

Noted that the condition

$$\|\Delta\mathbf{M}_1\mathbf{M}_0^{-1}\| < 1 \tag{9}$$

guarantee the convergence of Eq. (6) and when $\Delta\mathbf{b}$ is sufficiently small, first order perturbation can be used in the practical engineering.

3. The gain matrix of the deterministic control system

As an approximation of the uncertain system, the feedback gain matrix can be designed based on the deterministic control system. In the pole allocation method, to guarantee asymptotic stability, the closed-loops poles can be selected in advance and the gains are determined so as to produce these poles.

Adding feedback force $\mathbf{u} = \mathbf{Gx}$ to the deterministic system, we obtain deterministic closed-loop vibration control system

$$\dot{\mathbf{x}}_0(t) = (\mathbf{A}_0 + \mathbf{B}_0\mathbf{G})\mathbf{x}_0(t) \tag{10}$$

When the closed-loop eigenvalues of Eq. (10) are assigned to in advance, by using the pole allocation, the gain matrix $\mathbf{G}$ of the deterministic system (10) can be determined.

Suppose the left and the right modal matrices $\mathbf{U}_0 = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{2n_f}]$ and $\mathbf{V}_0 = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2n_f}]$ have been obtained, where $n_f\,(n_f << n)$ is the number of the retained modes. They satisfy the following equations

$$\mathbf{V}_0^T\mathbf{A}_0\mathbf{U}_0 = \mathbf{\Lambda}_0, \quad \mathbf{V}_0^T\mathbf{U}_0 = \mathbf{I} \tag{11}$$

where $\mathbf{\Lambda}_0 = diag(\lambda_1, \lambda_2, \ldots, \lambda_{2n_f})$ is the diagonal matrix of the eigenvalues of the deterministic system.

With the modal transformation

$$\mathbf{x}_0(t) = \mathbf{U}_0\xi_0(t) \tag{12}$$

Eq. (10) can be transferred into

$$\dot{\xi}(t) = (\mathbf{\Lambda}_0 + \mathbf{B}'\mathbf{G}')\xi_0(t) \tag{13}$$

where $\mathbf{D}' = \mathbf{V}_0^T\mathbf{D}$ and if the single input is used, $\mathbf{B}' = \mathbf{V}_0^T\mathbf{B} = (b_1', b_2', \ldots, b_{2n_f}')^T$ is a column vector, $\mathbf{G}' = \mathbf{GU}_0 = (g_1', g_2', \ldots, g_{2n_f}')$ is a row vector.

In Eq. (13), suppose the assigned eigenvalues of the retained modes are $\lambda_i^*(i = 1, 2, \ldots, 2n_f)$, the corresponding eigenvectors are $\mathbf{y}_i(i = 1, 2, \ldots, 2n_f)$, and they satisfy the following eigenproblem

$$(\mathbf{\Lambda}_0 + \mathbf{B}'\mathbf{G}')\mathbf{y}_i = \lambda_i^*\mathbf{y}_i \ (i = 1, 2, \ldots, 2n_f) \tag{14}$$

i.e.

$$\det(\mathbf{\Lambda}_0 + \mathbf{B}'\mathbf{G}' - \lambda_i^*\mathbf{I}) = 0 \tag{15}$$

Because $\mathbf{y}_i \neq \mathbf{0}$, then there exists

$$g_i' = \prod_{k=1}^{2n_f}(\lambda_k^* - \lambda_i) \Big/ b_i' \prod_{\substack{k=1 \\ k \neq i}}^{2n_f}(\lambda_k - \lambda_i) \qquad i = 1, 2, \ldots, 2n_f \tag{16}$$

thus obtaining the matrix $\mathbf{G}' = (g_1', g_2', \ldots, g_{2n}')$. From $\mathbf{G}' = \mathbf{GU}_0$, we have $\mathbf{G} = \mathbf{G}'\mathbf{V}_0^T$.

If the deterministic gain matrix $\mathbf{G}$ is applied to the uncertain system, there must exist some errors between the actual closed-loop eigenvalues and the assigned eigenvalues $\lambda_i^*(i = 1, 2, \ldots, 2n_f)$.

4. Interval characteristic matrices for structures with interval parameters

The classical formulation of the system matrices does not take into account the way the matrices are built for the system physical parameters. In this section we will present a way to build the interval matrices with the system physical parameters. Assume that the interval parameters of the structures are denoted by

$$\mathbf{b}^I = (\mathbf{b}_1^I, \mathbf{b}_2^I, \ldots, \mathbf{b}_m^I)^T = \mathbf{b}^c + \Delta\mathbf{b}^I \tag{17}$$

in which m is the number of the interval parameters.

For any $\mathbf{b} \in \mathbf{b}^I$, the stiffness and mass matrices of the element are

$$\mathbf{K}_i^e(\mathbf{b}) = \int_{V_e} \mathbf{E}^T\mathbf{F}(\mathbf{b})\mathbf{E}dV, \quad \mathbf{M}_i^e(\mathbf{b}) = \int_{V_e} \mathbf{N}^T\rho(\mathbf{b})\mathbf{N}dV \tag{18}$$

Where $\mathbf{E} = \mathbf{LN}$ and $\mathbf{L}$ is differential operation matrix; $\mathbf{N}$ is the shape function matrix; $\mathbf{F}(\mathbf{b})$ is the elastic matrix; $\rho(\mathbf{b})$ is the mass density matrix.

Using Taylor series and expanding the stiffness and mass matrices, $\mathbf{K}_i^e(\mathbf{b})$ and $\mathbf{M}_i^e(\mathbf{b})$ around the mean values, $\mathbf{b}^c$, one has

$$\begin{aligned} \mathbf{K}_i^e(\mathbf{b}) &= \mathbf{K}_i^e(\mathbf{b}^c) + \sum_{j=1}^{m} \frac{\partial \mathbf{K}_i^e(\mathbf{b}^c)}{\partial b_j}(b_j - b_j^c) \\ \mathbf{M}_i^e(\mathbf{b}) &= \mathbf{M}_i^e(\mathbf{b}^c) + \sum_{j=1}^{m} \frac{\partial \mathbf{M}_i^e(\mathbf{b}^c)}{\partial b_j}(b_j - b_j^c) \end{aligned} \tag{19}$$

in which

$$\frac{\partial \mathbf{K}_i^e(\mathbf{b}^c)}{\partial b_j} = \int_{V_e} \mathbf{E}^T \frac{\partial \mathbf{F}(\mathbf{b})}{\partial b_j}\mathbf{E}(dV), \quad \frac{\partial \mathbf{M}_i^e(\mathbf{b}^c)}{\partial b_j} = \int_{V_e} \mathbf{N}^T \frac{\partial \rho(\mathbf{b})}{\partial b_j}\mathbf{N}dV \tag{20}$$

Applying the natural interval extension (Moor 1979, Alefeld and Herzberger 1983) to Eq. (19), the interval matrices of the elements are obtained

$$\begin{aligned} \mathbf{K}_i^e(\mathbf{b}^I) &= \mathbf{K}_i^e(\mathbf{b}^c) + \sum_{j=1}^{m} \frac{\partial \mathbf{K}_i^e(\mathbf{b}^c)}{\partial b_j}(b_j^I - b_j^c) \\ \mathbf{M}_i^e(\mathbf{b}^I) &= \mathbf{M}_i^e(\mathbf{b}^c) + \sum_{j=1}^{m} \frac{\partial \mathbf{M}_i^e(\mathbf{b}^c)}{\partial b_j}(b_j^I - b_j^c) \end{aligned} \tag{21}$$

To carry out the calculation of $\partial \mathbf{K}_i^e(\mathbf{b}^c)/\partial b_j$ and $\partial \mathbf{M}_i^e(\mathbf{b}^c)/\partial b_j$ by using the above direct differential method is inconvenient. It is desirable to transform the differential approach into finite element perturbation. The approach for computing $\partial \mathbf{K}_i^e(\mathbf{b}^c)/\partial b_j$ and $\partial \mathbf{M}_i^e(\mathbf{b}^c)/\partial b_j$ are

$$\mathbf{K}_{i,j}^{C} = \frac{\Delta\mathbf{K}_{ij}^{e}}{\Delta B_j}, \quad \mathbf{M}_{i,j}^{C} = \frac{\Delta\mathbf{M}_{ij}^{e}}{\Delta B_j} \tag{22}$$

where $\Delta\mathbf{M}_{ij}^{e}$ and $\Delta\mathbf{K}_{ij}^{e}$ are the increment of $\mathbf{M}^e$ and $\mathbf{K}^e$ resulting from the $\Delta\mathbf{B}_j$, the increment of the jth parameter.

$$\begin{aligned} \Delta\mathbf{M}_{ij}^{e} &= \mathbf{M}_i^e(b_1^c, \ldots, b_j^c + \Delta B_j, \ldots, b_n^c) - \mathbf{M}_i^e(b_1^c, \ldots, b_j^c, \ldots, b_n^c) \\ \Delta\mathbf{K}_{ij}^{e} &= \mathbf{K}_i^e(b_1^c, \ldots, b_j^c + \Delta B_j, \ldots, b_n^c) - \mathbf{K}_i^e(b_1^c, \ldots, b_j^c, \ldots, b_n^c) \end{aligned} \tag{23}$$

With Eq. (23), the interval stiffness and mass matrices of the elements expressed in Eq. (21) become

$$\mathbf{K}_i^e(\mathbf{b}^I) = \mathbf{K}_i^e(\mathbf{b}^c) + \sum_{j=1}^{m} \mathbf{K}_{i,j}^{c}\Delta b_j e_j, \quad \mathbf{M}_i^e(\mathbf{b}^I) = \mathbf{M}_i^e(\mathbf{b}^c) + \sum_{j=1}^{m} \mathbf{M}_{i,j}^{c}\Delta b_j e_j \tag{24}$$

in which $e_j = [-1, 1]$

Using the natural interval extension and the rule of the finite element analysis, the interval global stiffness matrix of the structure can be expressed as

$$\begin{aligned} \mathbf{K}(\mathbf{b}^I) &= \sum_{i=1}^{N} \mathbf{K}_i^e(\mathbf{b}^I) = \sum_{i=1}^{N}\left[\mathbf{K}_i^e(\mathbf{b}^c) + \sum_{j=1}^{m}\frac{\Delta\mathbf{K}_{ij}^{e}}{\Delta B_j}\Delta b_j e_j\right] \\ &= \mathbf{K}(\mathbf{b}^c) + \sum_{i=1}^{N}\sum_{j=1}^{m}\frac{\Delta\mathbf{K}_{ij}^{e}}{\Delta B_j}\Delta b_j e_j \\ &= \mathbf{K}(\mathbf{b}^c) + \Delta\mathbf{K}(\mathbf{b}^I) \end{aligned} \tag{25}$$

in which

$$\mathbf{K}(\mathbf{b}^c) = \sum_{i=1}^{N}\mathbf{K}_i^e(\mathbf{b}^c), \quad \Delta\mathbf{K}(\mathbf{b}^I) = \sum_{i=1}^{N}\sum_{j=1}^{m}\frac{\Delta\mathbf{K}_{ij}^{e}}{\Delta B_j}\Delta b_j e_j \tag{26}$$

where N is the total number of the elements. $\mathbf{K}(\mathbf{b}^c)$ is the mid-matrix of the stiffness of the structure, $\Delta\mathbf{K}(\mathbf{b}^I)$ is the interval increment matrix, respectively. The similar expression exists for $\mathbf{M}(\mathbf{b}^I)$.

It should be noted that Rayleigh damping is used in practical engineering

$$\mathbf{C}(\mathbf{b}) = \alpha\mathbf{M}(\mathbf{b}) + \beta\mathbf{K}(\mathbf{b}) \tag{27}$$

in which α and β are assumed to be deterministic constants.
Similarly, we can obtain the expressions for $\mathbf{M}(\mathbf{b}^I)$ and $\mathbf{C}(\mathbf{b}^I)$.

Now we can obtain the expression for $\mathbf{A}(\mathbf{b}^I)$ and $\mathbf{B}(\mathbf{b}^I)$

$$\mathbf{A}(\mathbf{b}^I) = \mathbf{A}(\mathbf{b}^c) + \Delta\mathbf{A}(\mathbf{b}^I), \quad \mathbf{B}(\mathbf{b}^I) = \mathbf{B}(\mathbf{b}^c) + \Delta\mathbf{B}(\mathbf{b}^I) \tag{28}$$

in which

$$\mathbf{A}(\mathbf{b}^c) = \begin{bmatrix} -\left[\sum_{i=1}^{N}\mathbf{M}_i^e(\mathbf{b}^c)\right]^{-1}\left[\sum_{i=1}^{N}\mathbf{C}_i^e(\mathbf{b}^c)\right] & -\left[\sum_{i=1}^{N}\mathbf{M}_i^e(\mathbf{b}^c)\right]^{-1}\left[\sum_{i=1}^{N}\mathbf{K}_i^e(\mathbf{b}^c)\right] \\ \mathbf{I} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{B}(\mathbf{b}^c) = \begin{Bmatrix} \left[\sum_{i=1}^{N}\mathbf{M}_i^e(\mathbf{b}^c)\right]^{-1}\overline{\mathbf{B}} \\ \mathbf{0} \end{Bmatrix} \tag{29}$$

and

$$\Delta\mathbf{A}(\mathbf{b}^I) = \sum_{j=1}^{m}\begin{bmatrix} \Delta\mathbf{A}_{11} & \Delta\mathbf{A}_{12} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} b_j e_j, \quad \Delta\mathbf{B}(\mathbf{b}^I) = \sum_{j=1}^{m}\begin{Bmatrix} \Delta\mathbf{B}_1 \\ \mathbf{0} \end{Bmatrix} b_j e_j \tag{30}$$

in Eq. (30)

$$\Delta\mathbf{A}_{11} = \left[\sum_{i=1}^{N}\mathbf{M}_i^e(\mathbf{b}^c)\right]^{-1}\left\{\left[\sum_{i=1}^{N}\frac{\Delta\mathbf{M}_{ij}^e}{\Delta B_j}\right]\left[\sum_{i=1}^{N}\mathbf{M}_i^e(\mathbf{b}^c)\right]^{-1}\left[\sum_{i=1}^{N}\mathbf{C}_i^e(\mathbf{b}^c)\right] - \left[\sum_{i=1}^{N}\frac{\Delta\mathbf{C}_{ij}^e}{\Delta B_j}\right]\right\}$$

$$\Delta\mathbf{A}_{12} = \left[\sum_{i=1}^{N}\mathbf{M}_i^e(\mathbf{b}^c)\right]^{-1}\left\{\left[\sum_{i=1}^{N}\frac{\Delta\mathbf{M}_{ij}^e}{\Delta B_j}\right]\left[\sum_{i=1}^{N}\mathbf{M}_i^e(\mathbf{b}^c)\right]^{-1}\left[\sum_{i=1}^{N}\mathbf{K}_i^e(\mathbf{b}^c)\right] - \left[\sum_{i=1}^{N}\frac{\Delta\mathbf{K}_{ij}^e}{\Delta B_j}\right]\right\} \tag{31}$$

and

$$\Delta\mathbf{B}_1 = -\left[\sum_{i=1}^{N}\mathbf{M}_i^e(\mathbf{b}^c)\right]^{-1}\left[\sum_{i=1}^{N}\frac{\Delta\mathbf{M}_{ij}^e}{\Delta B_j}\right]\left[\sum_{i=1}^{N}\mathbf{M}_i^e(\mathbf{b}^c)\right]^{-1}\overline{\mathbf{B}} \tag{32}$$

For the linear closed-loop system with interval parameters

$$\dot{\mathbf{x}}(t) = [\mathbf{A}(\mathbf{b}^I) + \mathbf{B}(\mathbf{b}^I)\mathbf{G}]\mathbf{x}(t) = \mathbf{D}(\mathbf{b}^I)\mathbf{x}(t) \tag{33}$$

To simplify the expressions, we define $\mathbf{D}(\mathbf{b}^c)$ and $\Delta\mathbf{D}(\mathbf{b}^I)$ as follows

$$\begin{aligned} \mathbf{D}(\mathbf{b}^c) &= \mathbf{A}(\mathbf{b}^c) + \mathbf{B}(\mathbf{b}^c)\mathbf{G} \\ \Delta\mathbf{D}(\mathbf{b}^I) &= \Delta\mathbf{A}(\mathbf{b}^I) + \Delta\mathbf{B}(\mathbf{b}^I) \\ &= \sum_{j=1}^{m}\left(\begin{bmatrix} \Delta\mathbf{A}_{11} & \Delta\mathbf{A}_{12} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{Bmatrix} \Delta\mathbf{B}_1 \\ \mathbf{0} \end{Bmatrix}\mathbf{G}\right) b_j e_j \end{aligned} \tag{34}$$

in which $\mathbf{D}(\mathbf{b}^c)$ is the mean value of the state matrix and $\Delta\mathbf{D}(\mathbf{b}^I)$ is the interval increment state matrix, respectively. From Eqs. (29)-(34), the interval matrices of structure in state-space coordinates with the physical parameters can be obtained.

5. Interval complex eigenvalues of the stability of closed-loop systems

5.1 Matrix perturbation analysis for eigenvalues

Consider the eigenproblem

$$\mathbf{D}_0\mathbf{u}_{k0} = \lambda_{k0}^*\mathbf{u}_{k0}, \quad \mathbf{D}_0^T\mathbf{v}_{k0} = \lambda_{k0}^*\mathbf{v}_{k0} \tag{35}$$

where $\mathbf{D}_0 = \mathbf{A}_0 + \mathbf{B}_0\mathbf{G}$, $\mathbf{D}_0$ is the state matrix of the deterministic system, λ_{k0}^* is the kth eigenvalue and $\mathbf{u}_{k0}$ is the kth eigenvector, $\mathbf{v}_{k0}^T$ is the corresponding left eigenvector. When the small changes of the parameters are introduced into the state matrix $\mathbf{D}_0$, the eigenvalue problem becomes

$$(\mathbf{D}_0 + \Delta\mathbf{D})(\mathbf{u}_{k0} + \Delta\mathbf{u}_k) = (\lambda_{k0}^* + \Delta\lambda_k^*)(\mathbf{u}_{k0} + \Delta\mathbf{u}_k) \tag{36}$$

where $\Delta\mathbf{D}$ are the increment of $\mathbf{D}_0$.

According to the matrix perturbation method (Chen 1999), we have

$$\lambda_k^* = \lambda_{k0}^* + \lambda_{k1}^*, \quad \lambda_{k1}^* = \mathbf{v}_k^T\Delta\mathbf{D}\mathbf{u}_k/\mathbf{v}_k^T\mathbf{u}_k \tag{37}$$

where $k = 1, 2, 3, \ldots, 2n_f$, λ_{k0}, $\mathbf{u}_{k0}$ and $\mathbf{v}_k$ are the kth original eigenvalue and corresponding right and left eigenvectors; λ_{k1} is the first-order perturbation of the kth eigenvalue.

5.2 Interval analysis and robustness analysis of the closed-loop system

Consider the linear closed-loop system with interval parameters

$$\dot{\mathbf{x}}(t) = (\mathbf{A} + \mathbf{B}\mathbf{G})\mathbf{x}(t) = \mathbf{D}\mathbf{x}(t) \tag{38}$$

For any $\mathbf{b} \in \mathbf{b}^I$, the eigenproblem is

$$\mathbf{D}(\mathbf{b})\mathbf{u} = \lambda^*\mathbf{u} \tag{39}$$

Using interval extension, Eq. (39) can be expressed as

$$\mathbf{D}(\mathbf{b}^I)\mathbf{u} = (\lambda^*)^I\mathbf{u} \tag{40}$$

Eq. (40) is called interval eigenvalue problem. It is the basic problem for given interval state matrix $\mathbf{D}(\mathbf{b}^I)$, to find the interval eigenvalue $(\lambda^*)^I$, which is not only the smallest interval but enclose all possible eigenvalues λ^*, satisfying $\mathbf{D}\mathbf{u} = \lambda^*\mathbf{u}$. In other words, we seek a hull

$$\Gamma = \{\lambda^*:[\mathbf{D}(\mathbf{b}) - \lambda^*\mathbf{I}]\mathbf{u} = 0,\ [\mathbf{D}(\mathbf{b}) - \lambda^*\mathbf{I}]^T\mathbf{v} = 0,\ \mathbf{D}(\mathbf{b}) \in \mathbf{D}(\mathbf{b})^I\} \tag{41}$$

to the set

$$\underline{\lambda}^* = \min\lambda^*(\mathbf{D}(\mathbf{b})), \quad \underline{\lambda}^* = \max\lambda^*(\mathbf{D}(\mathbf{b})) \tag{42}$$

It should be noted that the number of the eigensolutions satisfying Eq. (40) may be infinite and thus it is difficult to solve using the standard methods.

In terms of the interval expression, the interval matrix $\mathbf{D}(\mathbf{b}^I)$ can be expressed as

$$\mathbf{D}(\mathbf{b}^I) = \mathbf{D}(\mathbf{b}^c) + \Delta\mathbf{D}(\mathbf{b}^I) \tag{43}$$

where the expressions for $\mathbf{D}(\mathbf{b}^c)$ and $\Delta\mathbf{D}(\mathbf{b}^I)$ are presented in Eq. (34). Thus, the interval eigenproblem Eq. (40) can be written as

$$[\mathbf{D}(\mathbf{b}^c) + \Delta\mathbf{D}(\mathbf{b}^I)]\mathbf{u} = (\lambda^*)^I\mathbf{u} \tag{44}$$

For any $\mathbf{b} \in \mathbf{b}^I$, there is a group of $\delta\mathbf{D}(\mathbf{b})$ which satisfies

$$\delta\mathbf{D}(\mathbf{b}) \in \Delta\mathbf{D}(\mathbf{b}^I) \tag{45}$$

The corresponding eigenproblem is

$$[\mathbf{D}(\mathbf{b}^c) + \delta\mathbf{D}(\mathbf{b})]\mathbf{u} = \lambda^*\mathbf{u} \tag{46}$$

According to the matrix perturbation theory, we have

$$\lambda_k^* = \lambda_{k0}^* + \lambda_{k1}^*, \quad \lambda_{k1}^* = \mathbf{v}_k^T\delta\mathbf{D}\mathbf{u}_k/\mathbf{v}_k^T\mathbf{u}_k \tag{47}$$

Applying natural interval extension to Eq. (47), one can obtain the interval eigenvalues, i.e.

$$\lambda_k^* = \lambda_{k0}^* + (\lambda_{k1}^*)^I \tag{48}$$

In Eq. (48)

$$(\lambda_{k1}^*)^I = \mathbf{v}_k^T\Delta\mathbf{D}(\mathbf{b}^I)\mathbf{u}_k/\mathbf{v}_k^T\mathbf{u}_k \tag{49}$$

Substituting Eq. (34) into Eq. (49) yields

$$\begin{aligned}(\lambda_{k1}^*)^I &= \mathbf{v}_k^T\sum_{j=1}^{m}\left(\begin{bmatrix}\Delta\mathbf{A}_{11} & \Delta\mathbf{A}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} + \mathbf{G}\begin{Bmatrix}\Delta\mathbf{B}_1\\ \mathbf{0}\end{Bmatrix}\right)b_j e_j\mathbf{u}_k\Big/\mathbf{v}_k^T\mathbf{u}_k\\ &= \sum_{j=1}^{m}(|(\lambda^*)_R^j| + i|(\lambda^*)_I^j|)e_j\end{aligned} \tag{50}$$

Denote $\Delta(\lambda^*)_R^k$ the real parts of the interval incremental complex eigenvalues and $\Delta(\lambda^*)_I^k$ the imaginary parts of the interval incremental complex eigenvalues, respectively.

$$\Delta(\lambda^*)_R^k = \sum_{j=1}^{m}|(\lambda^*)_R^j|, \quad \Delta(\lambda^*)_I^k = \sum_{j=1}^{m}|(\lambda^*)_I^j| \tag{51}$$

We can obtain

$$(\lambda^*)_k^I = (\lambda^*)_k^c + (\lambda^*)_k^I = (\lambda^*)_k^c + \Delta(\lambda^*)_R^k e_j + i\Delta(\lambda^*)_I^k e_j \tag{52}$$

Letting

$$(\lambda^*)_k^I = [\underline{(\lambda^*)}_{kR} + i\underline{(\lambda^*)}_{kI}, \overline{(\lambda^*)}_{kR} + i\overline{(\lambda^*)}_{kI}] \tag{53}$$

the lower and upper bounds of the real parts and imaginary parts of the complex eigenvalues can be obtained

$$\begin{aligned} \underline{(\lambda^*)}_{kR} &= (\lambda^*)_{kR}^c - \Delta(\lambda^*)_R^k, \quad \overline{(\lambda^*)}_{kR} = (\lambda^*)_{kR}^c + \Delta(\lambda^*)_R^k \\ \underline{(\lambda^*)}_{kI} &= (\lambda^*)_{kI}^c - \Delta(\lambda^*)_I^k, \quad \overline{(\lambda^*)}_{kI} = (\lambda^*)_{kI}^c + \Delta(\lambda^*)_I^k \end{aligned} \tag{54}$$

In vibration control engineering, the real and imaginary parts of the complex eigenvalues of close-loop system represent the damping and frequency characters of the system, respectively. Now the bounds of real and imaginary parts of eigenvalues have been obtained, we can also estimate the stability robustness of the closed-loop system by the bounds of the real parts of eigenvalues.

The following condition can be used to estimate the stability robustness

$$0 < \left|(\lambda^*)_{kR}^c\right| - \left|\Delta(\lambda^*)_R^k\right| \le |\alpha_k| \le \left|(\lambda^*)_{kR}^c\right| + \left|\Delta(\lambda^*)_R^k\right|, \quad (k = 1, 2, \ldots, 2n_f) \tag{55}$$

where α_k is the real part of the kth eigenvalue of uncertain closed-loop system. It is obvious that if $(\lambda^*)_{kR}^c$ $(k = 1, 2, \ldots, 2n_f)$ are large enough in designing the feedback control law, the stability of the uncertain closed-loop system will be remained.

6. Numerical examples

In order to demonstrate the application and effectiveness of the method given in this paper, we consider the vibration control system of a frame structure shown in Fig. 1. The finite element model of the given structure consists of 10 nodes and 12 beam elements with 30 degrees of freedom. The structural parameters are as follows: the Young's elastic modulus of the material $E^c = 2.1 \times 10^8$ N/m^2, the mass density $\rho^c = 7.8 \times 10^3$ kg/m^3, the width of the cross section of beam element $B^c = 5.5 \times 10^{-2}$ m. If the damping coefficient $\alpha = 0.75$, $\beta = 3.4E-04$, then the first 10 eigenvalues λ_i $(i = 1, 2, \ldots, 10)$ of the open-loop system are obtained and listed in Table 1.

To control the vibration and guarantee asymptotic stability, it is necessary to impart the open-loop system larger damping. To this problem, the real parts of the first 10 eigenvalues can be assigned as -5.00000. The eigenvalues of the closed-loop system λ_i^* $(i = 1, 2, \ldots, 10)$ are also listed in Table 1. Assume that the feedback force $\mathbf{u}(t)$ applied to the X axis of the node 9 as shown in Fig. 1. The first 10 modal vectors of the modal matrix corresponding to the first 10 eigenvalues are used by modal truncation, so we can reduce the dimensions of the state matrices and save CPU time of the operations. The feedback gain matrix $\mathbf{G}$ can be calculated by the pole allocation method introduced in section 3.

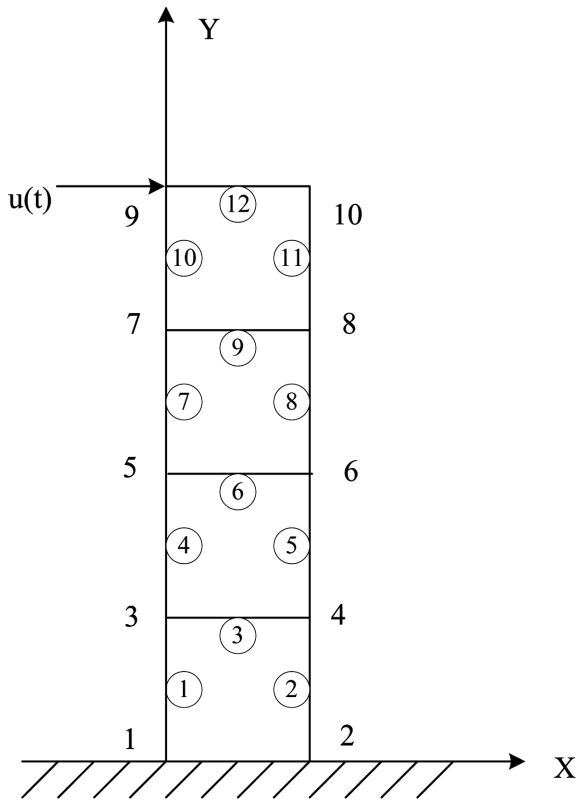

Fig. 1 A frame structure

Table 1 The first 10 eigenvalues of the open-loop and closed-loop systems for frame structures

i	λ_i (×1.0e+002)	λ_i^* (×1.0e+002)
1	−0.00376+0.06131i	−0.05000+0.06131i
2	−0.00376−0.06131i	−0.05000−0.06131i
3	−0.00382+0.20451i	−0.05000+0.20451i
4	−0.00382−0.20451i	−0.05000−0.20451i
5	−0.00400+0.38534i	−0.05000+0.38534i
6	−0.00400−0.38534i	−0.05000−0.38534i
7	−0.00427+0.55557i	−0.05000+0.55557i
8	−0.00427−0.55557i	−0.05000−0.55557i
9	−0.00529+0.95054i	−0.05000+0.95054i
10	−0.00529−0.95054i	−0.05000−0.95054i

If **G** is applied to the actual uncertain system with interval parameters given by $E_i^I = E_i^c + \Delta E e_i$; $B_i^I = B_i^c + \Delta B e_i$; $(i = 1, 2, \ldots, 12)$, where i is the element number, the closed-loop eigenvalues will have some errors. The lower and upper bounds and the mean value of the eigenvalues of the closed-loop system are obtained and listed in Tables 2, 3. In the Tables, k is the mode number; $\underline{(\lambda^*)}_{kR}$ is the lower bound of the real part of complex eigenvalue; $\underline{(\lambda^*)}_{kI}$ is the lower bound of the imaginary part; $(\lambda^*)^C_{kR}$ and $(\lambda^*)^C_{kI}$ are the middle values of the imaginary part and the real part;

Table 2 The lower and upper bounds of complex eigenvalues ($\Delta E_i = 10E_i/100$ for all elements)

k	$\underline{(\lambda^*)}_{kR}$	$\underline{(\lambda^*)}_{kI}$	$(\lambda^*)^C_{kR}$	$(\lambda^*)^C_{kI}$	$\overline{(\lambda^*)}_{kR}$	$\overline{(\lambda^*)}_{kI}$	$\left\|\frac{\Delta(\lambda^*)^k_R}{(\lambda^*)^c_{kR}}\right\|\%$	$\left\|\frac{\Delta(\lambda^*)^k_I}{(\lambda^*)^c_{kI}}\right\|\%$
1	−5.08355	5.85136	−5.00000	6.13068	−4.91644	6.40999	1.67110	4.55602
3	−5.02737	19.46985	−5.00000	20.45119	−4.97262	21.43253	0.54751	4.79844
5	−5.00151	36.64234	−5.00000	38.53378	−4.99848	40.42521	0.03030	4.90852
7	−5.06112	52.76402	−5.00000	55.55724	−4.93888	58.35047	1.22236	5.02765
9	−5.02742	90.28615	−5.00000	95.05422	−4.97258	99.82230	0.54838	5.01616

Table 3 The lower and upper bounds of complex eigenvalues ($\Delta B_i = 5B_i/100$ for all elements)

k	$\underline{(\lambda^*)}_{kR}$	$\underline{(\lambda^*)}_{kI}$	$(\lambda^*)^C_{kR}$	$(\lambda^*)^C_{kI}$	$\overline{(\lambda^*)}_{kR}$	$\overline{(\lambda^*)}_{kI}$	$\left\|\frac{\Delta(\lambda^*)^k_R}{(\lambda^*)^c_{kR}}\right\|\%$	$\left\|\frac{\Delta(\lambda^*)^k_I}{(\lambda^*)^c_{kI}}\right\|\%$
1	−5.07248	5.84963	−5.00000	6.13068	−4.92752	6.41173	1.44953	4.58430
3	−5.01650	19.49270	−5.00000	20.45119	−4.98350	21.40967	0.33008	4.68670
5	−5.01998	36.63369	−5.00000	38.53378	−4.98002	40.43386	0.39967	4.93095
7	−5.07728	52.71714	−5.00000	55.55724	−4.92272	58.39735	1.54567	5.11204
9	−5.65980	93.33555	−5.00000	95.05422	−4.34020	96.77291	13.19603	1.80810

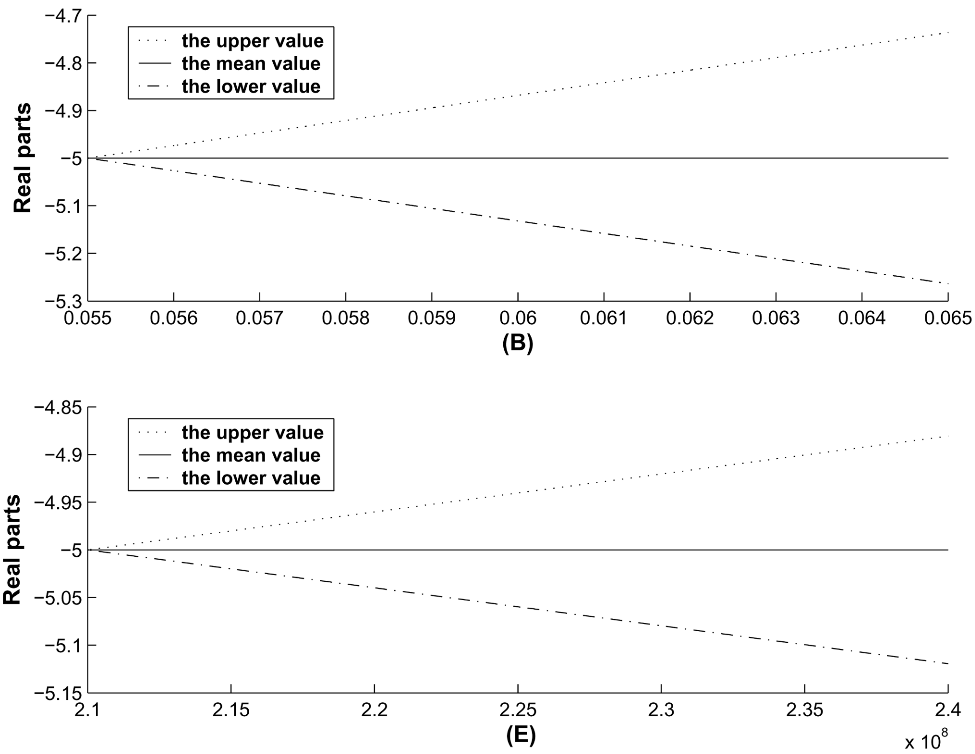

Fig. 2 The upper and lower bounds of the real part of the first eigenvalue obtained by the change of the width of the cross section of beam element B and Young's elastic modulus of the material E

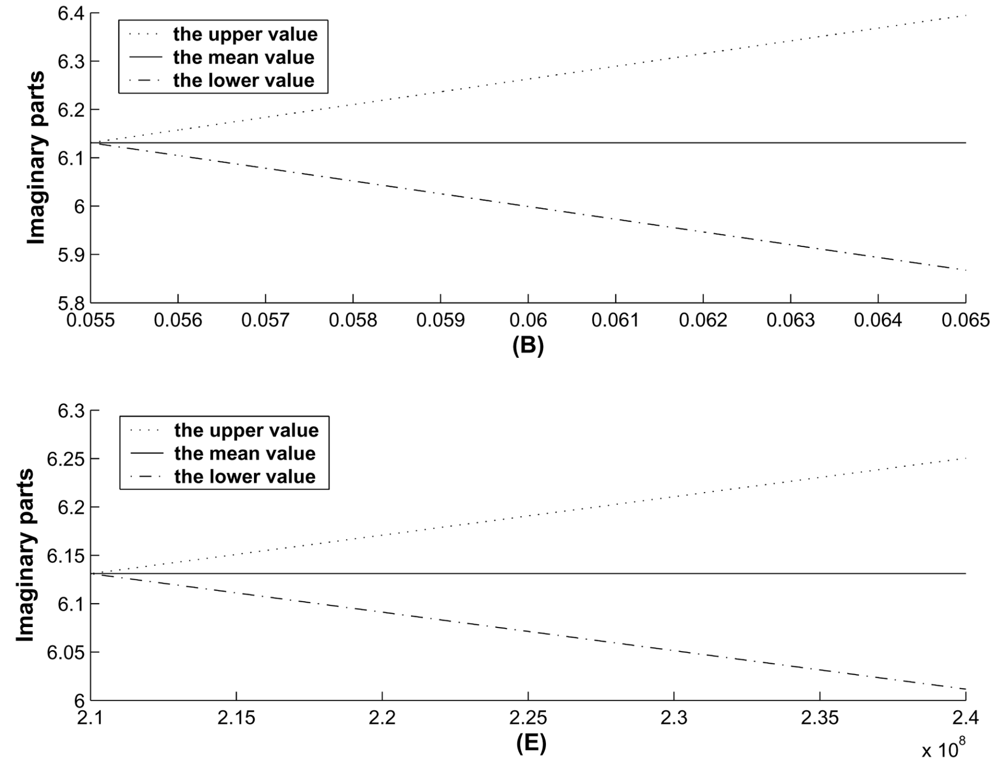

Fig. 3 The upper and lower bounds of the imaginary part of the first eigenvalue obtained by the change of the width of the cross section of beam element *B* and Young's elastic modulus of the material *E*

$\overline{(\lambda^*)}_{kR}$ is the upper bound of the real part; $\overline{(\lambda^*)}_{kI}$ is the upper bound of the imaginary part. $\left|\Delta(\lambda^*)^k_R/(\lambda^*)^c_{kR}\right|$% and $\left|\Delta(\lambda^*)^k_I/(\lambda^*)^c_{kI}\right|$% are the relative uncertainties of the real and imaginary parts, respectively.

The curves of upper and lower bounds of the 1st eigenvalue of the closed-loop are shown in Fig. 2 and Fig. 3 describing how the changes of the uncertain parameters affect the real and imaginary parts of eigenvalues, respectively. From the results, it can be seen that the effect of changes of the parameters on the imaginary parts of the eigenvalues is large than that of the real parts. At the same time we can see that the effect of the change of the width of the cross section of beam element *B* on the closed-loop eigenvalues is much greater than that of Young's elastic modulus of the material *E*. And the stability condition (55) can be satisfied.

For the sake of comparison, the upper and lower bounds of eigenvalues can be obtained by the random perturbation. In the sense of the probability, the lower and upper bounds of the complex eigenvalues can be considered to be

$$\begin{aligned} \underline{(\lambda^*)}_{kR} &= (\lambda^*)^c_{kR} - 3\sigma_{kR}, \quad \overline{(\lambda^*)}_{kR} = (\lambda^*)^c_{kR} + 3\sigma_{kR} \\ \underline{(\lambda^*)}_{kI} &= (\lambda^*)^c_{kI} - 3\sigma_{kI}, \quad \overline{(\lambda^*)}_{kI} = (\lambda^*)^c_{kI} + 3\sigma_{kI} \end{aligned} \tag{56}$$

where σ_{kR} and σ_{kI} are the standard deviations of the real and imaginary parts, respectively (Chen *et al.* 2004).

Table 4 The lower and upper bounds of complex eigenvalues ($\Delta B_i = 3B_i/100$, $\Delta E_i = 4E_i/100$ for all elements) using the interval method and the random perturbation method

k	$\underline{(\lambda^*)}_{kR}$	$\underline{(\lambda^*)}_{kI}$	$(\lambda^*)^C_{kR}$	$(\lambda^*)^C_{kI}$	$\overline{(\lambda^*)}_{kR}$	$\overline{(\lambda^*)}_{kI}$
1	−5.07691	5.85032	−5.00000	6.13068	−4.92309	6.41103
	(−5.05484)	(5.92839)			(−4.94515)	(6.33296)
3	−5.02085	19.48355	−5.00000	20.45119	−4.97914	21.41881
	(−5.01476)	(19.75490)			(−4.98523)	(21.14747)
5	−5.01260	36.63715	−5.00000	38.53378	−4.98740	40.43040
	(−5.01200)	(37.16552)			(−4.98799)	(39.90203)
7	−5.07082	52.73588	−5.00000	55.55724	−4.92918	58.37859
	(−5.05242)	(53.51955)			(−4.94757)	(57.59493)
9	−5.40685	92.11578	−5.00000	95.05422	−4.59315	97.99266
	(−5.39603)	(92.88606)			(−4.60396)	(97.22238)

The results obtained by the present method in contrast with the random perturbation method are listed in Table 4 and the latter enclosed in parentheses. In the computation, the uncertain parameters are assumed to be $\Delta b_i = 3\sigma_{ki}$, $(i = 1, 2, \ldots, 12)$, σ_{ki} are the standard deviations of the uncertain parameters b_i. It can be seen that the results obtained by the present method are nearby those obtained by the random perturbation method. This proves that the present method is valid.

In order to justify the computational efficiency of the present interval method and the random perturbation, Eq. (54) can be changed into the following form

$$\underline{(\lambda^*)}_{kR} = (\lambda^*)^c_{kR} - \sum_{i=1}^{m} \left|[(\lambda^*)_{kR}]_{,i}\Delta b_i\right| \tag{57}$$

where $[(\lambda^*)_{kR}]_{,i}$ is the sensitivity of $(\lambda^*)_{kR}$ with respect to b_i.

If the random parameters are probabilistic independent, the standard deviations of eigenvalues are

$$\sigma_{kR} = \left(\sum_{i=1}^{m} [(\lambda^*)_{kR}]^2_{,i}\sigma^2_{ki}\right)^{\frac{1}{2}} \tag{58}$$

If $\Delta b_i = 3\sigma_{ki}$, $(i = 1, 2, \ldots, 12)$, Eq. (57) becomes

$$\underline{(\lambda^*)}_{kR} = (\lambda^*)^c_{kR} - 3\sum_{i=1}^{m} \left|[(\lambda^*)_{kR}]_{,i}\sigma_{ki}\right| \tag{59}$$

and Eq. (56) becomes

$$\underline{(\lambda^*)}_{kR} = (\lambda^*)^C_{kR} - 3\left(\sum_{j=1}^{m} [(\lambda^*)_{kR}]^2_{,j}\sigma^2_{kj}\right)^{\frac{1}{2}} \tag{60}$$

The similar expressions exist for $\overline{(\lambda^*)}_{kR}$, $\underline{(\lambda^*)}_{kI}$, and $\overline{(\lambda^*)}_{kI}$.

From Eqs. (59) and (60), it is obvious that the computational efficiency of the interval method is higher than that of the random perturbation.

7. Conclusions

In this paper, with the interval finite element method and matrix perturbation theory, the vibration control problem of structures with uncertain parameters was discussed. The uncertain parameters are modeled to be an interval set rather than a probabilistic set. This does not require the probabilistic distribution descriptions of the uncertain parameters. The algorithm to estimate the upper and lower bounds of the real parts and imaginary parts of the complex eigenvalues of vibration control systems was developed. The results can be used to estimate the stability robustness of the closed-loop systems. In addition, the present procedure is easy to implement on the computer and incorporate to the finite element code and can be applied to large finite element model. From the numerical results, it can be seen that the assigned real parts of eigenvalues of the closed-loop system are large enough, the stability robustness of the uncertain close-loop system will be remained.

Acknowledgements

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References

Alefeld, G. and Herzberber, J. (1983), *Introductions to Interval Computations.* Academic Press, New York.
Alefeld, G. (1991), *Advanced Matrix Theory for Scientists and Engineers, 2nd Edition*, Abacus Press.
Ben-Haim, Y. and Elishakoff, I. (1990), *Convex Models of Uncertainty in Applied Mechanics,* Elsevier, New York.
Chen, R. and Ward, A.C. (1997), "Generalizing interval matrix operations for design", *J. Mech. Des*., **119**, 655-672.
Chen, S.H. (1999), *Matrix Perturbation Theory in Structural Dynamic Designs,* Science Press, Beijing. (in Chinese).
Chen, S.H., Lian, H.D. and Yang, X.W. (2002), "Dynamic response analysis for structures with interval parameters", *Struct. Eng. Mech.*, **13**(3), 299-312.
Chen, S.H., Liu, C., and Chen, Y.D. (2004), "The standard deviations for eigenvalues of the closed-loop systems with random parameters", *Struct. Eng. Mech.*, **18**(3), 331-342.
Chen, S.H. and Wu, J. (2004), "Interval optimization of dynamic response for uncertain structures with natural frequency constraints", *Eng. Struct*., **26**, 221-232.
Dimarogonas, A.D. (1995), "Interval analysis of vibration systems", *J. Sound Vib.*, **183**, 739-749.
Ferrara, A. and Giacomini, L. (2000), "Control of a class of mechanical systems with uncertainties Via a constructive adaptive/Second order VSC approach", *Trans. ASME J. Dyn. Syst.*, **122**(1), 33-39.
Ganzerli, S. and Pantelides, C.P. (2000), "Optimum structural design via convex model superposition", *Comput. Struct*., **74**(6), 639-647.
Ganzerli, S. and Pantelides, C.P. (1999), "Load and resistance convex models for optimum design", *Struct. Optimization*, **17**(4), 259-268.
Krodkiewski, J.M. (2000), "Stabilization of motion of helicoptor rotor blades using delayed feedback-modelling. computer simulation and experimental verification", *J. Sound Vib.,* **234**, 591-610.
Li, Y.Y. and Yarn, L.H. (2001), "Robust vibration control of uncertain systems using variable parameter feedback and model-based fuzzy strategies", *Comput. Struct*., **79**(1), 109-119.

Moore, R.E.(1979), *Methods and Applications of Interval Analysis*, SIAM, Philadelphia.
Pantelides, C.P. and Booth, B.C. (Jan. 2000), "Computer-aided design of optimal structures with uncertainty", *Comput. Struct.,* **74**(3), 293-307.
Pantelides, C.P. and Ganzerli, S. (Mar. 1998), "Design of trusses under uncertain loads using convex models", *J. Struct. Eng*., ASCE, **124**(3), 318-329.
Shin, Y.S. and Grandhi, R.V. (2001), "A global structural optimization technique using an interval method", *Struct. Multidisc. Optim*, **22**, 351-363.
Spencer, B.F. and Sain, M.K. Jr. (1992), "Probabilistic stability measures for controlled structures subject to real parameter uncertainties", *Smart Master Struct*., **1**, 294-305.
Venini, P. (1998), "Robust control of uncertain structures", *Comput. Struct.*, **67**, 165-174.