Structural Engineering and Mechanics, Vol. 26, No. 1 (2007) 15-30
DOI: http://dx.doi.org/10.12989/sem.2007.26.1.015

Duality in non-linear programming for limit analysis of not resisting tension bodies

A. Baratta† and O. Corbi‡

Department of Structural Engineering, University of Naples "Federico II", via Claudio 21, 80125 Naples, Italy

(Received January 28, 2005, Accepted November 1, 2006)

Abstract. In the paper, one focuses on the problem of duality in non-linear programming, applied to the solution of no-tension problems by means of Limit Analysis (LA) theorems for Not Resisting Tension (NRT) models. In details, one demonstrates that, starting from the application of the duality theory to the non-linear program defined by the static theorem approach for a discrete NRT model, this procedure results in the definition of a dual problem that has a significant physical meaning: the formulation of the kinematic theorem.

Keywords: structural analysis; operational research; NRT model; duality; non-linear programming; limit analysis.

1. Introduction

Masonry behaviour is often modelled by the Not Resisting Tension (NRT) assumption. As well known, basic NRT models exhibit a simple linear elastic behaviour under compression stress states and no resistance in tension, thus resulting in an overall fragile non-linear behaviour. It has been shown (see Heyman 1969) that the loading capacity of NRT structures can be investigated by means of the tools of the Limit Analysis (LA) theory, according to some extensions to NRT solids of the static and kinematic LA theorems (see e.g. Baratta 1996, Como and Grimaldi 1983, Del Piero 1998).

On the basis of the static theorem, the collapse multiplier definitely limiting the loading capacity of the structure is recognized as the upper bound of the class of statically admissible multipliers, i.e. its maximum value, while, by means of the kinematical approach, it is recognized as the lower bound of the class of kinematically sufficient multipliers, i.e. its minimum value.

Anyway both approaches, particularized to some specific cases (such as the evaluation of the loading capacity of a masonry wall loaded by in-plane forces and modelled by finite elements with *constant stress/constant strain*), lead to constrained extremum problems, governed by linear objective functions under linear and non-linear constraint conditions, thus resulting in non-linear programming problems, whose solution can be numerically pursued by means of Operational Research methods.

† Full Professor, E-mail: alessandro.baratta@unina.it
‡ Associate Professor, Corresponding author, E-mail: ottavia.corbi@unina.it

On the other side, since one deals with non-linear programming problems (Bazaraa *et al.* 1993, Jahn 1996), they may be approached by means of the tools of duality theory (see Mangasarian 1969); one should emphasize that application of duality to non-linear programming is related to the reciprocal principles of the calculus of variations, which have been known since as far back as 1927 (see Mond and Hansons 1967); therefore it yields interesting results when applied to the solution of problems, such as the evaluation of the loading capacity of masonry panels, which obey NRT LA theorems (Baratta and Corbi 2003, 2004).

In the following, one demonstrates that, starting from the application of the duality theory to the non-linear program defined by the static theorem approach for the above mentioned discrete masonry model, this procedure results in the definition of a dual problem that has a significant physical meaning: the expression of the kinematic theorem based on the associated flow law statement, despite it is, by no way, included in the origin static approach.

2. Limit analysis for not resisting tension bodies

2.1 The no-tension material

The basic assumption of *no-tension* masonry model coincides with the hypotheses that the tensile resistance is null, and that the behaviour in compression is indefinitely linear elastic; under these hypotheses, no-tension stress fields are selected by the body through the activation of an additional strain field, the *fractures* (see Heyman 1969). Solution stress and strain fields are proven to satisfy classical energetic principles, like the minimum principles of Complementary and Total Energy functionals (see e.g. Del Piero 1988, Di Pasquale 1982, Baratta and Toscano 1982, Baratta and Corbi, I. 2004, Baratta *et al.* 2004, Baratta and Corbi, O. 2003).

In details, in a NRT solid the equilibrium against external loads is required to be satisfied by *admissible* stress fields, which imply pure compression everywhere in the solid. Assuming stability of the material in the Drucker's sense, compatibility of the strain field can be ensured by superposing to the elastic strain field an additional *fracture* field, that does not admit contraction in any point and along any direction; that is to say that the stress tensor $\boldsymbol{\sigma}$ must be negative semi-definite everywhere in the solid (i.e., $\boldsymbol{\sigma}$ must be an element of the set of negative semi-definite

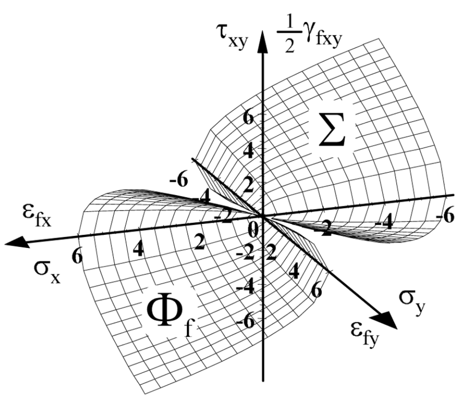

Fig. 1 Admissibility domains for stress (Σ) and fracture strain (Φ_f)

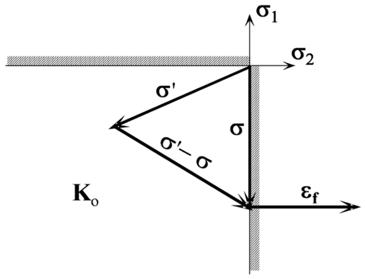

Fig. 2 Admissibility principal plane for stress (K_o) and normality Drucker's law

stress tensors Σ), while the fracture strain field $\boldsymbol{\varepsilon}_f$ is required to be positive semi-definite (i.e., $\boldsymbol{\varepsilon}_f$ must be an element of the set of positive semi-definite fracture strain tensors Φ_f) (Fig. 1).

The material should, hence, satisfy the following conditions

$$\text{SEMI-DEFINITE}\begin{cases}\boldsymbol{\varepsilon}_f \ \text{POSITIVE} \\ \boldsymbol{\sigma} \ \text{NEGATIVE}\end{cases} \rightarrow \begin{cases}\varepsilon_{fa} \geq 0 \\ \sigma_a \leq 0\end{cases} \quad \forall a \in r_a, \ \ \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_e + \boldsymbol{\varepsilon}_f = \mathbf{C}\boldsymbol{\sigma} + \boldsymbol{\varepsilon}_f \tag{1}$$

where r_a is the set of directions through the generic point in the solid, a is one of such directions, $\boldsymbol{\varepsilon}_f$ is the fracture strain that is assumed to superpose to the elastic strain $\boldsymbol{\varepsilon}_e$ in order to anneal tensile stresses if possible, and **C** denotes the tensor of elastic constants.

The material admissibility conditions for strain and stress reported in Eq. (1) can be synthetically referred to by the set of inequalities $\mathbf{h}_\varepsilon(\boldsymbol{\varepsilon}_f) \geq \mathbf{0}$ and $\mathbf{h}_\sigma(\boldsymbol{\sigma}) \leq \mathbf{0}$ respectively.

Moreover, one assumes that the classical Drucker's postulate holds for the fracture strain (Fig. 2).

With reference to the admissible domain reported in Fig. 2, the normality Drucker's law for no-tension material can thus be written as

$$\begin{gathered}(\boldsymbol{\sigma}' - \boldsymbol{\sigma}) \cdot \boldsymbol{\varepsilon}_f \leq 0 \qquad \forall \boldsymbol{\sigma}' \in \boldsymbol{\Sigma} \\ \{\, \sigma_1 \varepsilon_{f1} = 0, \ \ \sigma_2 \varepsilon_{f2} = 0 \} \Rightarrow \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}_f = 0\end{gathered} \tag{2}$$

where $\boldsymbol{\sigma}'$ is any admissible stress state other than the effective one $\boldsymbol{\sigma}$, and the principal tensors' components relevant to the principal directions are denoted by $(\cdot)_1$ and $(\cdot)_2$ respectively. From Eq. (2) one can infer that the internal fracture work $\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}_f$ is always equal to zero.

2.2 Identification of the collapse multiplier for no-tension bodies

Let consider the body and surface forces, **F** acting on the volume V and **p** acting on the free surface S_p, the displacement field **u**, the imposed displacement field $\mathbf{u}_o$ characterizing the constrained part of the solid surface S_u, the above mentioned strain field $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_e + \boldsymbol{\varepsilon}_f = \mathbf{C}\boldsymbol{\sigma} + \boldsymbol{\varepsilon}_f$, the stress field $\boldsymbol{\sigma}$.

As clear from the above, fracture strains $\boldsymbol{\varepsilon}_f$ can be developed at the considered point only if the stress situation can be represented by a stress tensor $\boldsymbol{\sigma}$ laying on the surface of the material plasticity domain, which is defined for NRT bodies by $\mathbf{h}_\sigma(\boldsymbol{\sigma}) \leq \mathbf{0}$; obviously if some fracture does exist, it is developed according to the NRT material inequalities $\mathbf{h}_\varepsilon(\boldsymbol{\varepsilon}_f) \geq \mathbf{0}$.

After assuming the applied loads as given by the sum of a fixed component ($\mathbf{F}_o$, $\mathbf{p}_o$) and a variable component ($s\mathbf{F}_v$, $s\mathbf{p}_v$) depending on the value assumed by the multiplier s (actually one thus assumes that only the portion $\mathbf{F}_v$, $\mathbf{p}_v$, may be destabilizing and should be controlled)

$$\begin{cases} \mathbf{F} = \mathbf{F}_o + s\mathbf{F}_v = \mathbf{0} & \text{in } V \\ \mathbf{p} = \mathbf{p}_o + s\mathbf{p}_v & \text{on } S_p \end{cases} \tag{3}$$

one can define two fundamental classes of load multipliers s for NRT bodies: the class of *statically admissible multipliers* β and the class of *kinematically sufficient multipliers* γ.

Denoted by $\boldsymbol{\alpha}_n$ the unit outgoing vector normal to the surface S_p, load multipliers β are defined to be statically admissible if the following relations hold

$$\begin{cases} \text{div}\boldsymbol{\sigma}^{\beta} + \mathbf{F}_o + \beta\mathbf{F}_v = \mathbf{0} & \text{in } V \\ \boldsymbol{\sigma}^{\beta}\boldsymbol{\alpha}_n = \mathbf{p}_o + \beta\mathbf{p}_v & \text{on } S_p \end{cases} \tag{4}$$

$$\mathbf{h}_{\sigma}(\boldsymbol{\sigma}^{\beta}) \le \mathbf{0} \tag{5}$$

that is to say, if it does exist a stress field $\boldsymbol{\sigma}^{\beta}$ equilibrating the applied loads with $s = \beta$ and satisfying the NRT material admissibility conditions. Any of such stress fields is qualified as *statically admissible*.

On the other side, load multipliers γ are defined to be kinematically sufficient if the following relations hold

$$\begin{cases} \boldsymbol{\varepsilon}_f^{\gamma} = \text{Grad}(\mathbf{u}_f^{\gamma}) & \text{in } V \\ \mathbf{u}_f^{\gamma} = \mathbf{0} & \text{on } S_p \end{cases} \tag{6}$$

$$\mathbf{h}_{\varepsilon}(\boldsymbol{\varepsilon}_f) \ge \mathbf{0} \tag{7}$$

$$\int_V \mathbf{F}_o \cdot \mathbf{u}_f^{\gamma} dV + \int_{S_p} \mathbf{p}_o \cdot \mathbf{u}_f^{\gamma} dS + \gamma \int_V \mathbf{F}_v \cdot \mathbf{u}_f^{\gamma} dV + \gamma \int_{S_p} \mathbf{p}_v \cdot \mathbf{u}_f^{\gamma} dS > 0 \tag{8}$$

that is to say, if it does exist a displacement field $\mathbf{u}_f^{\gamma}$ (a *collapse mechanism*) directly compatible with a NRT admissible fracture strain $\boldsymbol{\varepsilon}_f^{\gamma}$ apart from any elastic strain field, and such that the condition stated by Eq. (8) is also satisfied. It is understood that the body is stable under the basic load pattern ($\mathbf{F}_o$, $\mathbf{p}_o$), and that inequality Eq. (8) cannot be satisfied by any fracture strain field for $\gamma = 0$. In other terms it is assumed that the basic loads are suitably chosen in way that they cannot produce collapse.

Extensions to NRT continua of the fundamental *static and kinematic theorems* of Limit Analysis allow individuating the value $\overline{s}$ of the load multipliers s, limiting the loading capacity of the body.

On the basis of the *static theorem*, one can state that "*the collapse multiplier* $\overline{s}$ *represents the maximum of the statically admissible multipliers* β"

$$\overline{s} = \max\{\beta \in B_o\} \tag{9}$$

where B_o is the class of statically admissible multipliers.

On the basis of the *kinematic theorem*, one can state that "*the collapse multiplier* $\overline{s}$ *represents the minimum of the kinematically sufficient multipliers* γ"

$$\overline{s} = \min\{\gamma \in \Gamma_o\} \tag{10}$$

where Γ_o is the class of kinematically sufficient multipliers.

Therefore, by means of the static theorem, one can search for the collapse multiplier by implementing the problem

$$\text{Find } \max_{\beta,\, \boldsymbol{\sigma}^{\beta}}\{\beta\} \qquad \text{Sub} = \begin{cases} \begin{cases} \operatorname{div}\boldsymbol{\sigma}^{\beta} + \mathbf{F}_o + \beta\mathbf{F}_v = \mathbf{0} & \text{in } V \\ \boldsymbol{\sigma}^{\beta}\boldsymbol{\alpha}_n = \mathbf{p}_o + \beta\mathbf{p}_v & \text{on } S_p \end{cases} \\ \mathbf{h}_{\sigma}(\boldsymbol{\sigma}^{\beta}) \le \mathbf{0} \end{cases} \tag{11}$$

Or otherwise, by means of the kinematic theorem, by solving the problem

$$\text{Find } \min_{\gamma,\, \boldsymbol{\varepsilon}_f^{\beta},\, \mathbf{u}_f^{\beta}}\{\gamma\} \qquad \text{Sub} = \begin{cases} \begin{cases} \boldsymbol{\varepsilon}_f^{\gamma} = \operatorname{Grad}(\mathbf{u}_f^{\gamma}) & \text{in } V \\ \mathbf{u}_f^{\gamma} = \mathbf{0} & \text{on } S_p \end{cases} \\ \mathbf{h}_{\varepsilon}(\boldsymbol{\varepsilon}_f^{\gamma}) \ge \mathbf{0} \\ \int_V \mathbf{F}_o \cdot \mathbf{u}_f^{\gamma} dV + \int_{S_p} \mathbf{p}_o \cdot \mathbf{u}_f^{\gamma} dS + \gamma \int_V \mathbf{F}_v \cdot \mathbf{u}_f^{\gamma} dV + \gamma \int_{S_p} \mathbf{p}_v \cdot \mathbf{u}_f^{\gamma} dS > 0 \end{cases} \tag{12}$$

3. Finite element model of a NRT plane structure

3.1 Definitions of the variables governing the problem

Let consider a plane structure modelled by means of the NRT assumption, loaded by in-plane forces. As well known, the structure can be easily reduced to a model characterized by a finite number of elements, occupying small regions of the original surface, after making some suitable assumptions. Therefore, the structure can be viewed as given by the union of a number M of plane adjacent elements, jointed to each other at a number N of defined points, the nodes, which coincide with the edges of the considered elements. For simplicity's sake, reference is made to a mesh assembled by constant stress/strain elements.

After reducing the loads acting on the structure to simple nodal loads collected in the overall nodal load vector $\mathbf{q}$, in order to evaluate the loading capacity of the structure, $\mathbf{q}$ is assumed as the sum of a fixed component $\mathbf{q}_o$ and a variable component $s\mathbf{q}_v$ depending on the value assumed by the load multiplier s.

When the load multiplier s reaches the collapse value $\overline{s}$, a failure mechanism is activated (characterized by fracture displacements) and the structure collapses.

One then introduces the other quantities governing the problem, presented in vector form, in details, the nodal fracture (or mechanism) displacement $\mathbf{u}_f$ directly compatible with the fracture strain $\boldsymbol{\varepsilon}_f$ and the stress $\boldsymbol{\sigma}$.

All of these quantities relevant to the original structure, i.e. loads $\mathbf{q}$, displacements $\mathbf{u}_f$, strains $\boldsymbol{\varepsilon}_f$ and stresses $\boldsymbol{\sigma}$, can be built up by suitably collecting the analogous quantities $\mathbf{q}_e$, $\mathbf{u}_{fe}$, $\boldsymbol{\varepsilon}_{fe}$ and $\boldsymbol{\sigma}_e$

relevant to the single elements e.

Under the constant stress/strain assumption, one assumes that both the strain vector $\boldsymbol{\varepsilon}_{fe}$ and stress vector $\boldsymbol{\sigma}_e$ are kept constant in the generic element.

One can, thus, write with reference to the overall structure

$$\mathbf{q}_{[2N\times 1]} = \begin{bmatrix} q_1 \\ \vdots \\ q_{2N} \end{bmatrix}, \quad \mathbf{u}_{f[2N\times 1]} = \begin{bmatrix} u_{f1} \\ \vdots \\ u_{f2N} \end{bmatrix}, \quad \boldsymbol{\varepsilon}_{f[3M\times 1]} = \begin{bmatrix} \varepsilon_{f1} \\ \vdots \\ \varepsilon_{f3M} \end{bmatrix}, \quad \boldsymbol{\sigma}_{[3\times 1]} = \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_{3M} \end{bmatrix} \tag{13}$$

and, with reference to the single element

$$\mathbf{q}_{e[2N_e\times 1]} = \begin{bmatrix} q_1^{(e)} \\ \vdots \\ q_{2N_e}^{(e)} \end{bmatrix}, \quad \mathbf{u}_{fe[2N_e\times 1]} = \begin{bmatrix} u_{f1}^{(e)} \\ \vdots \\ u_{f2N_e}^{(e)} \end{bmatrix}, \quad \boldsymbol{\varepsilon}_{fe[3\times 1]} = \begin{bmatrix} \varepsilon_{f1}^{(e)} \\ \varepsilon_{f2}^{(e)} \\ \varepsilon_{f3}^{(e)} \end{bmatrix} = \begin{bmatrix} \varepsilon_{fxe} \\ \varepsilon_{fye} \\ \gamma_{fe} \end{bmatrix}, \quad \boldsymbol{\sigma}_{e[3\times 1]} = \begin{bmatrix} \sigma_1^{(e)} \\ \sigma_2^{(e)} \\ \sigma_3^{(e)} \end{bmatrix} = \begin{bmatrix} \sigma_{xe} \\ \sigma_{ye} \\ \tau_e \end{bmatrix} \tag{14}$$

where N_e is the number of nodes characterizing the element e; $u_{fr}^{(e)}, q_r^{(e)}$, with $r = 1\ldots 2N_e$, denote respectively the in-plane components (along the two plane co-ordinate directions x, y) of the element displacement and load vectors; finally $\varepsilon_{f\ell}^{(e)}, \sigma_\ell^{(e)}$ with $\ell = 1\ldots 3$ represent respectively the in-plane strain and stress element components in the co-ordinate reference axes, i.e., the normal ε_{fke}, σ_{ke} ($k = x, y$) and tangential components τ_{fe}, γ_{fe} of stress and strain.

One should notice that the components $\varepsilon_{f\ell}^{(e)}, \sigma_\ell^{(e)}$ of the element strain and stress vectors correspond to the components ε_{ft}, σ_t of the overall vectors, according to the relation between indexes $t = 3(e-1) + \ell$.

Under these assumptions, one can write general equalities and inequalities governing the whole problem, i.e., all of those equations that should be satisfied by the solution $\{\bar{\mathbf{u}}_f, \bar{\boldsymbol{\varepsilon}}_f, \bar{\boldsymbol{\sigma}}\}$ associated to the collapse value $\bar{s}$.

3.2 Compatibility, equilibrium and admissibility conditions

For the discrete model, one should consider compatibility equations between elements' fracture strains and nodal displacements on one side, and equilibrium conditions between elements' stresses and nodal loads on the other side. Therefore, compatibility conditions may be expressed in the following matrix and scalar forms

$$\mathbf{B}\mathbf{u}_f = \boldsymbol{\varepsilon}_f, \qquad \sum_{j=1}^{2N} b_{ij} u_{fj} = \varepsilon_{fi}, \quad i = 1\ldots 3M \tag{15}$$

where $\mathbf{B}$ is the compatibility matrix with elements b_{ij} and dimension $[3M \times 2N]$, while, analogously, equilibrium is given by

$$\mathbf{A}\boldsymbol{\sigma} = \mathbf{q} = \mathbf{q}_o + s\mathbf{q}_v, \qquad \sum_{j=1}^{3M} a_{ij}\sigma_j = q_{oi} + s q_{vi}, \quad i = 1\ldots 2N \tag{16}$$

where $\mathbf{A}$ is the equilibrium matrix with elements a_{ij}, whose dimension is $[2N\times 3M]$. The two matrices are related by the transpose operation $\mathbf{B} = \mathbf{A}^T$.

Therefore, since $b_{ij} = a_{ji}$, compatibility and equilibrium conditions can be rewritten in the form

$$\mathbf{f}_\varepsilon(\mathbf{u}_f, \boldsymbol{\varepsilon}_f) = \mathbf{B}\mathbf{u}_f - \boldsymbol{\varepsilon}_f = \mathbf{0}, \quad \sum_{j=1}^{2N} b_{ij}u_{fj} - \varepsilon_{fi} = \sum_{j=1}^{2N} u_{fj}a_{ji} - \varepsilon_{fi} = 0, \; i = 1\ldots3M \tag{17}$$

$$\mathbf{f}_\sigma(s, \boldsymbol{\sigma}) = -\mathbf{q}_o - s\mathbf{q}_v + \mathbf{A}\boldsymbol{\sigma} = \mathbf{0}, \quad q_{oi} + sq_{vi} - \sum_{j=1}^{3M} a_{ij}\sigma_j = 0, \; i = 1\ldots2N \tag{18}$$

Since, for the NRT material, compatibility requires that the additional fracture field does not admit contraction in any point and along any direction, as mentioned in the above, this would imply that the fracture strain field $\boldsymbol{\varepsilon}_f$ is positive semi-definite and, therefore, with reference to the finite element (FE) model, that the element strain components satisfy the inequalities

$$\mathbf{R}_\varepsilon(\boldsymbol{\varepsilon}_f) = \begin{cases} \varepsilon_{xe} \geq 0 \\ \varepsilon_{ye} \geq 0 \\ \varepsilon_{f\,xe}\varepsilon_{f\,ye} - \frac{1}{4}\gamma_{fe}^2 \geq 0 \end{cases}, \quad e = 1\ldots M \tag{19}$$

synthetically expressed by the set of inequalities $\mathbf{h}_\varepsilon(\boldsymbol{\varepsilon}_f) \geq \mathbf{0}$.

Moreover since the basic assumption of *no-tension* models is that purely compressive stresses are activated at any point of the body, which implies the negative definition of the stress tensor all over the body, admissibility for the discrete model imposes that the single element stress vector satisfies the relations

$$\mathbf{h}_\sigma(\boldsymbol{\sigma}) = \begin{cases} \sigma_{xe} \leq 0 \\ \sigma_{ye} \leq 0 \\ \tau_e^2 - \sigma_{xe}\sigma_{ye} \leq 0 \end{cases}, \quad e = 1\ldots M \tag{20}$$

synthetically expressed by the set of inequalities $\mathbf{h}_\sigma(\boldsymbol{\sigma}) \leq \mathbf{0}$.

Finally the problem is governed by compatibility equalities Eq. (17), equilibrium equalities Eq. (18), and admissibility inequalities Eqs. (19), (20).

Moreover two further conditions should characterize the mechanism

$$\begin{cases} \mathbf{q}_o \cdot \mathbf{u}_f \leq 0 \\ (\mathbf{q}_o + s\mathbf{q}_v) \cdot \mathbf{u}_f = \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}_f \end{cases} \tag{21}$$

The first representing the condition that $\mathbf{q}_o$ is the stabilizing part of the load, the second ensuring that the work produced by the load during the mechanism is balanced by the internal fracture work (i.e., the energy dissipated in correspondence of the fractures, which is equal to zero).

3.3 The static theorem approach

On the basis of the relations governing the problem introduced in the previous section, one can search for the value $\bar{s}$ (and, thus, the related mechanism $\bar{\mathbf{u}}_f$, $\bar{\boldsymbol{\varepsilon}}_f$ and stress $\bar{\boldsymbol{\sigma}}$) of the load multiplier

s, limiting the loading capacity of the structure, by applying respectively the LA static and kinematic theorems.

In details, the NRT static problem can be set up for the discrete plane model, by defining the class of statically admissible multipliers.

As mentioned in the above, a load multiplier, in order to be statically admissible should comply with Eqs. (18) and (20).

Therefore the problem of searching for the collapse multiplier of the applied loads $\bar{s}$, i.e. the maximum value of the parameter s equilibrated by NRT admissible stress fields, can be set as a constrained extremum problem, in the form

$$\text{Find} \quad \min_{\mathbf{z} \in Z} \theta(\mathbf{z}) = -s \qquad \text{Sub} \;\; \mathbf{g}(\mathbf{z}) = \begin{cases} \mathbf{f}(\mathbf{z}) \; = \; \mathbf{f}_{\sigma}(s, \boldsymbol{\sigma}) \; = \; \mathbf{0} \\ \mathbf{h}(\mathbf{z}) \; = \; \mathbf{h}_{\sigma}(\boldsymbol{\sigma}) \leq \mathbf{0} \end{cases} \tag{22}$$

where θ represents the objective function, $\mathbf{z}^T = [s \;\; \boldsymbol{\sigma}]$ is the unknown vector, Z is the *statically admissible* **z** *field*, made by all vectors **z** satisfying the constraints **g**.

Eq. (22) represent a non-linear programming problem (the objective function is linear but one of the constraints is non-linear) that can be numerically solved by means of the operational research tools. In Sect.5, after introducing some elements of duality theory (Sect.4), a physical interpretation of the dual problem of Eq. (22) is demonstrated.

3.4 The kinematic theorem approach

Analogously to what developed in the previous section, one can search for the collapse multiplier $\bar{s}$, by applying the LA kinematical theorem to the considered NRT problem.

After defining the class of kinematically sufficient multipliers, the collapse value $\bar{s}$ is identified as its minimum.

A load multiplier, in order to be kinematically sufficient should comply with Eqs. (17) and (19), which represent compatibility between associated fracture strain and displacement and NRT compatibility conditions relevant to the fracture strain. Conditions Eq. (21), accounting for balancing between internal and external energy dissipated during the mechanism, should also be satisfied.

Therefore the problem of searching for $\bar{s}$ can be set as a constrained minimum problem, which again represents a non-linear programming problem, since some of the described constraints are non-linear.

4. Duality in non-linear programming

4.1 A short introduction

Duality plays a crucial role in the theory and computational algorithms of linear programming. Duality in non-linear programming is of a somewhat later development of duality theory in linear programming, beginning with the duality results of quadratic programming (see Mangasarian 1969).

However, duality theory in non-linear programming is related to the reciprocal principles of the calculus of variations, which have been known since as far back as 1927 (see Mond and Hansons 1967).

This feature, makes its application to specific structural problems (Baratta and Corbi 2004) particularly interesting as regards the possibility of demonstrating the physical and theoretical meaning of the related dual problems, deduced by applying duality theory.

In the following, one introduces the minimization problem and its dual and shortly presents some basic duality results of non-linear programming, selected according to the specific subsequent application to NRT panels.

4.2 Primal minimization problem and related dual maximization problem

Let Z^o be an open set in R^n, θ be a numerical function on Z^o, and let **f** and **h** be respectively a λ-dimensional and a μ-dimensional vector function, both defined on Z^o.

Let consider *the (primal) minimization problem* (MP) that consists of searching for $\bar{\mathbf{z}}$, if it exists, such that

$$\theta(\bar{\mathbf{z}}) = \min_{\mathbf{z} \in Z} \theta(\mathbf{z}) \qquad \bar{\mathbf{z}} \in Z = \{\mathbf{z} \in Z^o, \mathbf{f}(\mathbf{z}) = \mathbf{0}, \mathbf{h}(\mathbf{z}) \le \mathbf{0}\} \tag{23}$$

The *dual (maximization) problem* (DP) of Eq. (23) is given as follows. Let θ, **f** and **h** be differentiable on Z^o. Find a $\hat{\mathbf{z}}$ and $\hat{\mathbf{v}} \in R^{\lambda}$, $\hat{\mathbf{w}} \in R^{\mu}$, if they exist, such that

$$\begin{cases} \psi(\hat{\mathbf{z}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}) = \max\limits_{(\mathbf{z}, \mathbf{v}, \mathbf{w}) \in Z^*} \psi(\mathbf{z}, \mathbf{v}, \mathbf{w}) \\ (\hat{\mathbf{z}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}) \in Z^* = \{(\mathbf{z}, \mathbf{v}, \mathbf{w}) | \mathbf{z} \in Z^o, \mathbf{v} \in R^{\lambda}, \mathbf{w} \in R^{\mu}, \nabla_{\mathbf{z}} \psi(\mathbf{z}, \mathbf{v}, \mathbf{w}) = \mathbf{0}, \mathbf{w} \ge \mathbf{0}\} \\ \psi(\mathbf{z}, \mathbf{v}, \mathbf{w}) = \theta(\mathbf{z}) + \mathbf{v} \cdot \mathbf{f}(\mathbf{z}) + \mathbf{w} \cdot \mathbf{h}(\mathbf{z}) \end{cases} \tag{24}$$

or equivalently

$$\begin{cases} \theta(\hat{\mathbf{z}}) + \hat{\mathbf{v}} \cdot \mathbf{f}(\hat{\mathbf{z}}) + \hat{\mathbf{w}} \cdot \mathbf{h}(\hat{\mathbf{z}}) = \max\limits_{(\mathbf{z}, \mathbf{v}, \mathbf{w}) \in Z^*} [\theta(\mathbf{z}) + \mathbf{v} \cdot \mathbf{f}(\mathbf{z}) + \mathbf{w} \cdot \mathbf{h}(\mathbf{z})] \\ (\hat{\mathbf{z}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}) \in Z^* = \{(\mathbf{z}, \mathbf{v}, \mathbf{w}) | \mathbf{z} \in Z^o, \mathbf{v} \in R^{\lambda}, \mathbf{w} \in \boldsymbol{R}^{\mu}, \nabla\theta(\mathbf{z}) + \nabla\mathbf{f}(\mathbf{z})\mathbf{v} + \nabla\mathbf{h}(\mathbf{z})\mathbf{w} = \mathbf{0}, \mathbf{w} \ge \mathbf{0}\} \end{cases} \tag{25}$$

The vectors **v** and **w** obviously represent additional unknown vectors respectively collecting the multipliers of the equality and inequality constraint functions, $f(\mathbf{z})$ and $h(\mathbf{z})$, in the dual problem, as shown in the expression of the dual objective function $\psi(\mathbf{z}, \mathbf{v}, \mathbf{w})$. One should emphasize that only multipliers associated to constraints represented by inequalities (the components of **w**) are required to be non-negative ($\mathbf{w} \ge \mathbf{0}$), while no condition applies to those (i.e., **v**) associated to equality constraints.

The duality results relate solutions $\bar{\mathbf{z}}$ of the primary problem MP and $(\hat{\mathbf{z}}, \hat{\mathbf{v}}, \hat{\mathbf{w}})$ of the dual problem DP to each other. The objective functions θ and ψ are also related by means of duality (see Mangasarian 1969).

5. Duality theory in non-linear programming and NRT discrete problems

5.1 Problem set up

After shortly introducing some elements of duality theory in non-linear programming, one can get back to the non-linear problem given in Eq. (22), which represents the explicit expression of the static problem of limit analysis for plane structures loaded by in-plane forces and made by NRT material.

The problem can be rewritten in a form analogous to the one given for generally expressing the primal problem Eq. (23), i.e.

$$\theta(\bar{\mathbf{z}}) = \min_{\mathbf{z} \in Z} \theta(\mathbf{z}) \qquad \bar{\mathbf{z}} \in Z = \{\mathbf{z} \in Z^o, \mathbf{f}(\mathbf{z}) = \mathbf{0}, \mathbf{h}(\mathbf{z}) \le \mathbf{0}\} \tag{26}$$

with

$$\theta(\mathbf{z}) = -s, \quad \mathbf{z}^T = [s \ \ \boldsymbol{\sigma}^T]^T \tag{27}$$

and

$$\mathbf{g}(\mathbf{z}) = \begin{cases} \mathbf{f}(\mathbf{z}) = \mathbf{f}_\sigma(s, \boldsymbol{\sigma}) = \mathbf{0} \\ \mathbf{h}(\mathbf{z}) = \mathbf{h}_\sigma(\boldsymbol{\sigma}) \le \mathbf{0} \end{cases} = \begin{cases} q_{oi} + s q_{vi} - \sum_{j=1}^{3M} a_{ij}\sigma_j = 0, \ \ i = 1 \ldots 2N \\ \begin{cases} \sigma_{xe} \le 0 \\ \sigma_{ye} \le 0 \qquad , \qquad e = 1 \ldots M \\ \tau_e^2 - \sigma_{xe}\sigma_{ye} \le 0 \end{cases} \end{cases} \tag{28}$$

the dimension of **f** is, for the specific case, $\lambda = 2N$, while the dimension of **h** is $\mu = 3M$, being the number of equilibrium equations equal to $2N$ and the number of admissibility conditions equal to $3M$.

Eqs. (26)-(28) thus define the primal *static* problem, whose dual can be given, according to Eq. (24), as

$$\begin{cases} \psi(\hat{\mathbf{z}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}) = \max_{(\mathbf{z}, \mathbf{v}, \mathbf{w}) \in Z^*} \psi(\mathbf{z}, \mathbf{v}, \mathbf{w}) \\ (\hat{\mathbf{z}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}) \in Z^* = \{(\mathbf{z}, \mathbf{v}) | \mathbf{z} \in Z^o, \mathbf{v} \in R^{2N}, \mathbf{w} \in R^{3M}, \nabla_{\mathbf{z}}\psi(\mathbf{z}, \mathbf{v}, \mathbf{w}) = \mathbf{0}, \mathbf{w} \ge \mathbf{0}\} \\ \psi(\mathbf{z}, \mathbf{v}, \mathbf{w}) = \theta(\mathbf{z}) + \mathbf{v} \cdot \mathbf{f}(\mathbf{z}) + \mathbf{w} \cdot \mathbf{h}(\mathbf{z}) \end{cases} \tag{29}$$

Since

$$\begin{aligned} \mathbf{z}^T_{[1 \times (1+3M)]} &= \mathbf{z}^T(s, \sigma_{xe}, \sigma_{ye}, \tau_e) = [s, \ \sigma_{x1} \ \sigma_{y1} \ \tau_1 \ \ldots \ \sigma_{xM} \ \sigma_{yM} \ \tau_M]^T \\ \mathbf{v}^T_{[1 \times 2N]} &= \mathbf{v}^T(\upsilon_i) = [\upsilon_1 \ldots \upsilon_{2N}]^T \\ \mathbf{w}^T_{[1 \times 3M]} &= \mathbf{w}^T(\rho_{xe}, \rho_{ye}, \xi_e) = [\rho_{x1} \ \rho_{y1} \ \xi_1 \ \ldots \ \rho_{xM} \ \rho_{yM} \ \xi_M]^T \end{aligned} \tag{30}$$

the final expression of the objective function is

$$\begin{aligned} \psi(\mathbf{z}, \mathbf{v}, \mathbf{w}) &= \psi(s, \sigma_{xe}, \sigma_{ye}, \tau_e, \upsilon_i, \rho_{xe}, \rho_{ye}, \xi_e) = \theta(\mathbf{z}) + \mathbf{v} \cdot \mathbf{f}(\mathbf{z}) + \mathbf{w} \cdot \mathbf{h}(\mathbf{z}) \\ &= -L_1^e(\mathbf{z}) - L_2^e(\mathbf{z}) + L^i(\mathbf{z}) \end{aligned} \tag{31}$$

with

$$L_1^e(\mathbf{z}) = s$$

$$L_2^e(\mathbf{z}) = -\sum_{i=1}^{2N}\left[\upsilon_i\left(q_{oi} + sq_{vi} - \sum_{j=1}^{3M} a_{ij}\sigma_j\right)\right] \quad (32)$$

$$L^i(\mathbf{z}) = \sum_{e=1}^{M}[\rho_{xe}\sigma_{xe} + \rho_{ye}\sigma_{ye} + \xi_e(\tau_e^2 - \sigma_{xe}\sigma_{ye})]$$

Furthermore, as previously emphasized, the unknown components of **w** are required to be non-negative ($\mathbf{w} \geq \mathbf{0}$) since they are associated to constraints represented by inequalities, while no condition applies for multiplier **v**, which are associated to equality constraints.

Thus, additional conditions on the unknown multipliers can be expressly given as

$$\rho_{xe} \geq 0, \quad \rho_{ye} \geq 0, \quad \xi_e \geq 0, \quad e = 1\ldots M \quad (33)$$

while no constraint should be applied on υ_i, $i = 1\ldots 2N$.

By developing, $\nabla_{\mathbf{z}}\psi(\mathbf{z}, \mathbf{v}, \mathbf{w}) = \mathbf{0}$ one gets

$$\frac{\partial\psi}{\partial s} = -1 + \sum_{i=1}^{2N} q_{vi}\upsilon_i = 0 \quad (34)$$

$$\frac{\partial\psi}{\partial\sigma_{xe}} = (\rho_{xe} - \xi_e\sigma_{ye}) - \sum_{i=1}^{2N}\upsilon_i a_{it} = 0, \quad t = 3(e-1) + \ell, \quad \ell = 1, \quad e = 1\ldots M \quad (35)$$

$$\frac{\partial\psi}{\partial\sigma_{ye}} = (\rho_{ye} - \xi_e\sigma_{xe}) - \sum_{i=1}^{2N}\upsilon_i a_{it} = 0, \quad t = 3(e-1) + \ell, \quad \ell = 2, \quad e = 1\ldots M \quad (36)$$

$$\frac{\partial\psi}{\partial\tau_e} = 2\xi_e\tau_e - \sum_{i=1}^{2N}\upsilon_i a_{it} = 0, \quad t = 3(e-1) + \ell, \quad \ell = 3, \quad e = 1\ldots M \quad (37)$$

Eq. (29) with the associated Eqs. (30)-(37) give a complete representation of the dual problem.

5.2 Solution of the primal and dual problem

By means of duality, one can assert that the solution $\hat{\mathbf{z}}$ of the dual problem Eq. (29) coincides with the solution $\bar{\mathbf{z}}$ of the primal problem Eq. (26).

5.3 A physical interpretation of the expression of the dual problem

Once set up the primal and dual problems as shown in the previous section, where the primal problem represents the expression of the static theorem of Limit Analysis for NRT discrete plane models loaded by in-plane forces, one can figure out some interesting considerations on the associated dual problem.

Looking at the expression of the dual objective function $\psi(\mathbf{z}, \mathbf{v}, \mathbf{w})$ in Eqs. (31), (32) as a primal problem, it can be shown that the dual problem Eqs. (29)-(37) represents the expression of the kinematic problem of Limit Analysis for the considered NRT model.

Thus, in the following, one starts from the problem Eqs. (29)-(37), trying to understand, by means of algebraic operations and overall demonstrations, the physical meaning of each term.

5.3.1 Stress admissibility and equilibrium

First of all, since the dual function is maximum in solution, any variation $\delta\psi$ associated to any admissible $\delta\mathbf{w}$ should be negative, i.e., $\delta\psi \le 0 \ \forall \ \delta\mathbf{w}$ such that $(\mathbf{w} + \delta\mathbf{w}) \ge \mathbf{0}$.

Thus, let consider the third term $L^i(\mathbf{z})$ on the right hand side of Eq. (31), given in Eq. (32).

Let analyse the contribution of the first term $\rho_{xe}\sigma_{xe}$. On the multiplier ρ_{xe} constraints Eq. (33) apply, so that $\rho_{xe} \ge 0$; thus, for any variation $\delta\rho_{xe}$ one should have a negative variation of the dual function, i.e., $\delta\psi \le 0$. Two cases should be considered depending on the initial value of ρ_{xe}: i) $\rho_{xe} > 0$; ii) $\rho_{xe} = 0$.

In the first case i) in order to have $\delta\psi \le 0$ for any $\delta\rho_{xe} \ne 0$ such that $(\rho_{xe} + \delta\rho_{xe}) \ge 0$, one should have that $\sigma_{xe} = 0$; in the second case ii), since one can have only $\delta\rho_{xe} > 0$, in order to have $\delta\psi \le 0$ one should have $\sigma_{xe} < 0$. After combining the two conditions one gets that $\sigma_{xe} \le 0$.

By applying the same reasoning to the other two terms in the expression of $L^i(\mathbf{z})$, i.e., $\rho_{ye}\sigma_{ye}$ and $\xi_e(\tau_e^2 - \sigma_{xe}\sigma_{ye})$, one can make analogous considerations, finally getting

$$\begin{cases} \sigma_{xe} \le 0 \\ \sigma_{ye} \le 0 \\ \tau_e^2 - \sigma_{xe}\sigma_{ye} \le 0 \end{cases} \quad , \qquad e = 1 \ldots M \tag{38}$$

which represent NRT stress admissibility conditions in the single element.

On the other side, by considering the second expression in Eq. (31), $L_2^e(\mathbf{z})$, given in the second of Eq. (32), since the coefficients υ_i are not constrained in sign, the quantity in parenthesis is forced to be zero, otherwise positive variations of the objective function could be produced. The result is that the stresses σ_i are admissible and in equilibrium with the applied load, and the term $L_2^e(\mathbf{z})$ is null in solution.

5.3.2 Strain admissibility and compatibility

Under the above deduced stress admissibility, strain compatibility can be individuated.

Let introduce the element and overall vectors, $\boldsymbol{\eta}_e^T = [\eta_{1e} \ \eta_{2e} \ \eta_{3e}]^T$ and $\boldsymbol{\eta}^T = [\eta_1 \ldots \eta_{3M}]^T$, such that the component $\eta_{\ell e}$ of $\boldsymbol{\eta}_e$ (with $\ell = 1, 2, 3$) is related to the component η_ℓ of $\boldsymbol{\eta}$ according to the relation $t = 3(e-1) + \ell$. Then, in Eqs. (34)-(37) let assume that

$$\begin{cases} \rho_{xe} - \xi_e\sigma_{ye} = \eta_{1e} \\ \rho_{ye} - \xi_e\sigma_{xe} = \eta_{2e}, \quad e = 1 \ldots M \\ 2\xi_e\tau_e = \eta_{3e} \end{cases} \tag{39}$$

Again with reference to the term $L^i(\mathbf{z})$, one can make additional considerations.

Let first consider the third term of $L^i(\mathbf{z})$, that is $\xi_e(\tau_e^2 - \sigma_{xe}\sigma_{ye})$ for the single element e. Because of NRT stress admissibility, one has two basic cases with reference to the element e: i) $\tau_e^2 < \sigma_{xe}\sigma_{ye}$; ii) $\tau_e^2 = \sigma_{xe}\sigma_{ye}$.

In the first <u>case i)</u> one has $\tau_e^2 < \sigma_{xe}\sigma_{ye}$; since also $\sigma_{xe} \le 0$ and $\sigma_{ye} \le 0$ one deduces that $\sigma_{xe} < 0$ and $\sigma_{ye} < 0$, which in turn, involves (for what stated in the previous paragraph) that $\rho_{xe} = 0$ and $\rho_{ye} = 0$. On the other side, the same condition i) imposes that $\xi_e = 0$ (again as stated in the previous paragraph). Therefore, when $\boldsymbol{\sigma}^{(e)}$ is strictly negatively defined, then $\boldsymbol{\eta}_e = \mathbf{0}$, and the relevant contribution to $L^i(\mathbf{z})$ is null.

In the second <u>case ii)</u> one has $\tau_e^2 = \sigma_{xe}\sigma_{ye}$ for the element e; the contribution to $L^i(\mathbf{z})$ can be written as

$$\rho_{xe}\sigma_{xe} + \rho_{ye}\sigma_{ye} \tag{40}$$

On the other hand, one can check that

$$\begin{aligned} \eta_{1e}\sigma_{xe} + \eta_{2e}\sigma_{ye} + \eta_{3e}\tau_e &= (\rho_{xe} - \xi_e\sigma_{ye})\sigma_{xe} + (\rho_{ye} - \xi_e\sigma_{xe})\sigma_{ye} + 2\xi_e\tau_e^2 \\ &= \rho_{xe}\sigma_{xe} + \rho_{ye}\sigma_{ye} + 2\xi_e(\tau_e^2 - \sigma_{xe}\sigma_{ye}) = \rho_{xe}\sigma_{xe} + \rho_{ye}\sigma_{ye} \end{aligned} \tag{41}$$

For what stated in the above, the considered case implies that $\xi_e > 0$ and, thus, because of Eq. (41), $\eta_{3e} > 0$; the term $\eta_{3e}\sigma_{3e}$, gives again a null contribution to $L^i(\mathbf{z})$.

Moreover, for the other two terms in Eq. (41), i.e., $\rho_{xe}\,\sigma_{xe}$ and $\rho_{ye}\,\sigma_{ye}$, one can make considerations that are analogous to the previous ones (now one can have both $\sigma_{xe} \le 0$ and $\sigma_{ye} \le 0$, with $\rho_{xe} \ge 0$ and $\rho_{ye} \ge 0$), thus demonstrating that both contributions are equal to zero.

Finally, for the considered case ii), from Eq. (41), one has that the contribution of the element e to $L^i(\mathbf{z})$ can be written as $\boldsymbol{\sigma}^{(e)} \cdot \boldsymbol{\eta}_e$ for $\boldsymbol{\eta}_e \ne \mathbf{0}$ (which means that some of the elements of $\boldsymbol{\eta}_e$ may be equal to zero), which is null since it is given by the sum of null quantities.

Hence, one can combine the considered cases i) and ii), synthesizing that

$$\begin{cases} \text{if } \boldsymbol{\eta}_e = \mathbf{0} \rightarrow \boldsymbol{\sigma}^{(e)} \cdot \boldsymbol{\eta}_e = 0 \\ \text{if } \boldsymbol{\eta}_e \ne \mathbf{0} \rightarrow \boldsymbol{\sigma}^{(e)} \cdot \boldsymbol{\eta}_e = 0 \end{cases} \rightarrow L^i(\mathbf{z}) = \sum_{e=1}^{M} \boldsymbol{\sigma}^{(e)} \cdot \boldsymbol{\eta}_e = \boldsymbol{\sigma} \cdot \boldsymbol{\eta} = 0 \tag{42}$$

Eq. (42) makes clear that in solution $L^i(\mathbf{z})$ represents the internal fracture work for NRT material $\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}_f$, that is to say that $\boldsymbol{\eta} = \boldsymbol{\varepsilon}_f$

$$\begin{cases} \rho_{xe} - \xi_e\sigma_{ye} = \eta_{1e} = \varepsilon_{fxe} \\ \rho_{ye} - \xi_e\sigma_{xe} = \eta_{2e} = \varepsilon_{fye}, \quad e = 1 \ldots M \\ 2\xi_e\tau_e = \eta_{3e} = \gamma_{fe} \end{cases} \tag{43}$$

and in the reference of principal axes 1, 2

$$\begin{cases} \text{if } \sigma_i \ne 0 \rightarrow \varepsilon_{fi} = 0 \\ \text{if } \sigma_i = 0 \rightarrow \varepsilon_{fi} \ne 0 \end{cases} \rightarrow L^i(\mathbf{z}) = \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}_f = 0, i = 1, 2 \tag{44}$$

which exactly represents the condition for the development of fractures in NRT materials, as shown in the general section devoted to the introduction of the NRT model.

Therefore, after substituting Eq. (44) in Eqs. (34)-(36), one gets

$$\begin{cases} \varepsilon_{fxe} - \sum_{i=1}^{2N} a_{it}\upsilon_i = \varepsilon_{fxe} - \sum_{i=1}^{2N} b_{ti}\upsilon_i = 0, \quad \ell = 1 \\ \varepsilon_{fye} - \sum_{i=1}^{2N} a_{it}\upsilon_i = \varepsilon_{fye} - \sum_{i=1}^{2N} b_{ti}\upsilon_i = 0, \quad \ell = 2, \quad t = 3(e-1)+\ell, \quad e = 1\ldots M \\ \gamma_{fe} - \sum_{i=1}^{2N} a_{it}\upsilon_i = \gamma_{fe} - \sum_{i=1}^{2N} b_{ti}\upsilon_i = 0, \quad \ell = 1 \end{cases} \tag{45}$$

which express compatibility between displacements and fractures, under the assumption that $\upsilon_i = u_{fi}$, $i = 1\ldots2N$. Thus the multipliers υ_i physically represent compatible displacements.

Compatibility for NRT materials also requires that the fracture strain tensor is positive semi-definite, that is to say that Eq. (19) $\mathbf{h}_\varepsilon(\boldsymbol{\varepsilon}_f) \geq \mathbf{0}$ are satisfied.

This can be demonstrated, simply looking at the expressions of the components of $\boldsymbol{\varepsilon}_f$, which are shown to be given by a sum of non-negative elements. Therefore, it can be easily checked, because of Eq. (33) and of the NRT stress admissibility Eq. (38), that

$$\begin{cases} \varepsilon_{fxe} = \rho_{xe} - \xi_e\sigma_{ye} \geq 0 \\ \varepsilon_{fye} = \rho_{ye} - \xi_e\sigma_{xe} \geq 0 \\ \varepsilon_{fxe}\varepsilon_{fye} - \frac{1}{4}\gamma_{fe}^2 = (\rho_{xe} - \xi_e\sigma_{ye})(\rho_{ye} - \xi_e\sigma_{xe}) - \xi_e\tau_e^2 \\ \qquad = \xi_e^2(\tau_e^2 - \sigma_{xe}\sigma_{ye}) - \xi_e\rho_{xe}\sigma_{xe} - \xi_e\rho_{ye}\sigma_{ye} + \rho_{xe}\rho_{ye} \geq 0 \end{cases} \tag{46}$$

5.3.3 Discussion on the objective function

As shown in the previous section, the dual objective function Eq. (31) is made of three main components; the third one $L^i(\mathbf{z})$ has been shown to have a precise physical meaning in solution: it represents the internal fracture work $\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}_f$ between the NRT admissible stress $\boldsymbol{\sigma}$ and the NRT compatible fracture strain $\boldsymbol{\varepsilon}_f$.

Consequently, as shown in the above, Eqs. (35)-(37) assume the meaning of compatibility equations between fractures and displacements, thus identifying the multipliers υ_i as compatible displacements u_{fi}, $i = 2\ldots2N$.

In any case, in solution both $L_2^e(\mathbf{z})$ and $L^i(\mathbf{z})$ are zero. The objective function turns out, in solution and hence at its maximum admissible value, to be equal to -s, i.e., after changing the sign, to the minimum value of the load factor compatible with the constraints.

5.3.4 Physical interpretation of the dual problem

The developments in the above prove that the dual problem in solution obeys some basic conditions. First of all, the σ-variables can be interpreted as admissible compressive stresses in equilibrium with the applied loads (Sec. 5.3.1); secondly, the ρ- and ξ-variables, suitably combined to form the η-variables, are correlated to the ε-variables that can be interpreted as a set of fracture strain components (Sec. 5.3.2); the latter are admissible fracture strains compatible with the υ-variables that can therefore be interpreted as a set of displacements $\mathbf{u}_f$. The result is that $\mathbf{u}_f$ is a collapse mechanism. Moreover the internal work $\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}_f$ is null in solution (Eq. (44)), as prescribed for NRT solids.

Since $L_2^e = L^i = 0$, the optimal value of the objective function in solution yields the minimum value of the load factor, reversed in sign, allowing for the existence of a collapse mechanism associated to a positive work performance by the variable load components, conventionally normalized to unity (Eq. (34)).

This result coincides with the definition of the safety factor as the smallest kinematically sufficient factor; the coincidence of the optimal values in both the primal and the dual problem (Eq. (27)), proves moreover that the largest statically admissible factor coincides with the smallest kinematically sufficient factor, their common value uniquely yielding the actual safety factor.

6. Conclusions

In the paper the problem of duality in non-linear programming applied to NRT structures is approached. Actually, in dealing with no-tension structures, a big variety of applications lead to the set up of constrained extremum problems, mainly characterized by linear objective functions and linear and non-linear constraint. All of these problems, as non-linear programming problems, can, thus, be numerically solved by means of Operational Research tools.

On the other side, application of duality principles to the treated cases may be of particular interest. In the paper, duality theory is applied for the evaluation of the loading capacity of NRT structures, by means of Limit Analysis theorems explicitly formulated for NRT models.

In details, one demonstrates that, starting from the application of the duality theory to the non-linear program defined by the static LA theorem approach for a discrete NRT model, this procedure results in the definition of a dual problem that represents the application of the kinematic LA theorem and yields the basic contiguity theorem of the two classes, the statically admissible and kinematically sufficient load factors.

By the way, the duality analysis proves that the assumption of an associated flow law (i.e., the Drucker's postulate is assumed to hold) is a necessary condition for the kinematic approach to yield separate results, contiguous to the static method. In other words, the static criterion for collapse has its own validity independently of the flow law, whilst the kinematic criterion needs some more detailed specification concerning the development of inelastic strain.

References

Baratta, A. (1996), "Structural analysis of masonry buildings", In: *Seismic Risk of Historic Centers. A Preliminary Approach to the Naples Case*, A. Baratta and T. Colletta Eds., La Città del Sole B.C., Napoli, 76-122.

Baratta, A. and Corbi, I. (2004), "Plane of elastic non-resisting tension material under foundation structures", *Int. J. Numer. Anal. Meth. Geom.*, **28**, 531-542.

Baratta, A. and Corbi, O. (2003), "Limit analysis of no tension bodies and non-linear programming", *Proc. 9th Int. Conf. Civil and Structural Engineering Computing*. Egmond Aan Zee, Netherlands.

Baratta, A. and Corbi, O. (2003), "The no tension model for the analysis of masonry-like structures strengthened by fiber reinforced polymers", *Int. J. Masonry Int.*, *British Masonry Society*, **16**(3), 89-98.

Baratta, A. and Corbi, O. (2004), "Applicability of duality theory to L.A. problems with undefined flow law", *Proc. 1st Int. Conf. RRTEA on Restoration, Recycling, Rejuvenation Technology for Engineering and*

Architecture Application. Cesena, Italy.
Baratta, A. and Toscano, R. (1982), "Stati tensionali in pannelli di materiale non reagente a trazione", *Proc. 6th Nat. Conf. AIMETA*, Genova, **2**, 291-301.
Baratta, A., Zuccaro, G. and Binetti, A. (2004), "Strength capacity of a no-tension portal arch-frame under combined seismic and ash loads", *J. Volcanol. Geoth. Res.*, **133**, 369-376.
Bazaraa, M.S., Shetty, C.M. and Sherali, H.D. (1993), "Non-linear programming: theory and algorithms", (Wiley Series in Discrete Mathematics & Optimization), John Wiley and Sons Eds., March 1993. ISBN: 0471557935.
Como, M. and Grimaldi, A. (1983), "A unilateral model for limit analysis of masonry walls", In: *Unilateral Problems in Structural Analysis*, Ravello, 25-46.
Del Piero, G. (1988), "Constitutive equations and compatibility of the external loads for linearly elastic masonry-like materials", *J. Meccanica*, **24**(3), 150-162.
Del Piero, G. (1998), "Limit analysis and no-tension materials", *Int. J.* Plasticity, **14**, 259-271.
Di Pasquale, S. (1982) "Questioni di meccanica dei solidi", *Proc. 6th Nat. Conf. AIMETA*, Genova, **2**, 251-263.
Heyman, J. (1969), "The safety of masonry arches", *J. Mech. Sci.*, **2**, 363-384.
Jahn, J. (1996), "Introduction to the theory of nonlinear optimization", Springer-Verlag Berlin and Heidelberg GmbH & Co. K, September 1996.
Mangasarian, O.L. (1969), *Nonlinear Programming*, McGraw-Hill, USA.
Mond, B. and Hansons, M.A. (1967), "Duality for variational problems", *J. Math. Anal. Appl.*, **18**, 355-364.