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Frictionless contact problem for a layer on an elastic half plane loaded by means of two dissimilar rigid punches

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Abstract. The contact problem for an elastic layer resting on an elastic half plane is considered according to the theory of elasticity with integral transformation technique. External loads P and Q are transmitted to the layer by means of two dissimilar rigid flat punches. Widths of punches are different and the thickness of the layer is h. All surfaces are frictionless and it is assumed that the layer is subjected to uniform vertical body force due to effect of gravity. The contact along the interface between elastic layer and half plane will be continuous, if the value of load factor, λ, is less than a critical value, λ_{cr}. However, if tensile tractions are not allowed on the interface, for $\lambda > \lambda_{cr}$ the layer separates from the interface along a certain finite region. First the continuous contact problem is reduced to singular integral equations and solved numerically using appropriate Gauss-Chebyshev integration formulas. Initial separation loads, λ_{cr}, initial separation points, x_{cr}, are determined. Also the required distance between the punches to avoid any separation between the punches and the layer is studied and the limit distance between punches that ends interaction of punches, is investigated. Then discontinuous contact problem is formulated in terms of singular integral equations. The numerical results for initial and end points of the separation region, displacements of the region and the contact stress distribution along the interface between elastic layer and half plane is determined for various dimensionless quantities.

Keywords: continuous contact; discontinuous contact; separation; elastic layer; rigid punch; asymmetric loading; integral equation; elasticity.

1. Introduction

A great many situations in engineering require the transmission of loads through contacts between different component and parts of structures. Contact phenomena are abundant in everyday life and play a very important role in engineering structures. Contact problem of beams and plates resting on an elastic half-plane has attracted the attention of several researchers due to its applicability to foundation-superstructures interaction such as railways ballasts, foundation grillages, continuous foundation beams, runaways, liquid tanks resting on the ground and grain silos.

While the initial contact is determined by geometric features of the bodies, the extent of the contact generally changes when the bodies are deformed by applied forces. If the body forces due to gravity were neglected, the application of any compressive load to free surface of an elastic layer

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resting on a half plane, no matter how small it is, cause layer to bend and the contact area between the layer and the subspace will diminish to a finite size (Civelek and Erdogan 1975). On the other hand, if the effect of gravity was taken into account, the normal stress along the layer subspace interface will be compressive and the contact is maintained through the interface unless the compressive applied load exceeds a certain critical value. When the magnitude of the compressive external load exceeds a certain critical value a separation will take place between the layer and the foundation. The length of separation region along the interface is unknown and the problem becomes a discontinuous contact problem.

Intensity of the applied force and material properties play a very important role in the formation of the initial separation point and stress distribution on the contact surface. The separation is not desirable since the resulting tensile stress can not be met (Civelek *et al*. 1978, Cakiroglu and Cakiroglu 1991, Birinci and Erdol 2001).

A long layer and/or a punch resting on a rigid or elastic foundation have been one of the most widely studied problems in contact mechanics. The importance of the problem lies in the fact that their geometry approximates a very common structural component (Dhaliwal 1970, Adams and Bogy 1976, Gladwell 1976, Gecit and Gökpınar 1985, Klarbring *et al*. 1991, Jaffar 1993, Shanahan 2000, Porter and Hills 2002, Jaffar 2002).

In most of previous works, frictionless layer is pressed locally against the foundation and the effect of gravity neglected, in this case contact shrinks. As mentioned earlier, a contact is described to be receding if the loaded contact area is completely contained within the unloaded contact area. (Keer *et al.* 1972, Civelek and Erdogan 1974, Gecit 1986, Comez *et al.* 2004). Furthermore, some example studies taking the effect of gravity into consideration is as follows; Civelek and Erdogan (1975) solved the contact problem between a rigid half plane and a beam where separation caused by tensile point load. They also solved the same problem for the case of a compressive single load (Civelek and Erdogan 1976). The frictionless contact problem for a layer which is resting on a rigid foundation and acted upon by an axisymmetric line load, a uniform clamping pressure and a vertical body force due to gravity is considered by Gecit and Erdogan (1978). Continuous and discontinuous contact positions are studied between semi-infinite plane and a layer by Cakiroglu and Cakiroglu (1991). Symmetrical finite distributed loads were applied to the layer. The frictionless contact problem for a layered composite which consists of two elastic layers having different elastic constants and heights on two simple supports is examined by Birinci and Erdol (2001). Also the authors are considered the plane contact problem for two infinite elastic layers lying on a Winkler foundation for various elastic constants and heights (Birinci and Erdol 2003).

Besides, contact problem deals with interaction of two circular punches on an elastic layer is solved by Lan *et al.* (1996). They examined both the normal indentation and a relaxed tangential indentation problems. The solution of contact problem for two plane punches is summarized in (Artan and Omurtag 2000). The two punches are resting on an elastic half space and the interface between punches and elastic space is frictionless.

The problem of a flat-ended rigid punch has important applications in soil mechanics, particularly in estimating the safety of foundations. The two dissimilar punches problem for an elastic layer resting on an elastic half plane does not appear to be investigated. The application of this problem in soil mechanics is obvious e.g., the punches can be taken as footings erected on a layered soil (Dhaliwal 1970).

In this study, contact problem of the two dissimilar punches for an elastic layer resting on an elastic half plane is considered according to the theory of elasticity with integral transformation technique. The external loads P and Q are transmitted to the layer by means of two rigid flat punches. The width of punches can be different and thickness of the layer is constant, h. The intensity of the body force acting on elastic layer due to gravity is $\rho_1 g$. All surfaces are frictionless. First, the continuous contact problem between layer and half plane is examined where the magnitude of load factor λ is less than a critical value, λ_{cr} $(\lambda = P/\rho_1 gh^2)$. If $\lambda > \lambda_{cr}$, a separation takes place between the layer and the half plane and the contact is discontinuous. An integration procedure is performed numerically for the solution of the problems and different parameters are researched for various dimensionless quantities for both continuous and discontinuous contact cases. Numerical results are analyzed and conclusions are drawn.

2. General expressions for stresses and displacements

Consider a frictionless elastic layer of thickness h lying on an elastic half plane. The geometry and coordinate system are shown in Fig. 1. The governing equations are

$$\mu_k \nabla^2 u_k + \frac{2\mu_k}{(\kappa_k - 1)}\frac{\partial}{\partial x}\left(\frac{\partial u_k}{\partial x} + \frac{\partial v_k}{\partial y}\right) = 0 \tag{1a}$$

$$(k = 1, 2)$$

$$\mu_k \nabla^2 v_k + \frac{2\mu_k}{(\kappa_k - 1)}\frac{\partial}{\partial y}\left(\frac{\partial u_k}{\partial x} + \frac{\partial v_k}{\partial y}\right) = \rho_k g \tag{1b}$$

where $\rho_k g$ is the intensity of the body force acting vertically in which ρ_k and g are mass density and gravity acceleration. u_k and v_k are the x and y component of the displacement vector, μ_k and κ_k represent shear modulus and elastic constant of the layer and the half plane, respectively. $\kappa_k = (3 - \gamma_k)/(1 + \gamma_k)$ for plane stress and $\kappa_k = (3 - 4\gamma_k)$ for plane strain. γ_k is the Poisson's ratio $(k = 1, 2)$. Subscript 1 indicates the elastic layer and subscript 2 indicates the elastic half plane.

u_p and v_p represent the displacements for the case in which gravity forces are considered. u_h and v_h are the displacements when the gravity forces are ignored and total field of displacements may be expressed as

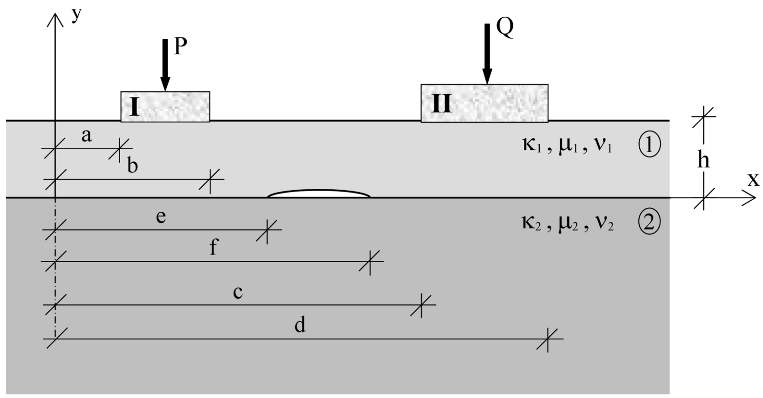

Fig. 1 Elastic layer resting on an elastic half plane loaded by means of two dissimilar flat punches

$$u = u_p + u_h \tag{2a}$$

$$v = v_p + v_h \tag{2b}$$

Considering Fourier transformation in x, homogeneous part of displacements for the elastic layer may be written in the following form:

$$u_{1_h}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} U_1(\alpha,y)e^{i\alpha x}d\alpha \tag{3a}$$

$$v_{1_h}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} V_1(\alpha,y)e^{i\alpha x}d\alpha \tag{3b}$$

where, $i = \sqrt{-1}$ and $U_1(\alpha,y)$ and $V_1(\alpha,y)$ are inverse Fourier transforms of $u_1(x,y)$ and $v_1(x,y)$ in x and y, respectively. Substituting (3) in (1) and solving second order differential equations, the following expressions may be obtained for displacements.

$$u_{1_h}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\{ie^{-|\alpha|y}\left[-A\frac{|\alpha|}{\alpha} + B\left(\frac{\kappa_1}{\alpha} - \frac{|\alpha|}{\alpha}y\right)\right] + ie^{|\alpha|y}\left[C\frac{|\alpha|}{\alpha} + D\left(\frac{\kappa_1}{\alpha} + \frac{|\alpha|}{\alpha}y\right)\right]\right\}e^{i\alpha x}d\alpha \tag{4a}$$

$$v_{1_h}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\{[A + By]ie^{-|\alpha|y} + |C + Dy|ie^{|\alpha|y}\}e^{i\alpha x}d\alpha \tag{4b}$$

Using Hooke's Law and Eq. (4), stress component which do not include the effect of gravity force may be expressed as follows:

$$\sigma_{y_{1_h}}(x,y) = \frac{\mu_1}{2\pi}\int_{-\infty}^{+\infty} e^{i\alpha x}\{e^{-|\alpha|y}\{-2A|\alpha| + B[(\kappa_1 - 1) - 2|\alpha|y]\}$$

$$+ e^{|\alpha|y}\{2C|\alpha| + D[(\kappa_1 - 1) + 2|\alpha|y]\}\}d\alpha \tag{5a}$$

$$\sigma_{x_{1_h}}(x,y) = -\frac{\mu_1}{2\pi}\int_{-\infty}^{+\infty} e^{i\alpha x}\{e^{-|\alpha|y}\{-2A|\alpha| + B[(\kappa_1 + 3) - 2|\alpha|y]\}$$

$$+ e^{|\alpha|y}\{2C|\alpha| + D[(\kappa_1 + 3) + 2|\alpha|y]\}\}d\alpha \tag{5b}$$

$$\tau_{xy_{1_h}}(x,y) = \frac{i\mu_1}{2\pi}\int_{-\infty}^{+\infty} e^{i\alpha x}\left\{e^{-|\alpha|y}\left\{2A\alpha + B\left[-\frac{|\alpha|}{\alpha}(\kappa_1 + 1) + 2\alpha y\right]\right\}\right.$$

$$\left.+ e^{|\alpha|y}\left\{2C\alpha + D\left[\frac{|\alpha|}{\alpha}(\kappa_1 + 1) + 2\alpha y\right]\right\}\right\}d\alpha \tag{5c}$$

For the case in which gravity force exist, particular part of the displacement components corresponding to $\rho_1 g$, the following expressions are obtained, i.e., special solution of the Navier equations for a layer with a height h.

$$u_{1_p} = \frac{3-\kappa_1}{8\mu_1}\frac{\rho_1 gh}{2}x \tag{6a}$$

$$v_{1_p} = \frac{\rho_1 gy}{2\mu_1}\left[\frac{(\kappa_1-1)}{(\kappa_1+1)}(y-h) - \frac{(\kappa_1+1)}{8}h\right] \tag{6b}$$

$$\sigma_{y_{1_p}} = \frac{\rho_1 g}{2\mu_1}(y-h) \tag{6c}$$

$$\sigma_{x_{1_p}} = \frac{\rho_1 g}{2\mu_1}\left(y-\frac{h}{2}\right)\frac{(1+\kappa_2)}{(1+\kappa_2)+2\mu_2(1+\kappa_1)} \tag{6d}$$

$$\tau_{xy_{1_p}} = 0 \tag{6e}$$

Considering the orthogonal axes shown in Fig. 1, displacements will be zero for $y = -\infty$ and if μ_2, ν_2 are the elastic constants of the half plane, then homogenous field of displacements and stresses of the elastic half plane may be obtained as

$$u_{2_h}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\{ie^{|\alpha|y}\left[E\frac{|\alpha|}{\alpha} + F\left(\frac{\kappa_2}{\alpha} + \frac{|\alpha|}{\alpha}y\right)\right]\right\}e^{i\alpha x}d\alpha \tag{7a}$$

$$v_{2_h}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\{[E+Fy]ie^{|\alpha|y}\}e^{i\alpha x}d\alpha \tag{7b}$$

$$\sigma_{y_{2_h}}(x,y) = \frac{\mu_2}{2\pi}\int_{-\infty}^{+\infty}e^{i\alpha x}\{e^{|\alpha|y}\{2E|\alpha| + F[(\kappa_2-1)+2|\alpha|y]\}\}d\alpha \tag{7c}$$

$$\sigma_{x_{2_h}}(x,y) = -\frac{\mu_2}{2\pi}\int_{-\infty}^{+\infty}e^{i\alpha x}\{e^{|\alpha|y}\{2E|\alpha| + F[(\kappa_2+3)+2|\alpha|y]\}\}d\alpha \tag{7d}$$

$$\tau_{xy_{2_h}}(x,y) = \frac{i\mu_1}{2\pi}\int_{-\infty}^{+\infty}e^{i\alpha x}\left\{e^{|\alpha|y}\left\{2E\alpha + F\left[\frac{|\alpha|}{\alpha}(\kappa_2+1)+2\alpha y\right]\right\}\right\}d\alpha \tag{7e}$$

Note that the body force acting in the foundation is neglected since it does not disturb the contact pressure distribution. A, B, C, D, E and F are the unknown constants which will be determined from the boundary and continuity conditions at $y = 0$ and $y = h$.

3. Case of continuous contact

An elastic layer with a height h resting on an elastic half plane without any friction and subjected to concentrated loads P and Q by means of two dissimilar rigid flat punches, shown in Fig. 1, is analyzed. The widths of the punches are $(b-a)$ and $(d-c)$. Especially, the initial separation point (x_{cr}) where the layer separated from the elastic half plane and the variation of the stress distribution between both of them is examined. Punches make different settlements depending on the material properties of the elastic media, the width of the punches and more importantly the magnitude of the external loads P and Q. This may cause a separation between punch I and the elastic layer if they are close enough. So critical distance between the two punches indicating the separation between the first punch and the elastic layer takes place is researched and limit distance between the two punches where no interaction occurs is investigated.

If load factor (λ) is sufficiently small then the contact along the layer-subspace, $y=0$ $-\infty < x < \infty$, will be continuous and A, B, C, D, E and F must be determined from the following boundary and continuity conditions.

$$\sigma_{y_1}(x,h) = \begin{cases} -p(x) & a<x<b \\ -q(x) & c<x<d \\ 0 & -\infty<x<a,\ b<x<c,\ d<x<\infty \end{cases} \tag{8a}$$

$$\tau_{xy_1}(x,h) = 0, \qquad -\infty<x<\infty \tag{8b}$$

$$\tau_{xy_1}(x,0) = 0, \qquad -\infty<x<\infty \tag{8c}$$

$$\tau_{xy_2}(x,0) = 0, \qquad -\infty<x<\infty \tag{8d}$$

$$\sigma_{y_1}(x,0) = \sigma_{y_2}(x,0), \qquad -\infty<x<\infty \tag{8e}$$

$$\frac{\partial}{\partial x}[v_2(x,0)-v_1(x,0)] = 0, \qquad -\infty<x<\infty \tag{8f}$$

$$\frac{\partial}{\partial x}[v_1(x,h)] = 0, \qquad a<x<b \tag{8g}$$

$$\frac{\partial}{\partial x}[v_1(x,h)] = 0, \qquad c<x<d \tag{8h}$$

where $p(x)$ and $q(x)$ are the unknown contact pressures under punch I and punch II, respectively and they have not been determined yet.

Equilibrium conditions of the problem may be expressed as

$$\int_a^b p(x)dx = P \tag{9a}$$

$$\int_c^d q(x)dx = Q \tag{9b}$$

Boundary and continuity conditions (8a-f) with (2), (4), (5), (6) and (7) give the constants A, B, C, D, E and F in terms of unknown functions $p(x)$ and $q(x)$ and by substituting values of these constants into Eqs. (8g,h), after some simple manipulations, one may obtain the following singular integral equations for $p(x)$ and $q(x)$ (Erdogan and Gupta 1972, Muskhelisvili 1958).

$$-\frac{1}{\pi\mu_1}\int_a^b\left[k_1(x,t)+\frac{(1+\kappa_1)}{4}\frac{1}{t-x}\right]p(t)dt-\frac{1}{\pi\mu_1}\int_c^d\left[k_1(x,t)+\frac{(1+\kappa_1)}{4}\frac{1}{t-x}\right]q(t)dt=0$$

$$a<x<b \tag{10a}$$

$$-\frac{1}{\pi\mu_1}\int_a^b\left[k_1(x,t)+\frac{(1+\kappa_1)}{4}\frac{1}{t-x}\right]p(t)dt-\frac{1}{\pi\mu_1}\int_c^d\left[k_1(x,t)+\frac{(1+\kappa_1)}{4}\frac{1}{t-x}\right]q(t)dt=0$$

$$c<x<d \tag{10b}$$

where,

$$k_1(x,t)=\int_0^\infty\{\{4\alpha^2(1+\kappa_1)[(1+\kappa_2)[-e^{-4\alpha h}+4\alpha h e^{-2\alpha h}+1]+\mu_2/\mu_1(1+\kappa_1)[e^{-4\alpha h}-2e^{-2\alpha h}+1]]\}/\Delta$$
$$-(1+\kappa_1)/4\}\{\sin\alpha(t-x)\}d\alpha \tag{11}$$

in which,

$$\Delta=16\alpha^2\{(1+\kappa_2)[e^{-4\alpha h}+e^{-2\alpha h}(-2-4\alpha^2h^2)+1]+\mu_2/\mu_1(1+\kappa_1)[-e^{-4\alpha h}+4\alpha h e^{-2\alpha h}+1]\} \tag{12}$$

If evaluated values of A, B, C, D in terms of $p(x)$ and $q(x)$ are substituted into (5a), the expression of the contact stress between elastic layer and half plane may be obtained as,

$$\sigma_{y_1}(x,0)=-\rho_1 gh-\frac{1}{\pi}\int_a^b k_2(x,t)p(t)dt-\frac{1}{\pi}\int_c^d k_2(x,t)q(t)dt,\quad -\infty<x<\infty \tag{13}$$

where,

$$k_2(x,t)=\int_0^\infty\{32\alpha^2\mu_2/\mu_1(1+\kappa_1)[e^{-3\alpha h}(-1+\alpha h)+e^{-\alpha h}(1+\alpha h)]\}/\{\cos\alpha(t-x)\}d\alpha \tag{14}$$

Designating the variables (x,t) on $y=h$ by (x_1,t_1) and (x_2,t_2) and defining the following dimensionless quantities with new variables such as r and s,

$$x_1=\frac{b-a}{2}r_1+\frac{b+a}{2},\quad t_1=\frac{b-a}{2}s_1+\frac{b+a}{2},\quad x_2=\frac{d-c}{2}r_2+\frac{d+c}{2}$$
$$t_2=\frac{d-c}{2}s_2+\frac{d+c}{2},\quad g_1(s_1)=p\left(\frac{b-a}{2}s_1+\frac{b+a}{2}\right)\Big/P/h \tag{15a-h}$$
$$g_2(s_2)=p\left(\frac{d-c}{2}s_2+\frac{d+c}{2}\right)\Big/P/h,\quad \lambda=P/\rho_1 gh^2,\quad \alpha=wh$$

from Eqs. (9), (10) and (13) we obtain,

$$\int_{-1}^{1}g_1(s_1)\frac{b-a}{2h}ds_1=1 \tag{16a}$$

$$\int_{-1}^{1}g_2(s_2)\frac{d-c}{2h}ds_2=Q/P \tag{16b}$$

$$-\frac{1}{\pi}\int_{-1}^{1} g_1(s_1)\frac{b-a}{2h}ds_1\left[m_1(r_1,s_1)+\frac{(1+\kappa_1)}{4}\frac{1}{\frac{b-a}{2}(s_1-r_1)}\right]-\frac{1}{\pi}\int_{-1}^{1} g_2(s_2)\frac{d-c}{2h}ds_2$$

$$\left[m_2(r_1,s_2)+\frac{(1+\kappa_1)}{4}\frac{1}{\left(\frac{d-c}{2}s_2+\frac{d+c}{2}\right)-\left(\frac{b-a}{2}r_1+\frac{b+a}{2}\right)}\right]=0, \quad -1<r_1<1 \tag{16c}$$

$$-\frac{1}{\pi}\int_{-1}^{1} g_1(s_1)\frac{b-a}{2h}ds_1\left[m_3(r_2,s_1)+\frac{(1+\kappa_1)}{4}\frac{1}{\left(\frac{b-a}{2}s_1+\frac{b+a}{2}\right)-\left(\frac{d-c}{2}r_2+\frac{d+c}{2}\right)}\right]$$

$$-\frac{1}{\pi}\int_{-1}^{1} g_2(s_2)\frac{d-c}{2h}ds_2\left[m_4(r_2,s_2)+\frac{(1+\kappa_1)}{4}\frac{1}{\frac{d-c}{2}(s_2-r_2)}\right]=0, \quad -1<r_2<1 \tag{16d}$$

$$\frac{\sigma_{y_1}(x,0)}{P/h} = -\frac{1}{\lambda}-\frac{1}{\pi}\int_{-1}^{1} m_5(r_1,s_1)g_1(s_1)\frac{b-a}{2h}ds_1-\frac{1}{\pi}\int_{-1}^{1} m_6(r_2,s_2)g_2(s_2)\frac{d-c}{2h}ds_{2s},$$
$$-1<r_1,r_2<1 \tag{16e}$$

where,

$$m_1(r_1,s_1) = k_1(x_1,t_1), \quad m_2(r_1,s_2)=k_1(x_1,t_2), \quad m_3(r_2,s_1)=k_1(x_2,t_1)$$
$$m_4(r_2,s_2) = k_1(x_2,t_2), \quad m_5(r_1,s_1)=k_2(x_1,t_1), \quad m_6(r_2,s_2)=k_2(x_2,t_2) \tag{17a-f}$$

The index of the integral equations in (16) is +1, so the function $g_i(s_i)$ may be expressed in the following form:

$$g_i(s_i) = G_i(s_i)(1-s_i^2)^{-1/2}, \qquad (-1<s_i<1), \qquad (i=1,2) \tag{18}$$

where $G(s)$ is a function bounded in $[-1,1]$ (Erdogan and Gupta 1972). Now one can make use of the Gauss-Chebyshev integration formula given in (Erdogan and Gupta 1972) and replace Eqs. (16a-d) by the following algebraic equations:

$$\sum_{i=1}^{n} \pi W_i\frac{b-a}{2h}G_1(s_{1_i}) = 1 \tag{19a}$$

$$\sum_{i=1}^{n} \pi W_i\frac{d-c}{2h}G_2(s_{2_i}) = Q/P \tag{19b}$$

$$-\sum_{i=1}^{n} W_i G_1(s_{1_i})\frac{b-a}{2h}\left[m_1(r_{1_j}, s_{1_i}) + \frac{(1+\kappa_1)}{4}\frac{1}{\frac{b-a}{2}(s_{1_i} - r_{1_j})}\right]$$

$$-\sum_{i=1}^{n} W_i G_2(s_{2_i})\frac{d-c}{2h}\left[m_2(r_{1_j}, s_{2_i}) + \frac{(1+\kappa_1)}{4}\frac{1}{\left(\frac{d-c}{2}s_{2_i} + \frac{d+c}{2}\right) - \left(\frac{b-a}{2}r_{1_j} + \frac{b+a}{2}\right)}\right] = 0$$

$$(j = 1, \ldots, n-1) \tag{19c}$$

$$-\sum_{i=1}^{n} W_i G_1(s_{1_i})\frac{b-a}{2h}\left[m_3(r_{2_j}, s_{1_i}) + \frac{(1+\kappa_1)}{4}\frac{1}{\left(\frac{b-a}{2}s_{1_i} + \frac{b+a}{2}\right) - \left(\frac{d-c}{2}r_{2_j} + \frac{d+c}{2}\right)}\right]$$

$$-\sum_{i=1}^{n} W_i G_2(s_{2_i})\frac{d-c}{2h}\left[m_4(r_{2_j}, s_{2_i}) + \frac{(1+\kappa_1)}{4}\frac{1}{\frac{d-c}{2}(s_{2_i} - r_{2_j})}\right] = 0, \quad (j = 1, \ldots, n-1) \tag{19d}$$

$$W_1 = W_n = \frac{1}{2n-2}, \quad W_i = \frac{1}{n-1} \qquad (i = 2, \ldots, n-1) \tag{20a}$$

$$s_{1_i} = s_{2_i} = \cos\left(\frac{i-1}{n-1}\pi\right) \qquad (i = 1, \ldots, n) \tag{20b}$$

$$r_{1_j} = r_{2_j} = \cos\left(\frac{2j-1}{2n-2}\pi\right) \qquad (j = 1, \ldots, n-1) \tag{20c}$$

Thus equations given by (19) constitutes a system of $2n$ equations for $2n$ unknowns $G_1(s_i)$ and $G_2(s_i)$ $(i = 1, \ldots, n)$. Once $G_1(s_i)$ and $G_2(s_i)$ are calculated, substituting the results into (16e) and using appropriate Gauss-Chebyshev integration formula, the contact stress $\sigma_{y_1}(x, 0)$ can be found everywhere. The critical load factor λ_{cr} and the corresponding location of interface separation x_{cr} can be determined through the use of Eq. (16e) for various dimensionless quantities as

$$-\frac{1}{\lambda_{cr}} - \sum_{i=1}^{n} W_i G_1(s_{1_i}) m_5(r_{1_j}, s_{1_i})\frac{b-a}{2h} - \sum_{i=1}^{n} W_i G_2(s_{2_i}) m_6(r_{2_j}, s_{2_i})\frac{d-c}{2h} = 0 \tag{21}$$

4. Case of discontinuous contact

When the value of λ exceeds λ_{cr} the interface separation takes place in the neighborhood of $x = x_{cr}$ on the contact plane $y = 0$, as shown in Fig. 1. Assuming that the separation region is described by $e < x < f$, $y = 0$, where e and f are unknowns and functions of λ boundary and continuity conditions for the discontinuous contact case are defined as follows

$$\sigma_{y_1}(x,h) = \begin{cases} -p(x) & a<x<b \\ -q(x) & c<x<d \\ 0 & -\infty<x<a,\ b<x<c,\ d<x<\infty \end{cases} \tag{22a}$$

$$\tau_{xy_1}(x,h) = 0, \qquad -\infty<x<\infty \tag{22b}$$

$$\tau_{xy_1}(x,0) = 0, \qquad -\infty<x<\infty \tag{22c}$$

$$\tau_{xy_2}(x,0) = 0, \qquad -\infty<x<\infty \tag{22d}$$

$$\sigma_{y_1}(x,0) = \sigma_{y_2}(x,0), \qquad -\infty<x<\infty \tag{22e}$$

$$\frac{\partial}{\partial x}[v_2(x,0)-v_1(x,0)] = \begin{cases} \varphi(x) & e<x<f \\ 0 & -\infty<x<e,\ f<x<\infty \end{cases}, \tag{22f}$$

$$\sigma_{y_1}(x,0) = \sigma_{y_2}(x,0) = 0, \qquad e<x<f \tag{22g}$$

$$\frac{\partial}{\partial x}[v_1(x,h)] = 0, \qquad a<x<b \tag{22h}$$

$$\frac{\partial}{\partial x}[v_1(x,h)] = 0, \qquad c<x<d \tag{22i}$$

After utilizing the boundary and continuity conditions defined in Eqs. (22a-f), new value for the constants A, B, C, D, E and F which appear in Eqs. (4), (5) and (7) may be obtained in terms of new unknown functions $p(x), q(x)$ and $\varphi(x)$. Eqs. (22g-i) give the following integral equations to determine $p(x), q(x)$ and $\varphi(x)$, after some straightforward manipulations

$$-\frac{1}{\pi\mu_1}\int_a^b\left[k_1(x,t)+\frac{(1+\kappa_1)}{4}\frac{1}{t-x}\right]p(t)dt-\frac{1}{\pi\mu_1}\int_c^d\left[k_1(x,t)+\frac{(1+\kappa_1)}{4}\frac{1}{t-x}\right]q(t)dt$$
$$+\frac{1}{\pi}\int_e^f k_2(x,t)\varphi(t)dt = 0, \qquad a<x<b \tag{23a}$$

$$-\frac{1}{\pi\mu_1}\int_a^b\left[k_1(x,t)+\frac{(1+\kappa_1)}{4}\frac{1}{t-x}\right]p(t)dt-\frac{1}{\pi\mu_1}\int_c^d\left[k_1(x,t)+\frac{(1+\kappa_1)}{4}\frac{1}{t-x}\right]q(t)dt$$
$$+\frac{1}{\pi}\int_e^f k_2(x,t)\varphi(t)dt = 0, \qquad c<x<d \tag{23b}$$

$$-\frac{1}{\pi}\int_a^b k_2(x,t)p(t)dt-\frac{1}{\pi}\int_c^d k_2(x,t)q(t)dt$$
$$-\frac{\mu_1}{\pi}\int_e^f\left[k_3(x,t)-\frac{4\mu_2/\mu_1}{(1+\kappa_2)+\mu_2/\mu_1(1+\kappa_1)}\frac{1}{t-x}\right]\varphi(t)dt-\rho_1 gh = 0, \qquad e<x<f \tag{23c}$$

where kernels $k_1(x,t)$ and $k_2(x,t)$ are given by (11) and (14) and

$$k_3(x,t) = \int_0^{\infty}\left\{\left\{\frac{64\alpha^2\mu_2}{\mu_1}\{e^{-2\alpha h}(2+4\alpha^2h^2)-e^{-4\alpha h}-1\}\right\}\Big/\Delta + \frac{4\mu_2/\mu_1}{(1+\kappa_2)+\mu_2/\mu_1(1+\kappa_1)}\right\}$$
$$\{\sin\alpha(t-x)\}\,d\alpha \tag{24}$$

in which Δ is given by Eq. (12).

The index of integral Eqs. (23a) and (23b) is +1. On the other hand, the index of the singular integral Eq. (23c) is −1 due to the physical requirement of smooth contact at the end points e and f (Muskhelishvili 1958). Thus, in solving the problem the two conditions which would account for the unknowns e and f are the consistency condition of integral Eq. (23c) and the single valuedness condition

$$\int_e^f \varphi(x)dx = 0 \tag{25}$$

Designating the variables (x, t) on $y = 0$ by (x_3, t_3) and defining following dimensionless quantities:

$$x_3 = \frac{f-e}{2}r_3 + \frac{f+e}{2}, \quad t_3 = \frac{f-e}{2}s_3 + \frac{f+e}{2} \tag{26a}$$

$$g_3(s_3) = \mu_1\varphi\left(\frac{f-e}{2}s_3 + \frac{f+e}{2}\right)\Big/P/h \tag{26b}$$

by making use of Eq. (15), the integral Eq. (23) may be expressed as follows

$$-\frac{1}{\pi}\int_{-1}^{1} g_1(s_1)\frac{b-a}{2h}ds_1\left[m_1^*(r_1,s_1) + \frac{(1+\kappa_1)}{4}\frac{1}{\frac{b-a}{2}(s_1-r_1)}\right]$$

$$-\frac{1}{\pi}\int_{-1}^{1} g_2(s_2)\frac{d-c}{2h}ds_2\left[m_2^*(r_1,s_2) + \frac{(1+\kappa_1)}{4}\frac{1}{\left(\frac{d-c}{2}s_2+\frac{d+c}{2}\right)-\left(\frac{b-a}{2}r_1+\frac{b+a}{2}\right)}\right]$$

$$+\frac{1}{\pi}\int_{-1}^{1} g_3(s_3)m_3^*(r_1,s_3)\frac{f-e}{2h}ds_3 = 0 \qquad -1 < r_1 < 1 \tag{27a}$$

$$-\frac{1}{\pi}\int_{-1}^{1} g_1(s_1)\frac{b-a}{2h}ds_1\left[m_4^*(r_2,s_1) + \frac{(1+\kappa_1)}{4}\frac{1}{\left(\frac{b-a}{2}s_1+\frac{b+a}{2}\right)-\left(\frac{d-c}{2}r_2+\frac{d+c}{2}\right)}\right]$$

$$-\frac{1}{\pi}\int_{-1}^{1} g_2(s_2)\frac{d-c}{2h}ds_2\left[m_5^*(r_2,s_2) + \frac{(1+\kappa_1)}{4}\frac{1}{\frac{d-c}{2}(s_2-r_2)}\right]$$

$$+\frac{1}{\pi}\int_{-1}^{1} g_3(s_3)m_6^*(r_2,s_3)\frac{f-e}{2h}ds_3 = 0 \qquad -1 < r_2 < 1 \tag{27b}$$

$$-\frac{1}{\pi}\int_{-1}^{1} g_1(s_1)m_7^*(r_3,s_1)\frac{b-a}{2h}ds_1 - \frac{1}{\pi}\int_{-1}^{1} g_2(s_2)m_8^*(r_3,s_2)\frac{d-c}{2h}ds_2$$

$$-\frac{1}{\pi}\int_{-1}^{1} g_3(s_3)\frac{f-e}{2h}ds_3\left[m_9^*(r_3,s_3) - \frac{4\mu_2/\mu_1}{(1+\kappa_2)+\mu_2/\mu_1(1+\kappa_1)}\frac{1}{\frac{f-e}{2h}(s_3-r_3)}\right] - \frac{1}{\lambda} = 0$$

$$-1 < r_3 < 1 \tag{27c}$$

where

$$\begin{aligned}
m_1^*(r_1,s_1) &= k_1(x_1,t_1), \quad m_2^*(r_1,s_2) = k_1(x_1,t_2), \quad m_3^*(r_1,s_3) = k_2(x_1,t_3)\\
m_4^*(r_2,s_1) &= k_1(x_2,t_1), \quad m_5^*(r_2,s_2) = k_1(x_2,t_2), \quad m_6^*(r_2,s_3) = k_2(x_2,t_3)\\
m_7^*(r_3,s_1) &= k_2(x_3,t_1), \quad m_8^*(r_3,s_2) = k_2(x_3,t_2), \quad m_9^*(r_3,s_3) = k_3(x_3,t_3)
\end{aligned} \tag{28a-i}$$

Similar to Eqs. (16a,b) additional condition (25) may be expressed as

$$\int_{-1}^{1} g_3(s_3)ds_3 = 0 \tag{29}$$

To solve the system of integral equations, it is found to be more convenient to assume that Eq. (27c) as well as Eqs. (27a) and (27b) has an index +1 (Civelek *et al.* 1978), consequently the function $g_i(s_i)$ $(i = 1, \ldots, 3)$ may be expressed in the form

$$g_i(s_i) = G_i(s_i)(1-s_i^2)^{-1/2}, \quad -1 < s_i < 1, \qquad (i = 1, \ldots, 3) \tag{30}$$

where $G_i(s_i)$ is a bounded function. In order to insure smooth contact at the end points of the separation region we then impose the following conditions on $G_3(s_3)$:

$$G_3(-1) = 0 \qquad G_3(1) = 0 \tag{31}$$

Eqs. (27), (19a,b) and (29) can easily be reduced to the following system of linear algebraic equations by employing the appropriate Gauss-Chebyshev integration formula (Erdogan and Gupta 1972)

$$-\sum_{i=1}^{n} W_i G_1(s_{1_i})\frac{b-a}{2h}\left[m_1^*(r_{1_j},s_{1_i}) + \frac{(1+\kappa_1)}{4}\frac{1}{\frac{b-a}{2}(s_{1_i}-r_{1_j})}\right]$$

$$-\sum_{i=1}^{n} W_i G_2(s_{2_i})\frac{d-c}{2h}\left[m_2^*(r_{1_j},s_{2_i}) + \frac{(1+\kappa_1)}{4}\frac{1}{\left(\frac{d-c}{2}s_{2_i}+\frac{d+c}{2}\right)-\left(\frac{b-a}{2}r_{1_j}+\frac{b+a}{2}\right)}\right]$$

$$+\sum_{i=2}^{n-1} W_i G_3(s_{3_i})\frac{f-e}{2h}m_3^*(r_{1_j},s_{3_i}) = 0 \qquad (j = 1, \ldots, n-1) \tag{32a}$$

$$-\sum_{i=1}^{n} W_i G_1(s_{1_i}) \frac{b-a}{2h}\left[m_4^*(r_{2_j}, s_{1_i}) + \frac{(1+\kappa_1)}{4} \frac{1}{\left(\frac{b-a}{2}s_{1_i} + \frac{b+a}{2}\right) - \left(\frac{d-c}{2}r_{2_j} + \frac{d+c}{2}\right)} \right]$$

$$-\sum_{i=1}^{n} W_i G_2(s_{2_i}) \frac{d-c}{2h}\left[m_5^*(r_{2_j}, s_{2_i}) + \frac{(1+\kappa_1)}{4} \frac{1}{\frac{d-c}{2}(s_{2_i} - r_{2_j})} \right]$$

$$+\sum_{i=2}^{n-1} W_i G_3(s_{3_i}) \frac{f-e}{2h} m_6^*(r_{2_j}, s_{3_i}) = 0 \qquad (j = 1, \ldots, n-1) \tag{32b}$$

$$-\sum_{i=1}^{n} W_i G_1(s_{1_i}) \frac{b-a}{2h} m_7^*(r_{3_j}, s_{1_i}) - \sum_{i=1}^{n} W_i G_2(s_{2_i}) \frac{b-c}{2h} m_8^*(r_{3_j}, s_{2_i})$$

$$-\sum_{i=2}^{n-1} W_i G_3(s_{3_i}) \frac{f-e}{2h}\left[m_9^*(r_{3_j}, s_{3_i}) - \frac{4\mu_2/\mu_1}{(1+\kappa_2) + \mu_2/\mu_1(1+\kappa_1)} \frac{1}{\frac{f-e}{2}(s_{3_i} - r_{3_j})} \right] - \frac{1}{\lambda} = 0$$

$$(j = 1, \ldots, n-1) \tag{32c}$$

$$\sum_{i=1}^{n} \pi W_i \frac{b-a}{2h} G_1(s_{1_i}) = 1 \tag{32d}$$

$$\sum_{i=1}^{n} \pi W_i \frac{d-c}{2h} G_2(s_{2_i}) = Q/P \tag{33a}$$

$$\sum_{i=2}^{n-1} \pi W_i G_3(s_{3_i}) = 0 \tag{33b}$$

where W_i, s_i and r_j are given by (20). It was shown in (Erdogan and Gupta 1972) that the consistency condition is automatically satisfied if the Gauss-Chebyshev integration formula is used for solving integral equations. Thus, Eqs (32) and (33) give $3n$ equation for $3n$ unknowns $G_1(s_i)$, $G_2(s_i)$, $G_3(s_j)$ $(i = 1, \ldots, n)$, $(j = 2, \ldots, n-1)$, e and f. The equations are linear in $G(s)$ but highly nonlinear in e and f. So, an interpolation scheme is required for the solution. Selected values of e and f are substituted into (32) and $G_1(s)$, $G_2(s)$ and $G_3(s)$ are obtained which must satisfy Eqs. (33a,b) simultaneously for known $\lambda > \lambda_{cr}$. If Eqs. (33a,b) are not satisfied, then solution must be repeated with new values of e and f until the Eqs (33a,b) are satisfied simultaneously.

It should be noted that Eq. (23c) gives the $\sigma_{y_1}(x, 0)$ outside as well as inside the separation region (e, f). Thus, once the functions $G_1(s)$, $G_2(s)$ and $G_3(s)$ and the constants e and f are determined, contact stress $\sigma_{y_1}(x, 0)$ may easily be evaluated. The displacement component $v^*(x, 0)$ in the separation region (e, f) referring to (22f) and (26b), may be obtained from

$$v^*(x, 0) = v_2(x, 0) - v_1(x, 0) = \int_e^x g_3(t)\,dt, \qquad e < x < f \tag{34a}$$

or

$$\frac{\mu_1}{P/h} v^*(x, 0) = \frac{f-e}{2h} \int_{-1}^{r_3} G_3(s_3) ds_3, \qquad -1 < r_3 < 1 \tag{34b}$$

where

$$x = \frac{f-e}{2} r_3 + \frac{f+e}{2} \tag{34c}$$

Also using appropriate Gauss-Chebyshev integration formula and taking +1 the index of Eq. (34b), the following expression may be written for the separation

$$\frac{\mu_1}{P/h} v^*(x, 0) = \frac{f-e}{2h} \sum_{i=2}^{k-1} W_i G_3(s_{3_i}) \qquad (k = 2, \ldots, n-1) \tag{35}$$

where W_i and s_i are given by Eq. (20).

5. Results and discussion

Some of the results obtained in the study for continuous and discontinuous contact cases are presented in Figs. 2-8 and Tables 1-4 for various dimensionless quantities such as $(b-a)/h$, $(d-c)/h$, Q/P, μ_2/μ_1, λ and $(c-b)/h$. It is assumed that Q/h is always greater than P/h or equal but not small. If $P/h > Q/h$ rather than $Q/h > P/h$ problem is symmetrical.

In the case of discontinuous contact only first separation region between elastic layer and elastic half plane is examined. However, there may be separation between first punch and elastic layer

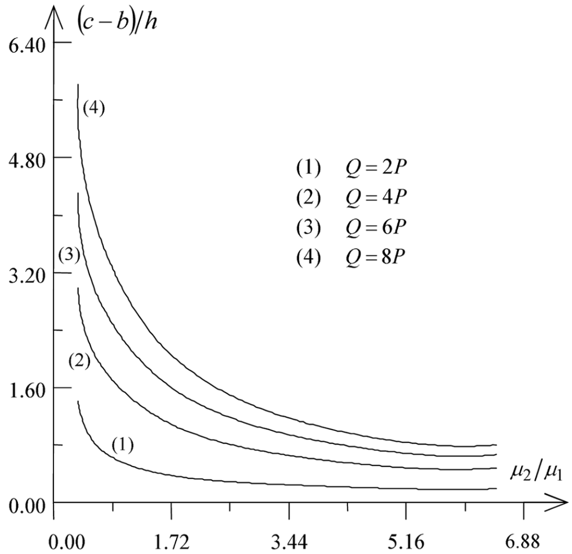

Fig. 2 Variation of minimum value of distance between two punches $(c-b)/h$ to avoid separation under first punch with μ_2/μ_1 for various values of Q/P ($a/h = 3$, $(b-a)/h = 1$, $(d-c)/h = 1$)

Table 1 Variation of minimum value of distance between two punches $(c-b)/h$ to avoid separation under first punch with $(b-a)/h$ and $(d-c)/h$ for various values of Q/P ($\mu_2/\mu_1 = 1.65$, $a/h = 3$)

$(b-a)/h$	$(d-c)/h$	$(c-b)/h$			
		$Q = 2P$	$Q = 4P$	$Q = 6P$	$Q = 8P$
0.5	0.5	0.2029	0.7253	1.1436	1.4679
	1.0	0.0668	0.5252	0.9392	1.2677
	1.5	no separation	0.3649	0.7654	1.0982
1.0	0.5	0.5488	1.3271	1.8474	2.3183
	1.0	0.3890	1.1231	1.6494	2.1195
	1.5	0.2259	0.9478	1.4819	1.9507
1.5	0.5	0.9002	1.7414	2.4417	3.2264
	1.0	0.6881	1.5415	2.2402	3.0117
	1.5	0.5047	1.3713	2.0669	2.8191

where $Q/h > P/h$ and there may be more than one separation region between elastic layer and elastic half plane depending on λ. These cases are not examined ,but the possibility of these cases are determined.

In Fig. 2 variation of critical distance between two punches with μ_2/μ_1 is presented for various values of Q/P where separation initiates between first punch and elastic layer. The widths of the punhes are the same and fixed. As μ_2/μ_1 increases, i.e., elastic half plane is getting stiffer than elastic layer, punches may be placed closer to each other without causing separation between punch I and elastic layer. It is hard to separate punch I from elastic layer compared with small values of μ_2/μ_1. As it can be seen from the figure that the distance between the two punches must be increased to avoid separation between punch I and elastic layer when Q/P increases.

Variation of critical distance between the two punches with the widths of the punches $(b-a)/h$ and $(d-c)/h$ for various values of Q/P is presented in Table 1, where contact pressure $p(x)h/P$ is zero under first punch at the edge b/h for this critical value of $(c-b)/h$. Ratio of elastic constants μ_2/μ_1 is fixed. If Q/P increases, distance between two punches must also be increased to avoid separation under first punch. As second punch width $(d-c)/h$ increases, two punches may be closer to each other without separation under first punch. As it can be seen from Table 1, for $Q = 2P$, $(b-a)/h = 0.5$ and $(b-c)/h = 1.5$, $(c-b)/h$ is zero, i.e., two punches may be placed side by side. On the other hand, any increase in the first punch width $(b-a)/h$ requires longer distance between two punches to avoid separation under first punch.

If the distance between the two punches is greater than a limit value, there is no need to consider the two punches together. In this case each punch can be considered separately. Tables 2-4 show the variation of distance between punches $(c-b)/h$ that ends interaction of punches with Q/P, μ_2/μ_1, $(b-a)/h$ and $(d-c)/h$. Tables 2-4 also shows the relations between these dimensionless quantities and the critical load factor, λ_{cr}, at which cause interface separation between the elastic layer and the half plane.

As it can be seen from Table 2, an increase in Q/P does not effect limit distance that ends the interaction of punches, $(c-b)/h$. Also the distance of initial separation point x_{cr} from the origin is not heavily changed with Q/P. In addition to this, the greater the value of Q/P, the smaller the critical load factor value λ_{cr}.

Table 2 Variation of distance between two punches $(c-b)/h$ that ends interaction of punches with Q/P ($\mu_2/\mu_1 = 2.75$, $a/h = 3$, $(b-a)/h = 0.5$, $(d-c)/h = 0.5$)

Q	$\frac{(c-b)}{h}$	PUNCH I		PUNCH II	
		$\lambda_{cr_{left}} = \lambda_{cr_{right}}$	$(a-x_{cr_{left}})/h = (x_{cr_{right}}-b)/h$	$\lambda_{cr_{left}} = \lambda_{cr_{right}}$	$(c-x_{cr_{left}})/h = (x_{cr_{right}}-d)/h$
P	8.1935	95.9916	2.092	95.9924	2.093
2P	8.1932	94.9805	2.094	48.2535	2.093
4P	8.1897	93.0150	2.098	24.1911	2.090
6P	8.1876	91.1270	2.101	16.1911	2.088
8P	8.1866	89.3141	2.105	12.1121	2.087

Table 3 Variation of distance between two punches $(c-b)/h$ that ends interaction of punches with μ_2/μ_1 ($Q = 6P$, $a/h = 3$, $(b-a)/h = 0.5$, $(d-c)/h = 0.5$)

μ_2/μ_1	$\frac{(c-b)}{h}$	PUNCH I		PUNCH II	
		$\lambda_{cr_{left}} = \lambda_{cr_{right}}$	$(a-x_{cr_{left}})/h = (x_{cr_{right}}-b)/h$	$\lambda_{cr_{left}} = \lambda_{cr_{right}}$	$(c-x_{cr_{left}})/h = (x_{cr_{right}}-d)/h$
0.36	11.8499	147.8050	3.666	30.6821	3.601
0.61	10.5952	137.9865	3.065	29.9632	3.043
1.65	9.09062	112.2659	2.406	20.5277	2.404
2.75	8.18769	91.12702	2.090	16.1420	2.090
6.48	6.47982	61.79429	1.749	10.5854	1.739

Table 4 Variation of distance between two punches $(c-b)/h$ that ends interaction of punches with $(b-a)/h$ and $(d-c)/h$ ($Q = 4P$, $\mu_2/\mu_1 = 0.36$, $a/h = 3$)

$\frac{(b-a)}{h}$	$\frac{(d-c)}{h}$	$\frac{(c-b)}{h}$	PUNCH I		PUNCH II	
			$\lambda_{cr_{left}} = \lambda_{cr_{right}}$	$(a-x_{cr_{left}})/h = (x_{cr_{right}}-b)/h$	$\lambda_{cr_{left}} = \lambda_{cr_{right}}$	$(c-x_{cr_{left}})/h = (x_{cr_{right}}-d)/h$
0.5	1.5	11.4729	158.2221	3.642	54.0767	3.391
0.5	1.0	11.6431	158.4436	3.640	48.9311	3.459
0.5	0.5	11.8479	158.5702	3.640	45.8572	3.602
1.0	1.0	11.2908	165.9662	3.470	48.9244	3.459
1.5	1.5	10.6821	176.2471	3.421	54.0838	3.393

As μ_2/μ_1 increases the elastic half plane gets stiffer compared to the elastic layer and limit distance between two punches, $(c-b)/h$, decreases which indicates the end of interaction of punches. This fact can be observed in Table 3. It is also seen that the initial separation point x_{cr} occurs at a shorter distance from the origin if the elastic half plane becomes stiffer than that of elastic layer. This results in a smaller load factor.

Variation of distance between two punches that ends interaction of punches with $(b-a)/h$ and

Table 5 Variation of load factor values with distance between two punches $(c-b)/h$ ($Q = 2P$, $\mu_2/\mu_1 = 1.65$, $a/h = 3$, $(b-a)/h = 1$, $(b-a)/h = 1$)

$\frac{(c-b)}{h}$	PUNCH I				PUNCH II			
	$\lambda_{cr_{left}}$	$x_{cr_{left}}$	$\lambda_{cr_{right}}$	$x_{cr_{right}}$	$\lambda_{cr_{left}}$	$x_{cr_{left}}$	$\lambda_{cr_{right}}$	$x_{cr_{right}}$
0.5	86.4843	0.7955					**61.6381**	**7.7679**
1	105.5191	0.7538					**64.4863**	**8.2798**
3	131.2629	0.6863					**68.1769**	**10.2961**
5	133.3809	0.6891	**48.3705**	**6.5977**	**48.3705**	**6.5977**	68.4457	12.2952
6	133.4365	0.6912	**59.7812**	**7.6306**	**59.7812**	**7.6306**	68.4507	13.2947
7	133.2261	0.6929	117.4795	6.3808	**66.2849**	**8.6937**	68.4215	14.2942
8.4183	132.6267	0.6977	132.6267	6.2957	**68.3435**	**10.1298**	**68.3435**	**15.7063**

$(d-c)/h$ is shown in Table 4. For fixed value of first punch width, an increase in the second punch width, $(d-c)/h$, ends interaction of punches in a shorter distance. If both of punch widths increase, limit distance between punches, $(c-b)/h$, decreases. It can be seen from Table 4 that critical load factor, λ_{cr} increases with an increase in punch width and initial separation point x_{cr} approaches to the edges of punches.

Table 5 shows the variation of critical load factor values with distance between two punches $(c-b)/h$. For small values of $(c-b)/h$ ($(c-b)/h < 5$) there may be two separation regions depending on λ. As mentioned earlier Q/h is greater than P/h, so first separation occurs on the right side of the second punch. It is clear that if load factor λ is great enough (if $\lambda > 86.4843$ for $(c-b)/h = 0.5$) there may be a second separation region on the left side of the first punch. If $(c-b)/h$

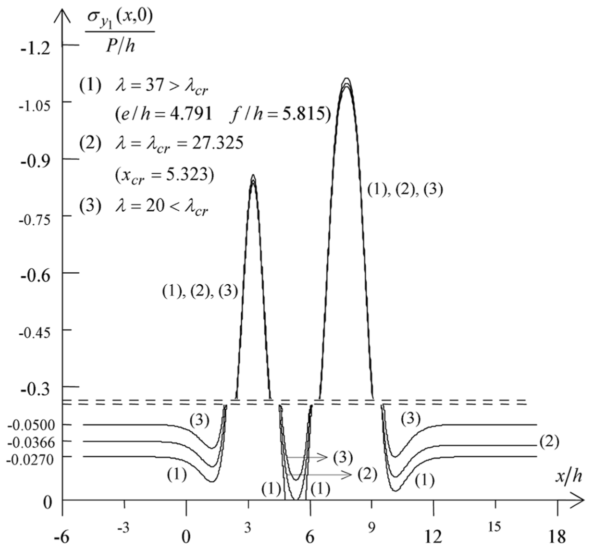

Fig. 3 Contact stress distribution between elastic layer and elastic half plane for the cases of continuous ($\lambda < \lambda_{cr}$) and discontinuous contact ($\lambda > \lambda_{cr}$) ($Q = 2P$, $\mu_2/\mu_1 = 6.48$, $a/h = 3$, $(c-b)/h = 3.5$, $(b-a)/h = 0.5$, $(d-c)/h = 0.5$)

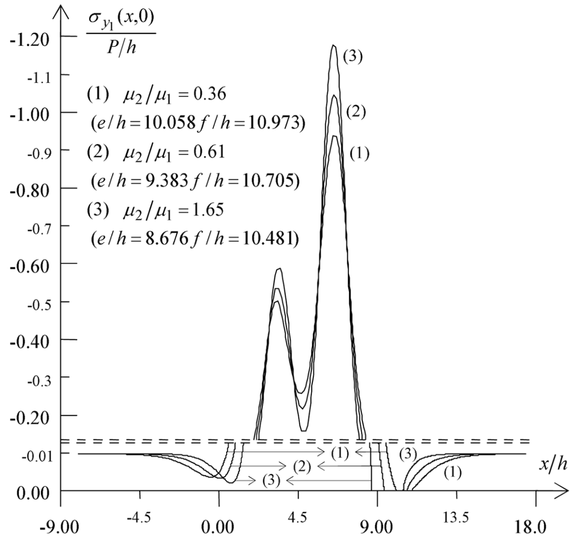

Fig. 4 Contact stress distribution between elastic layer and elastic half plane for the case of discontinuous contact ($Q = 2P$, $a/h = 3$, $(c-b)/h = 2$, $(b-a)/h = 1$, $(d-c)/h = 1$, $\lambda = 100 > \lambda_{cr}$)

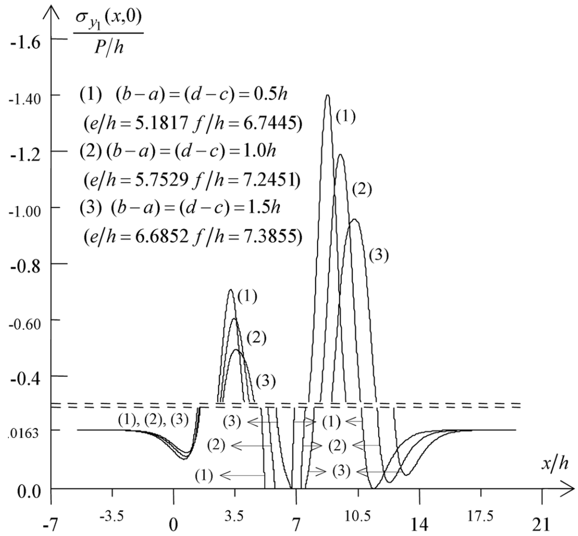

Fig. 5 Contact stress distribution between elastic layer and elastic half plane for the case of discontinuous contact ($Q = 2P$, $\mu_2/\mu_1 = 6.48$, $a/h = 3$, $(c-b)/h = 5$, $\lambda = 61.3503 > \lambda_{cr}$)

increases more, as it is seen from Table 5, besides left and right sides of first and second punches, also there may be separation between two punches which is the probable first separation region. When $(c-b)/h$ increases sufficiently there may be four separation regions depending on the load factor λ and first separation region occurs in the neighborhood of the second punch. For a certain value of $(c-b)/h$ interaction between punches disappears. For fixed values of Q/h, μ_2/μ_1,$(b-a)/h$

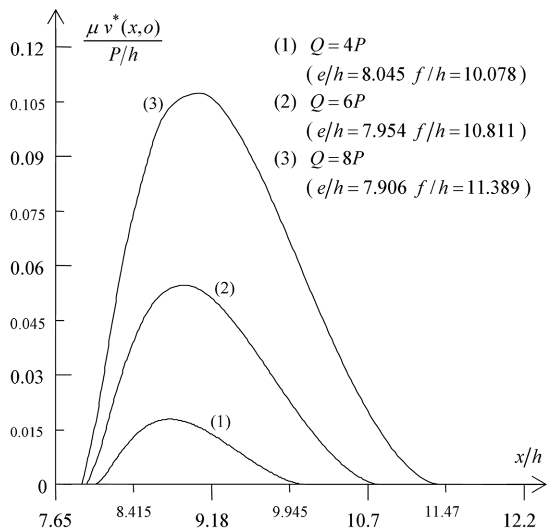

Fig. 6 Separation displacement $v^*(x, 0)$ between elastic layer and elastic half space as a function of x for various values of second punch load Q ($\mu_2/\mu_1 = 6.48$, $a/h = 3$, $(c-b)/h = 1$, $(b-a)/h = 1.5$, $(d-c)/h = 1.5$, $\lambda = 50 > \lambda_{cr}$)

and $(d-c)/h$, this certain value is determined and given in Table 5 as $(c-b)/h = 8.4183$. The value of the critical load factor increases and approaches to a certain value as $(c-b)/h$ increases.

In Figs. 3-5 the normalized contact stress distribution $\sigma_{y_1}(x, 0)h/P$ at the interface of elastic layer and half plane is given for the cases of the continuous and discontinuous contact described in sections 3 and 4. Different scales have been used for continuous and discontinuous contact cases in order to include the entire pressure distribution and to give sufficient details in compact forms.

Contact stress distribution $\sigma_{y_1}(x, 0)h/P$ for three selected values of the load factor λ, namely $\lambda < \lambda_{cr}$, $\lambda = \lambda_{cr}$ and $\lambda > \lambda_{cr}$ shown in Fig. 3. It can be observed in Fig. 3 that there will be two separation regions if λ takes values between $46.6306 < \lambda < 63.0922$ and there will be three separation regions if λ is greater than $\lambda = 63.0922$.

In Figs. 4 and 5, the variation of the normalized contact stress $\sigma_{y_1}(x, 0)h/P$ at the interface between elastic layer and half plane is given for discontinuous contact case. As it can be seen in the graphics, there are three regions in discontinuous contact between elastic layer and elastic half plane. These are continuous contact regions where the effects of external loads P and Q decreases and disappears infinitely, separation zone and continuous contact region. Contact pressure has peaks around the edges of the rigid punches.

Variation of contact stress $\sigma_{y_1}(x, 0)h/P$ with μ_2/μ_1 is shown in Fig. 4. As μ_2/μ_1 increases the elastic half plane gets stiffer compared to the layer and it becomes harder to bend. In this case, separation region increases while contact stress $\sigma_{y_1}(x, 0)h/P$ decreases. Fig. 5 shows that both separation region and $\sigma_{y_1}(x, 0)h/P$ increase with decrease in punches widths.

Some of the results giving the displacement $v^*(x, 0)$ in the separation region $e < x < f$ as a function of x are shown in Fig. 6. These results are calculated from (36). In Fig. 6, it is seen that separation region $e < x < f$ and displacement $v^*(x, 0)$ increase with increase in Q.

6. Conclusions

The numerical results obtained for continuous and discontinuous contact problem show that the rigid punch widths, the elastic properties, the distance between punches and applied loads play a very important role in the formation of critical load factor, the separation initiation distance, separation displacement, separation zone and the stress distribution on the contact surface. From this study following conclusions may be drawn:

- It is hard to separate punch I from elastic layer if μ_2/μ_1 or $(d-c)/h$ increases.
- It is also hard to separate punch I from elastic layer if Q/h or $(b-a)/h$ decreases.
- If applied load or μ_2/μ_1 decreases, it is difficult to separate elastic layer from elastic half plane. This results also in punch widths increase.
- Interaction of punches disappears in a shorter distance with increasing μ_2/μ_1 or punch widths. Increase in Q/h does not effect much the distance between two punches that ends interaction of punches.
- The greater the value of μ_2/μ_1, the shorter the distance of initial separation point from the origin. x_{cr} is not effected much with the increase in Q/h.
- Depending on λ and distance between two punches, first separation region occurs between two punches or in the region $d < x < \infty$.
- Contact pressures make peaks around the edges of rigid punches.
- Separation zone decreases as μ_2/μ_1 decreases or the widths of the punches increase.

The technique used in this paper of formulating the problem in terms of system of singular integral equations can be used without major modification to solve similar antisymmetric contact problems for multilayered materials and for more complex contact geometries.

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