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Technical Note

Effects of element distortions on the performance of enriched quadrilateral elements

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1. Introduction

It is well known that the accuracy of finite element solutions deteriorate in the presence of severe mesh distortions. But distortion is often unavoidable in mesh procedure involving complex geometry. Lee and Bathe (1993) studied the influence of mesh distortion on the serendipity and Lagrange quadrilateral elements. Lautersztajn and Samuelsson (2000) discussed the effects of geometric distortions on four-node isoparametric quadrilateral elements and concluded that the element performance can be rendered 'insensitive' to a particular type of mesh distortion by increasing the order of the interpolation functions for the displacement field. In order to overcome the influence of element distortions, unsymmetric 8-node (Rajendran and Liew 2003) and unsymmetric 20-node element (Ooi *et al.* 2004) are developed to reproducing any linear and quadratic displacement field under any admissible mesh distortions. However, they will produce an asymmetrical stiffness matrix, so these formulations require an asymmetrical solver to solve the resulting stiffness equations.

The goal of this paper is to discuss the effects of element distortions on the accuracy and efficiency of enriched quadrilateral elements with bubble functions. A bubble function is defined as a function that vanishes along the element boundaries. Bubble functions have been introduced to construct plate element models (Auricchio and Taylor 1995, Cook *et al.* 2002, Hong *et al.* 2001). They are employed to solve advection-diffusion problems by Brezzi, Franca and Farhat (Brezzi *et al.* 1992, Brezzi and Russo 1994, Franca and Farhat 1995). Furthermore, the limitation of bubble functions is discussed by Franca and Farhat (1994) and error analysis of residual-free bubbles is discussed by Brezzi *et al.* (1999) and Sangalli (2000).

2. Formulation of the enriched elements

In this paper, the enriched quadrilateral elements are constructed by adding the interior nodes of Lagrange basis to a serendipity basis. The serendipity basis is then corrected so that the Kronecker delta property is satisfied at the interior nodes.

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For the general case of an n-th order element of the family of enriched elements (herein designated nSmL element), the interior nodes of the m-th order Lagrange element are added to the n-th order serendipity elements. The shape functions of the nSmL element can be represented as

$$N_i(\xi, \eta) = \begin{cases} N^{(L)}_{(4m-4n+i)}(\xi, \eta) & \text{for} \quad i = 4n+1 \sim 4n+(m-1)^2 \\ N^{(S)}_i(\xi, \eta) - \sum_{j=4n+1}^{4n+(m-1)^2} a_{ij} N_j(\xi, \eta) & \text{for} \quad i = 1 \sim 4n \end{cases} \tag{1}$$

Here $a_{ij} = N^{(S)}_j(\xi_i, \eta_i)$, it enables N_i to satisfy the $N_i(\xi_j, \eta_j) = \delta_{ij}$ condition.

In general, all the interior freedoms can be made invisible to the user by applying a small dose of static condensation. The element coefficient matrix can be reduced at the element level and divided into several partitions

$$\begin{bmatrix} K_{ee} & K_{ei} \\ K_{ie} & K_{ii} \end{bmatrix} \begin{Bmatrix} u_e \\ u_i \end{Bmatrix} = \begin{Bmatrix} F_e \\ F_i \end{Bmatrix} \tag{2}$$

where u_e is the undetermined variables vector of the edge nodes and u_i is the undetermined variables vector of the interior nodes. K_{ee}, K_{ei}, K_{ie} and K_{ii} are submatrices. Because the interior nodes have no relation with other elements, we can use K_{ii} as pivoting to condense it. Let $\overline{K_{ee}} = K_{ee} - K_{ei}K_{ii}^{-1}K_{ie}$ and $\overline{F_e} = F_e - K_{ei}K_{ii}^{-1}F_i$, the coefficient matrix can be shown as

$$\begin{bmatrix} \overline{K_{ee}} & K_{ei} \\ K_{ie} & K_{ii} \end{bmatrix} \begin{Bmatrix} u_e \\ u_i \end{Bmatrix} = \begin{Bmatrix} \overline{F_e} \\ F_i \end{Bmatrix} \tag{3}$$

Then the $\overline{K_{ee}}$ and $\overline{F_e}$ are assembled to the global coefficient matrix and force. After u_e is solved, we get u_i as

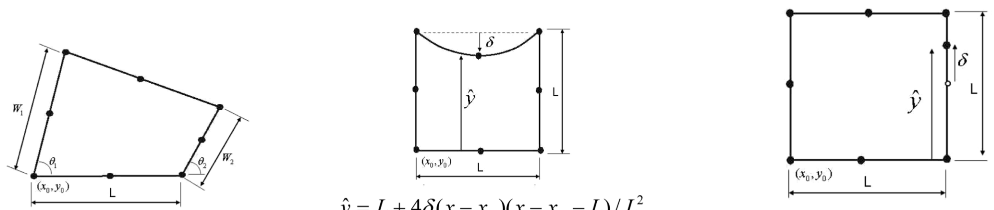
$$u_i = K_{ii}^{-1}F_i - K_{ii}^{-1}K_{ie}u_e \tag{4}$$

Therefore we can see that Gaussian elimination is used to produce the $\overline{K_{ee}}$ and $\overline{F_e}$, then store the parts $K_{ii}^{-1}F_i$ and $K_{ii}^{-1}K_{ie}$ which are needed to solve u_i.

In the coefficient matrix $\overline{K_{ee}} = K_{ee} - K_{ei}K_{ii}^{-1}K_{ie}$ it can be seen that K_{ee} applies partial factorization and this procedure can be regarded as a precondition (Farhat and Sobh 1989). Notably, this precondition is applied directly to the entities of the coefficient matrix, so the procedure for solving the system of equations can be applied to another precondition if the precondition conjugate gradient method is used.

3. Test problem: A cantilever beam subject to a linear bending moment

A plane stress problem is considered in this section to verify the accuracy of the enriched elements. Isotropic material is used in the numerical example. The degrees of freedom of the internal nodes are statically condensed out from the element stiffness matrix before assembling into the global stiffness matrix. The precondition conjugate gradient method (PCG) is used to solve the system of equations. The following elements are assessed under undistorted element and various distorted elements as shown in Fig. 1.



Angular distortion

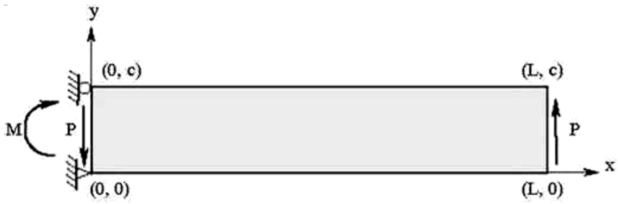
$$\hat{y} = L + 4\delta(x - x_0)(x - x_0 - L)/L^2$$

Curved-edge distortion

$$\hat{y} = y + 4\delta(y - y_0)(y - y_0 - L)/L^2$$

Unevenly-spaced-nodes distortion

Fig. 1 Element distortions classification

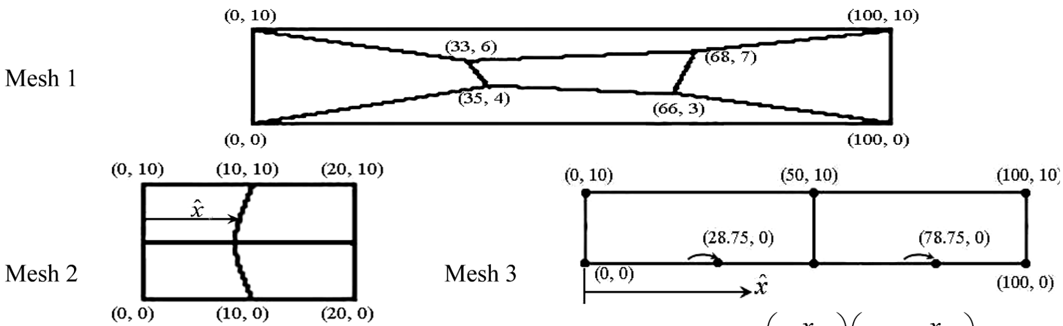


$P = \dfrac{20c^2}{L}$ is distributed as $f_x = \dfrac{120y}{L} - \dfrac{120y^2}{cL}$

$M = 20c^2$ is distributed as $f_x = \dfrac{240y}{c} - 120$

Young's modulus $E = 1.0 \times 10^7$, Poisson's ratio $\nu = 0.3$, Thickness $t = 1.0$

Fig. 2 Test problem : A cantilever beam subject to a linear bending moment



$$\hat{x} = 10 - 8\left(\frac{y}{10}\right)\left(1 - \frac{y}{10}\right) \qquad \hat{x} = x + 20\left(\frac{x}{100}\right)\left(1 - \frac{x}{100}\right)$$

Fig. 3 Meshes used for solving the demonstrative problems

1. 8- and 12-node serendipity element (2S and 3S elements)
2. 9- and 16-node Lagrange element (2L and 3L elements)
3. Enriched quadrilateral elements (2S3L, 2S4L, 3S2L and 3S4L elements)

The linear bending moment problem assesses the distortion sensitivity of the cubic element and is described in Fig. 2. Three difference meshes as shown in Fig. 3 are considered to assess the effects of mesh distortions. Mesh 1 consists of elements having severe angular distortions, while Mesh 2 consists of elements with curved-edge distortion. Mesh 3 consists of elements with unevenly-spaced-nodes distortion. The results show solution error as error energy norm $\|e\|_{eng}$ with the following definitions (Zienkiewicz and Taylor 1989):

Table 1 Results of enriched quadrilateral elements for test problem

	Gauss quadrature	Mesh1		Mesh2		Mesh3	
		$\|e\|_{eng}$	v	$\|e\|_{eng}$	v	$\|e\|_{eng}$	v
2S	3 × 3	3.9320E-01	5.576%	5.1852E-02	91.140%	2.5393E-01	11.031%
2L	3 × 3	1.9100E-01	78.180%	4.8523E-02	91.242%	2.5227E-01	11.075%
2S3L	4 × 4	1.9068E-01	78.335%	3.8015E-02	98.258%	2.5095E-01	11.226%
2S4L	5 × 5	1.9044E-01	78.470%	3.5421E-02	98.906%	2.4637E-01	11.547%
3S	4 × 4	3.8978E-01	7.167%	2.7383E-02	96.542%	1.0977E-01	65.107%
3L	4 × 4	4.9959E-05	100.000%	6.9796E-03	100.000%	8.8741E-02	80.065%
3S2L	4 × 4	1.8642E-01	78.874%	1.2264E-02	98.247%	9.0062E-02	79.968%
3S4L	5 × 5	4.9938E-05	100.000%	4.0930E-03	100.000%	8.7172E-02	80.223%

Error energy norm :
$$\|e\|_{eng} = \left(\iint_{\Omega}(\varepsilon-\varepsilon_h)^T(\sigma-\sigma_h)\,dx\,dy\right)^{\frac{1}{2}} \tag{5}$$

where σ and ε is the exact solution vector, stress vector and strain vector, respectively. σ_h and ε_h is the approximate solution vector, stress vector and strain vector, respectively. The normalized tip deflections v which are normalized with respect to the theoretical solution are also listed in the following table. The theoretical solution of mesh 1 and mesh 3 is 0.008046 and the theoretical solution of mesh 2 is 0.000366.

Table 1 shows the numerical result of the enriched quadrilateral elements. In the angular distortion and curved-edge distortion results of the second order cases, the 2L, 2S3L and 2S4L elements have higher resistance for distortion than the 2S element and the 2S3L and 2S4L elements are a little more accurate than the 2S element. In the unevenly-spaced-nodes distortion cases of the second order cases, all the second order elements are sensitive to unevenly-spaced-nodes distortion. In the third order cases, the results also show that the 3L and 3S4L elements are not affected by angular distortions whereas the 3S and 3S2L elements perform poorly when subjected to angular distortions. The 3S and 3S2L elements are badly affected by angular distortions but the result of the 3S2L element is more accurate than the result of the 3S element. Furthermore, the 3L and 3S4L elements are able to reproduce the third order displacement field exactly even subject to the angular distortions. These third order elements are affected by the unevenly-spaced-nodes distortion, but the 3S2L element, with one extra node in the center, is more accurate than the 2S element. Therefore, the additional shape functions can improve the accuracy in the element's interior when the elements are subjected to curved-edge distortion and unevenly-space-nodes distortion.

4. Conclusions

The effect of element distortions on the enriched quadrilateral elements, which are constructed by adding the interior nodes of Lagrange basis to a serendipity basis, has been discussed with both the completeness requirements test and numerical example in this paper. The enriched quadrilateral elements 2S3L and 2S4L are less sensitive to angular distortion than the second order serendipity and Lagrange elements. The 3S2L element and 3S4L element have higher resistance for angular and curved-edge distortions than the third order serendipity element. Although non-satisfaction of the completeness requirements leads to poorer performance under geometric distortions of the element, the additional interior nodes of enriched quadrilateral elements can improve the accuracy even under element distortions.

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