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# A simple finite element formulation for large deflection analysis of nonprismatic slender beams

Samir Z. AL-Sadder†

*Department of Civil Engineering, Faculty of Engineering, Hashemite University, Zarqa 13115, Jordan*

Ra'ad A. Othman

*Department of Civil Engineering, Faculty of Engineering, Baghdad University, Baghdad, Iraq*

Anis S. Shatnawi

*Department of Civil Engineering, Faculty of Engineering and Technology,*
*The University of Jordan, Amman 11942, Jordan*

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**Abstract.** In this study, an improved finite element formulation with a scheme of solution for the large deflection analysis of inextensible prismatic and nonprismatic slender beams is developed. For this purpose, a three-noded Lagrangian beam-element with two dependent degrees of freedom per node (i.e., the vertical displacement, $y$, and the *actual* slope, $dy/ds = \sin\theta$, where $s$ is the curved coordinate along the deflected beam) is used to derive the element stiffness matrix. The element stiffness matrix in the global $xy$-coordinate system is achieved by means of coordinate transformation of a highly nonlinear $(6 \times 6)$ element matrix in the local $sy$-coordinate. Because of bending with large curvature, highly nonlinear expressions are developed within the global stiffness matrix. To achieve the solution after specifying the proper loading and boundary conditions, an iterative quasi-linearization technique with successive corrections are employed considering these nonlinear expressions to remain constant during all iterations of the solution. In order to verify the validity and the accuracy of this study, the vertical and the horizontal displacements of prismatic and nonprismatic beams subjected to various cases of loading and boundary conditions are evaluated and compared with analytic solutions and numerical results by available references and the results by ADINA, and excellent agreements were achieved. The main advantage of the present technique is that the solution is directly obtained, i.e., non-incremental approach, using few iterations (3 to 6 iterations) and without the need to split the stiffness matrix into elastic and geometric matrices.

**Keywords:** ADINA; finite element; large deflection; nonprismatic beams; quasi-linearization.

† Assistant Professor, Corresponding author, E-mail: samzahi@hu.edu.jo

## 1. Introduction

Due to the fact that all physical structures are nonlinear, linear analysis is obviously inadequate for many structural simulations. A nonlinear structural problem is one in which the structure's stiffness changes as it deforms. Geometric nonlinearity occurs whenever the magnitude of the displacements affects the response of the structure. It includes the effects of large displacements and rotations (i.e., slopes), snap through, and load stiffening. As key source of nonlinearity in problems of elastica, the term "large displacement" refers to a branch of nonlinear bending analysis in which the deflections and rotations of the sections are large but the fiber extensions and angle changes between fibers are small (i.e., small strain theory). In which, a linear or nonlinear stress-strain relationship may take place.

When both large deflections and large rotations are concerned, the analysis becomes highly nonlinear due to the non-linearity appears in the exact expression of the curvature. Thus, if the displacements are large, the shape of the structure and its stiffness change. Afterward, the stiffness matrix of the structure has to be assembled and inverted many times during the course of the analysis, making it more expensive to solve than a linear analysis. For more details on large deflection theory one may refer to (Cook *et al.* 1989, Crisfield 1991, 1997, Hinton 1992, Oden 1972).

Slender beams have been widely used in many civil, mechanical and aerospace engineering applications, both in their stand-alone forms and as supporting structures. Development of reliable beam models with incorporating the effects of the large deflection of slender beams has received considerable interest. Early attempts to present a finite element formulation for a flexural behavior of slender beams were based on the closed form solutions using elliptic integrals that are extensively available but for certain loading and boundary conditions (Timoshenko and Gere 1961, Mattiasson 1981, Chucheepsakul *et al.* 1994, Bona and Zelenika 1997, Wang 1997, Chucheepsakul 1999, Coffin and Bloom 1999, Ohtsuki and Ellyin 2001). Thereafter, with the deepening of the research work, great attention was paid to the numerical integration solutions and finite differences modeling approaches, which have proved to be efficient for solving large deflection problems (Dado *et al.* 2004, Kooi and Kuipers 1984, Katsikadelis and Tsiatas 2003, Lee and Oh 2000, Lee 2001, Mau 1990, Schmidt 1977, Saje and Srpcic 1985, Srpcic and Saje 1986, Wang and Watson 1980, Wang 1981, Watson and Wang 1981, Wang and Watson 1982, Watson and Wang 1983). Also intensive and remarking research works (Williams 1964, Yang 1973, Oran and Kassimali 1976, Bathe and Bolourchi 1979, Argyris and Symeonidis 1981, Surana 1983, Golley 1984, Kooi 1985, Kassimali and Abbasnia 1991, Yang and Kuo 1994, Neuenhofer and Filippou 1997, Golley 1997, Chucheepsakul and Huang 1997, Neuenhofer and Filippou 1998, Taylor and Filippou 2003) have been made over the years to develop finite element models that can accurately represent the large deformable response of the slender beams.

The finite element formulation of nonlinear problems proceeds in much the same way as for linear problems. The main difference arises in the solution of the finite element algebraic equations. Once the finite element method is used for nonlinear analysis, there are three basic techniques that can be followed to solve the nonlinear stiffness matrix equations. A primary technique is the incremental or stepwise procedure in which the load is applied in several small steps, and the structure is assumed to respond linearly or quasi-linearly within each step with its stiffness recomputed on the basis of the structural geometry and member end actions at the end of the previous load step. This is a simple procedure that may require no or few iterations within each step, but errors are likely to increase after several steps due to possible deviations unless very fine steps are used. Bear in mind that, the incremental procedure divides the stiffness matrix into an initial (or linear) stiffness matrix and a tangential stiffness matrix (or geometrical stiffness matrix) that depends on the state of deformation existing in the element during the load increment.

Other technique is the iterative methods which seek an approximate solution to the algebraic equations by linearization. The direct iteration method, also known as the Picard method, is one of these iterative methods. Additional iterative method is the Newton-Raphson method (or modified Newton-Raphson method), which is based on the Taylor series expansion of the algebraic equations about known solution. In this iteration procedure, the total load is applied. The iteration is continued until the convergence is satisfied or the residual is less than a certain preselected value. Unfortunately, convergence may not be achieved using iterative methods, particularly with stiff nonlinearity.

A further technique is the mixed procedure in which a combination of the incremental and iterative schemes is utilized. The load is applied incrementally, and iterations are performed either within each load increment or after the application of all load increments. This procedure yields better accuracy but requires more computational effort.

The key objective of this paper is not only to develop the improved finite element formulation considering large deflection and large rotation effects, but also to propose an efficient, robust, and practical procedure in studying various large deflection problems of prismatic and nonprismatic slender beams subjected to continuous and discontinuous loadings and with different boundary conditions. The effect of the horizontal force on the deflection profile of the beam is also considered. Even though a significant amount of researches have been conducted on development of slender beams analyses, the significance of this study is the simplicity, as a non-incremental approach, and the small number of elements along with small number of iterations required to achieve an economical and accurate solution.

The main points presented in this study for applications on inextensible prismatic and nonprismatic slender beams are summarized as follows:

1. A general theory of flexural slender beams with large deflections and large rotations using a three-noded Lagrangian beam-element with two dependent degrees of freedom per node (i.e., the vertical displacement, $y$, and the *actual* slope, $dy/ds = \sin\theta$, where $s$ is the curved coordinate along the deflected beam) is developed.
2. Considering the actual curved length of the element and the actual moment-curvature relations, the strain-displacement and the stress strain relations are derived.
3. The element stiffness matrix, in $sy$-coordinate system, containing a highly nonlinear terms is obtained using the principle of Total Potential Energy. A transformation matrix, [**T**], is used in order to transform the element stiffness matrix into the global $xy$-coordinates.
4. The global (or assembled) stiffness matrix is obtained by assembling all element stiffness matrices. The nodal boundary conditions and the equivalent nodal vector, including the effect of the horizontal force on the deflection profile of the beam, are also specified for different problems.
5. To achieve the solution, an iterative quasi-linearization technique is adopted with utilizing successive corrections. The solution of the quasi-linearized equations is obtained by using the Gauss elimination technique (Gerald and Wheatley 1970) instead of the conventional inverse technique in order to reduce the computational time.
6. Finally, to verify the validity and the accuracy of this study, the vertical and the horizontal displacements of prismatic and nonprismatic beams subjected to various cases of loading and boundary conditions are evaluated and compared with analytic solutions and numerical results by available references and the results by ADINA (2000).

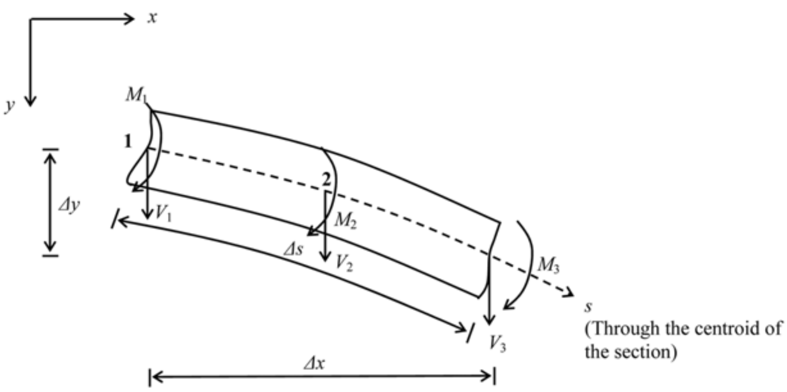



Fig. 1 Three-noded isoparametric beam element in the deformed configuration

## 2. Formulation

A small incremental length of a three-noded beam element of constant flexural rigidity $EI$ is considered in the deformed shape with actual curved length, $\Delta s$, horizontal projection, $\Delta x$, and lateral projection, $\Delta y$, as shown in Fig. 1. Vertical forces, $V_1$, $V_2$ and $V_3$ and bending moments, $M_1$, $M_2$ and $M_3$ act upon the element.

Neglecting the horizontal displacement, $u$, along the $x$-axis (as ineffective), there remain two degrees of freedom for each node, which are the vertical displacement, $y$, and the slope $dy/ds = \sin\theta$. The force vector associated with the two degrees of freedom, $y$, and $\sin\theta$ are the vertical force, $V$, and the equivalent bending moment, $M'$, which is equal to the bending moment $M$ multiplied by the ratio $\sin\theta/\theta$. The vertical displacement, $y$, can be expressed in terms of the deflected curved coordinate, $s$, and it has the following deflection polynomial:

$$y = a_o + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5 \tag{1}$$

where $s$ is the distance along the curved length, $\Delta s$ and $a_0$, $a_1$, $a_2$, $a_3$, $a_4$ and $a_5$ are constants that can be obtained by using the boundary conditions at the three nodes of the element.

The slope (i.e., $dy/ds = \sin\theta$) is obtained from Eq. (1) after differentiation as:

$$\frac{dy}{ds} = \sin\theta = a_1 + 2a_2 s + 3a_3 s^2 + 4a_4 s^3 + 5a_5 s^4 \tag{2}$$

In a matrix form, Eqs. (1) and (2) can be written as:

$$\begin{Bmatrix} y \\ \sin\theta \end{Bmatrix} = \begin{bmatrix} 1 & s & s^2 & s^3 & s^4 & s^5 \\ 0 & 1 & 2s & 3s^2 & 4s^3 & 5s^4 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} \tag{3}$$

The actual curved length of the element is $\Delta s$ (finite). Fig. 1 shows the three nodes of the beam element which are considered with the following boundary conditions:

At node (1): $s = 0$, $y = y_1$, $\left.\dfrac{dy}{ds}\right|_1 = \sin\theta|_1$

At node (2): $s = \Delta s/2$, $y = y_2$, $\left.\dfrac{dy}{ds}\right|_2 = \sin\theta|_2$

At node (3): $s = \Delta s$, $y = y_3$, $\left.\dfrac{dy}{ds}\right|_3 = \sin\theta|_3$

Then, in matrix form:

$$\begin{Bmatrix} y_1 \\ (\sin\theta)_1 \\ y_2 \\ (\sin\theta)_2 \\ y_3 \\ (\sin\theta)_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & \dfrac{\Delta s}{2} & \left(\dfrac{\Delta s}{2}\right)^2 & \left(\dfrac{\Delta s}{2}\right)^3 & \left(\dfrac{\Delta s}{2}\right)^4 & \left(\dfrac{\Delta s}{2}\right)^5 \\ 0 & 1 & 2\left(\dfrac{\Delta s}{2}\right) & 3\left(\dfrac{\Delta s}{2}\right)^2 & 4\left(\dfrac{\Delta s}{2}\right)^3 & 5\left(\dfrac{\Delta s}{2}\right)^4 \\ 1 & \Delta s & (\Delta s)^2 & (\Delta s)^3 & (\Delta s)^4 & (\Delta s)^5 \\ 0 & 1 & 2\Delta s & 3(\Delta s)^2 & 4(\Delta s)^3 & 5(\Delta s)^4 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} \tag{4}$$

By inversion

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\dfrac{23}{(\Delta s)^2} & -\dfrac{6}{(\Delta s)} & \dfrac{16}{(\Delta s)^2} & -\dfrac{8}{(\Delta s)} & \dfrac{7}{(\Delta s)^2} & -\dfrac{1}{(\Delta s)} \\ \dfrac{66}{(\Delta s)^3} & \dfrac{13}{(\Delta s)^2} & -\dfrac{32}{(\Delta s)^3} & \dfrac{32}{(\Delta s)^2} & -\dfrac{34}{(\Delta s)^3} & \dfrac{5}{(\Delta s)^2} \\ -\dfrac{68}{(\Delta s)^4} & -\dfrac{12}{(\Delta s)^3} & \dfrac{16}{(\Delta s)^4} & -\dfrac{40}{(\Delta s)^3} & \dfrac{52}{(\Delta s)^4} & -\dfrac{8}{(\Delta s)^3} \\ \dfrac{24}{(\Delta s)^5} & \dfrac{4}{(\Delta s)^4} & 0 & \dfrac{16}{(\Delta s)^4} & -\dfrac{24}{(\Delta s)^5} & \dfrac{4}{(\Delta s)^4} \end{bmatrix} \begin{Bmatrix} y_1 \\ (\sin\theta)_1 \\ y_2 \\ (\sin\theta)_2 \\ y_3 \\ (\sin\theta)_3 \end{Bmatrix} \tag{5}$$

Substituting Eq. (5) into Eq. (3) gives

$$\begin{Bmatrix} y \\ \sin\theta \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ s & 1 \\ s^2 & 2s \\ s^3 & 3s^2 \\ s^4 & 4s^3 \\ s^5 & 5s^4 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{23}{(\Delta s)^2} & -\frac{6}{(\Delta s)} & \frac{16}{(\Delta s)^2} & -\frac{8}{(\Delta s)} & \frac{7}{(\Delta s)^2} & -\frac{1}{(\Delta s)} \\ \frac{66}{(\Delta s)^3} & \frac{13}{(\Delta s)^2} & -\frac{32}{(\Delta s)^3} & \frac{32}{(\Delta s)^2} & -\frac{34}{(\Delta s)^3} & \frac{5}{(\Delta s)^2} \\ -\frac{68}{(\Delta s)^4} & -\frac{12}{(\Delta s)^3} & \frac{16}{(\Delta s)^4} & -\frac{40}{(\Delta s)^3} & \frac{52}{(\Delta s)^4} & -\frac{8}{(\Delta s)^3} \\ \frac{24}{(\Delta s)^5} & \frac{4}{(\Delta s)^4} & 0 & \frac{16}{(\Delta s)^4} & -\frac{24}{(\Delta s)^5} & \frac{4}{(\Delta s)^4} \end{bmatrix} \begin{Bmatrix} y_1 \\ (\sin\theta)_1 \\ y_2 \\ (\sin\theta)_2 \\ y_3 \\ (\sin\theta)_3 \end{Bmatrix} \tag{6}$$

Eq. (6) gives the lateral deflection and slope at any distance $s$ along the element in terms of the nodal deflections and slopes through the derived shape function $N$.

The bending strain-curvature relationship for a beam element is taken as:

$$\{\varepsilon\} = \eta\ \{\phi\} \tag{7}$$

where $\varepsilon$ is the bending strain vector, $\phi$ is the curvature and $\eta$ is distance from the neutral axis. According to Euler-Bernoulli beam theory, the exact curvature $\phi$ can be expressed as (Kooi and Kuipers 1984):

$$\phi = \frac{d^2y/ds^2}{[1-(dy/ds)^2]^{1/2}} \tag{8}$$

or

$$\phi = A\frac{d^2y}{ds^2} \tag{9}$$

where

$$A = \frac{1}{[1-(dy/ds)^2]^{1/2}} \tag{10}$$

During iteration, $A$ is considered constant for each element. From Eqs. (1) and (9)

$$\{\phi\} = A(2a_2 + 6a_3 s + 12a_4 s^2 + 20a_5 s^3) = A[0\ \ 0\ \ 2\ \ 6s\ \ 12s^2\ \ 20s^3]\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} \tag{11}$$

Substituting Eq. (5) into Eq. (11) yields

$$\{\phi\} = A[0\ \ 0\ \ 2\ \ 6s\ \ 12s^2\ \ 20s^3]\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{23}{(\Delta s)^2} & -\frac{6}{(\Delta s)} & \frac{16}{(\Delta s)^2} & -\frac{8}{(\Delta s)} & \frac{7}{(\Delta s)^2} & -\frac{1}{(\Delta s)} \\ \frac{66}{(\Delta s)^3} & \frac{13}{(\Delta s)^2} & -\frac{32}{(\Delta s)^3} & \frac{32}{(\Delta s)^2} & -\frac{34}{(\Delta s)^3} & \frac{5}{(\Delta s)^2} \\ -\frac{68}{(\Delta s)^4} & -\frac{12}{(\Delta s)^3} & \frac{16}{(\Delta s)^4} & -\frac{40}{(\Delta s)^3} & \frac{52}{(\Delta s)^4} & -\frac{8}{(\Delta s)^3} \\ \frac{24}{(\Delta s)^5} & \frac{4}{(\Delta s)^4} & 0 & \frac{16}{(\Delta s)^4} & -\frac{24}{(\Delta s)^5} & \frac{4}{(\Delta s)^4} \end{bmatrix}\begin{Bmatrix} y_1 \\ (\sin\theta)_1 \\ y_2 \\ (\sin\theta)_2 \\ y_3 \\ (\sin\theta)_3 \end{Bmatrix} \tag{12}$$

Simplifying gives

$$\{\phi\} = A[B]\begin{Bmatrix} y_1 \\ (\sin\theta)_1 \\ y_2 \\ (\sin\theta)_2 \\ y_3 \\ (\sin\theta)_3 \end{Bmatrix} \tag{13}$$

where $A[\mathbf{B}]$ is the strain-displacement transformation matrix and given by:

$$[B] = \begin{bmatrix} \left(-\frac{46}{(\Delta s)^2} + \frac{396s}{(\Delta s)^3} - \frac{816s^2}{(\Delta s)^4} + \frac{480s^3}{(\Delta s)^5}\right) \\ \left(-\frac{12}{(\Delta s)} + \frac{78s}{(\Delta s)^2} - \frac{144s^2}{(\Delta s)^3} + \frac{80s^3}{(\Delta s)^4}\right) \\ \left(\frac{32}{(\Delta s)^2} - \frac{192s}{(\Delta s)^3} + \frac{192s^2}{(\Delta s)^4}\right) \\ \left(-\frac{16}{(\Delta s)} + \frac{192s}{(\Delta s)^2} - \frac{480s^2}{(\Delta s)^3} + \frac{320s^3}{(\Delta s)^4}\right) \\ \left(\frac{14}{(\Delta s)^2} - \frac{204s}{(\Delta s)^3} + \frac{624s^2}{(\Delta s)^4} - \frac{480s^3}{(\Delta s)^5}\right) \\ \left(-\frac{2}{(\Delta s)} + \frac{30s}{(\Delta s)^2} - \frac{96s^2}{(\Delta s)^3} + \frac{80s^3}{(\Delta s)^4}\right) \end{bmatrix}^T \tag{14}$$

The bending stress-strain relationship for a linear elastic beam is given by:

$$\{\sigma\} = E\{\varepsilon\} \tag{15}$$

where $\{\sigma\}$ is the bending stress vector and $E$ is the modulus of elasticity.

The element stiffness matrix, [**K**] in local coordinate system is obtained using the principle of Total Potential Energy. Thus:

$$[K] = A^2 \int_0^{\Delta s} [B]^T (EI)[B] ds \tag{16}$$

Substituting Eqs. (14) and (15) into Eq. (16) and integrating, then:

$$[\mathbf{K}] = \frac{EI}{35(\Delta s)^3} A^2 \begin{bmatrix} 5092 & 1138\Delta s & -3584 & 1920\Delta s & -1508 & 242\Delta s \\ 1138\Delta s & 332(\Delta s)^2 & -896\Delta s & 320(\Delta s)^2 & -242\Delta s & 38(\Delta s)^2 \\ -3584 & -896\Delta s & 7168 & 0 & -3584 & 896\Delta s \\ 1920\Delta s & 320(\Delta s)^2 & 0 & 1280(\Delta s)^2 & -1920\Delta s & 320(\Delta s)^2 \\ -1508 & -242\Delta s & -3584 & -1920\Delta s & 5092 & -1138\Delta s \\ 242\Delta s & 38(\Delta s)^2 & 896\Delta s & 320(\Delta s)^2 & -1138\Delta s & 332(\Delta s)^2 \end{bmatrix} \tag{17}$$

Eq. (17) presents the element stiffness matrix in local $sy$-coordinates. The stiffness matrix is highly nonlinear due to the existence of the nonlinear expression $A^2$. In order to transform the element stiffness matrix into the global $xy$-coordinates, a transformation matrix [**T**] is used.

$$[\mathbf{K}]_g = [\mathbf{T}]^T [\mathbf{K}][\mathbf{T}] \tag{18}$$

where $[\mathbf{K}]_g$ is the element stiffness matrix in global coordinates and [**T**] is given by:

$$[\mathbf{T}] = \begin{bmatrix} \Delta x/\Delta s & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta x/\Delta s & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta x/\Delta s & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

The global (or assembled) stiffness matrix $[\mathbf{K}^*]$ of the whole beam is obtained by assembling all element stiffness matrices where the beam is considered as an assemblage of structures, each consisting of one element at a time. For a beam divided into $n$ elements, the dimensions of the global stiffness matrix are $(4n+2) \times (4n+2)$. Solution can be made after applying the proper loading and boundary conditions. It should be noted that when the finite element method is used, different element lengths might be used in the analysis. This may be beneficial in cases where loads are applied unevenly over the length of the beam.

## 3. Boundary conditions

Different boundary conditions can be easily represented by the finite element method. The dimensions of the global stiffness matrix are modified according to the type of the boundary condition. For a beam divided into $n$ elements, the application of the boundary conditions yields:

1) Fixed-fixed beam (zero deflection and zero slope at both ends)
   $(4n - 2) \times (4n - 2)$.
2) Fixed-hinged beam (zero deflection at both ends and zero slope at one end)
   $(4n - 1) \times (4n - 1)$.
3) Hinged-hinged (zero deflection at both ends)
   $(4n) \times (4n)$.
4) Fixed-free (zero deflection and zero slope at the fixed end)
   $(4n) \times (4n)$.

## 4. Loading

For the sake of analysis, loads shall be applied at the nodes. When a load on an element is given, it should be transformed into equivalent nodal forces. This can be done by utilizing the shape function matrix, [**N**]. The equivalent nodal forces and moments at the nodes of the element, $V^*_1$, $M^*_1$, $V^*_2$, $M^*_2$, $V^*_3$ and $M^*_3$ are calculated from:

$$\begin{bmatrix} V^*_1 \\ V^*_2 \\ V^*_3 \\ M^*_1 \\ M^*_2 \\ M^*_3 \end{bmatrix} = \int_0^{\Delta s} [\mathbf{N}]^T \begin{Bmatrix} q_o(s) \\ m_o(s) \end{Bmatrix} ds \tag{20}$$

where $q_o(s)$ and $m_o(s)$ are the distributed load and moment on the element, respectively.

## 5. Effect of horizontal force

Fig. 2 shows a section in the beam subjected to large deflection. In case of vertical force acting on the beam, the shearing force in the section at node $i$ is given by:

$$F_i = V_i \cos\theta_i \tag{21}$$

The shearing force corresponding to the constant horizontal force at the same node is:

$$F_i = -P\sin\theta_i \tag{22}$$

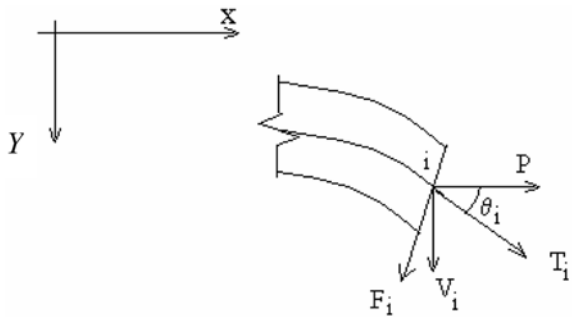



Fig. 2 A section in a beam element

Hence, the value of the vertical force $V_i$ that produces exactly the same shearing force as that of a horizontal force $P$ is:

$$V_i = -P\frac{\sin\theta_i}{\cos\theta_i} \tag{23}$$

Eq. (23) gives the value of the equivalent vertical force that should be applied to simulate the effect of a horizontal force. The solution has been carried out assuming that for the case of the applied horizontal force less than the critical buckling load with no lateral forces applied on the beam, the slopes become small, i.e., small deflection, $\sin\theta_i$ reaches zero and the equivalent vertical force due to a horizontal force will be equal to zero at the given section.

## 6. Solution sequence

The nonlinear coefficients $A$ in each element depend on the initially unknown deflections. They are needed to initiate a solution. The initial deflected profile may be taken from the linear small deflection analysis (or any properly estimated deflection shape). These deflections and slopes are considered for the initial solution. Thus, the stiffness matrix can be calculated. In matrix form:

$$[\mathbf{K}^*]^{(0)}\ \{\delta\}^{(1)} = \{\mathbf{F}\} \tag{24}$$

Solving for $\{\delta\}^{(1)}$ gives:

$$\{\delta\}^{(1)} = ([\mathbf{K}^*]^{(0)})^{-1}\{\mathbf{F}\} \tag{25}$$

From the new calculated $\{\delta\}^{(1)}$, improved values of A and hence $[\mathbf{K}^*]^{(1)}$ are obtained. After $r$ cycles

$$\{\delta\}^{(r)}=([\mathbf{K}^*]^{(r-1)})^{-1}\{\mathbf{F}\} \tag{26}$$

To achieve a solution, the process is repeated until in each element, the error vector of the form $\{\Delta\delta\}= (\{\delta\}^{(r)} - \{\delta\}^{(r-1)})/\{\delta\}^{(r)}$ is less than a specified small value (convergence criterion). A value of 0.001 is found to be appropriate.

## 7. Determination of horizontal projection length

The total length of the beam, $L$, remains constant as the axial extensions are neglected (inextensible beam). When large deformations take place, the horizontal projection, $L_x$, is required for analysis changes according to these deformations. In order to obtain an accurate value of this length, an iterative procedure is used.

A proper trial value of the horizontal length, $L_x$, is estimated. When the deflections are obtained, the length of each element, $\Delta s$, and then the total length of the beam are recalculated from the obtained horizontal projection and lateral deflections as:

$$L = \sum_{1}^{2n-1} [(y_{i+1} - y_i)^2 + (x_{i+1} - x_i)^2]^{1/2} \tag{27}$$

If the calculated length from Eq. (27) is not equal to the actual length, $L$, the estimated horizontal projection should be modified accordingly. A difference ratio less than 0.01% between the actual and computed length is assumed to be the accepted tolerance herein.

## 8. Non-dimensional parameters

For the purpose of analysis and generalization, the following normalized parameters are defined:

1) The normalized horizontal and vertical coordinates parameter $x_n$ and $y_n$ are

$$x_n = \frac{x}{L}; \quad y_n = \frac{y}{L} \tag{28}$$

2) The normalized horizontal and vertical concentrated loads are

$$P_{xn} = \frac{P_x L^2}{EI}; \quad P_{yn} = \frac{P_y L^2}{EI} \tag{29}$$

3) The normalized distributed loads intensity in horizontal and vertical are

$$q_{xn} = \frac{q_{xo} L^3}{EI}; \quad q_{yn} = \frac{q_{yo} L^3}{EI} \tag{30}$$

For prismatic beam $I = I$ and for non-prismatic members $I = I_1$. Where $I_1$ is the moment of inertia of the cross-section at the left support.

## 9. Computer programming

In order to apply the present method, a computer program is constructed using Qbasic language. The solution of the quasi-linearized equations is obtained by using the Gauss elimination technique (Gerald and Wheatley 1970) instead of the conventional inverse technique in order to reduce the computational time.

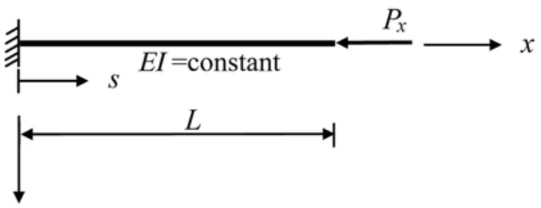



Fig. 3 Cantilever beam subjected to axial compressive load

Table 1 Normalized tip displacements parameters of prismatic cantilever beam subjected to a concentrated axial compressive load $P_{xn}$

| Present study | | | Timoshenko and Gere (1961) | |
|---|---|---|---|---|
| $P_{xn}$ | $x_t$ | $y_t$ | $x_t$ | $y_t$ |
| 2.4671 | 1 | 0 | 1 | 0 |
| 2.5049 | 0.9695 | 0.2195 | 0.9712 | 0.2201 |
| 2.6240 | 0.8794 | 0.4212 | 0.8811 | 0.4218 |
| 2.8412 | 0.7407 | 0.5925 | 0.7412 | 0.5929 |
| 3.1920 | 0.5594 | 0.7189 | 0.5603 | 0.7190 |
| 3.7458 | 0.3487 | 0.7914 | 0.3491 | 0.7918 |
| 4.6498 | 0.1210 | 0.8028 | 0.1213 | 0.8031 |

## 10. Numerical examples

Five representative numerical examples are investigated to demonstrate the versatility and efficiency of the present new finite element scheme. For all numerical examples presented in this section, the actual length of the beam is divided into ten elements to achieve excellent results.

### *10.1 Example 1: Post-buckling analysis of a prismatic cantilever beam subjected to a concentrated axial compressive load*

Fig. 3 shows a prismatic cantilever beam subjected to a concentrated axial compressive load. Table 1 shows the normalized tip displacements parameters $x_t$ and $y_t$ for different concentrated axial compressive load parameter $P_{xn}$. The present results shown in this table are compared with the results analyzed by Timoshenko and Gere (1961) using elliptic integrals. Table 1 shows excellent agreements between present scheme and elliptic integrals. Convergence with good accuracy was achieved by the present study with a number of iterations equal to 4.

### *10.2 Example 2: Post-buckling analysis of a prismatic cantilever beam subjected to a uniformly distributed axial compressive load*

Fig. 4 shows a prismatic cantilever beam subjected to uniformly distributed axial compressive load. Table 2 shows the normalized tip displacements parameters $x_t$ and $y_t$ for different uniformly

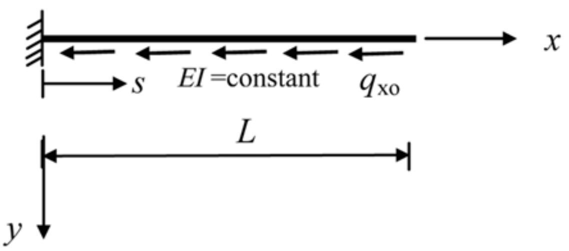



Fig. 4 Cantilever beam subjected to uniformly distributed axial compressive load

Table 2 Normalized tip displacements parameters of prismatic cantilever beam subjected to uniformly distributed axial compressive load $q_{xn}$

| | Present study | | Lee (2001) | |
|---|---|---|---|---|
| $q_{xn}$ | $x_t$ | $y_t$ | $x_t$ | $y_t$ |
| 7.8356 | 1 | 0 | 1 | 0 |
| 7.9448 | 0.9683 | 0.2249 | 0.9685 | 0.2255 |
| 8.2877 | 0.8754 | 0.4328 | 0.8761 | 0.4336 |
| 8.9101 | 0.7304 | 0.6079 | 0.7299 | 0.6082 |
| 9.907 | 0.5398 | 0.7354 | 0.5406 | 0.7358 |
| 11.4638 | 0.3209 | 0.8061 | 0.3216 | 0.8063 |
| 13.9685 | 0.0868 | 0.8129 | 0.0872 | 0.8130 |

distributed axial compressive load parameter $q_{xn}$. The present results shown in this table are compared with the recent results obtained by Lee (2001) using numerical integration procedure in connection with Butcher's fifth-order Runge-Kutta method. It can be seen from this table that the agreement between present numerical scheme and Lee (2001) is excellent. Convergence with good accuracy was achieved by the present study with a number of iterations equal to 4.

### *10.3 Example 3: Simply supported beam subjected to the combined action of a concentrated load located at the quarter span from left support and an axial compressive load*

Fig. 5 shows a prismatic simply supported beam subjected to the combined action of a concentrated load $P_{yn}$ located at the quarter span from left support and an axial compressive load

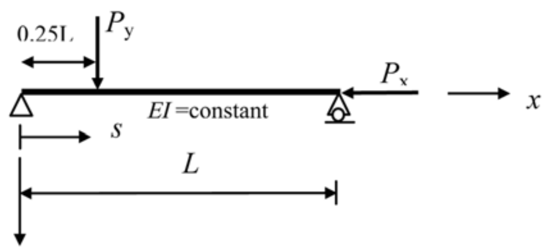



Fig. 5 Simply supported beam subjected to a concentrated load located at the quarter span from left support and axial compressive load

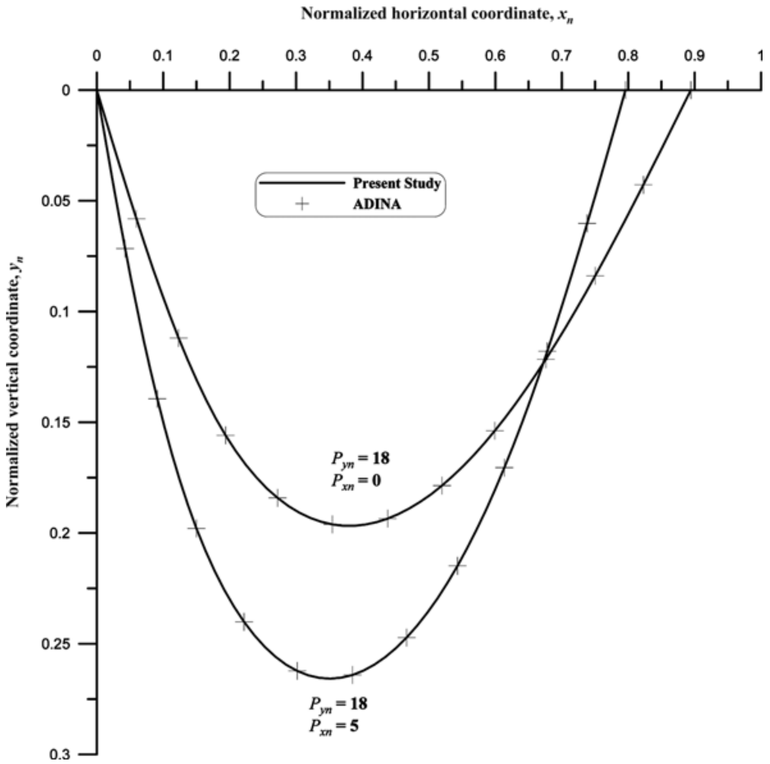


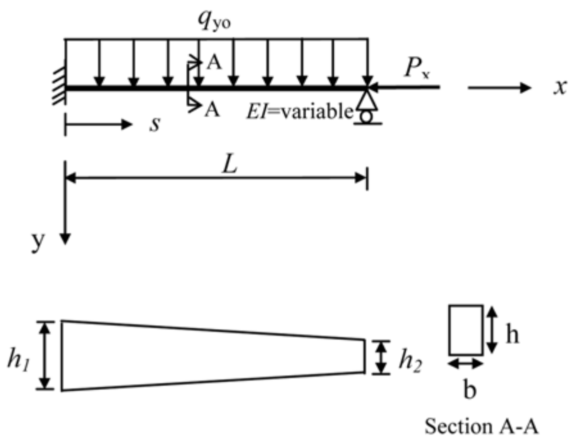

Fig. 6 Deformed configurations for Fig. 5

$P_{xn}$. Two cases are considered for the combined load. One with $P_{xn} = 0$ and $P_{yn} = 18$ and the other with $P_{xn} = 5$ and $P_{yn} = 18$. Fig. 6 shows the deformed configurations for the two cases mentioned above. Fig. 6 also shows results obtained via ADINA by dividing the beam into fifty two-dimensional beam elements. A comparison between the present scheme and ADINA reveals excellent agreement. The number of iterations required to achieve good accuracy were equal to 3.

### *10.4 Example 4: Large deflection analysis of a non-prismatic fixed-hinged beam of tapered rectangular section type subjected to combined load consisting of a concentrated compressive load and uniformly distributed transverse load*

Fig. 7 shows a non-prismatic fixed-hinged beam of tapered rectangular section type subjected to a combined load consisting of a concentrated compressive load $P_{xn}$ and uniformly distributed transverse load $q_{yn}$. The depth of the rectangular section of the beam at the fixed end is equal to twice that the free end. Two cases are considered for the combined load. One with $P_{xn} = 0$ and $q_{yn} = 20$ and the other with $P_{xn} = 6$ and $q_{yn} = 5$.

Fig. 8 shows the deformed configurations for the two cases. Fig. 8 also shows results obtained via ADINA. A comparison between the present scheme and ADINA reveals excellent agreement. The number of iterations required to achieve good accuracy were equal to 4.




$$EI(s) = EI_1\left[\frac{1}{2}\left(1-\frac{s}{L}\right)+\frac{1}{2}\right]^3 ; h_2 = \frac{h_1}{2}$$

Fig. 7 Fixed-hinged beam of tapered rectangular beam subjected to uniformly distributed load and axial compressive load

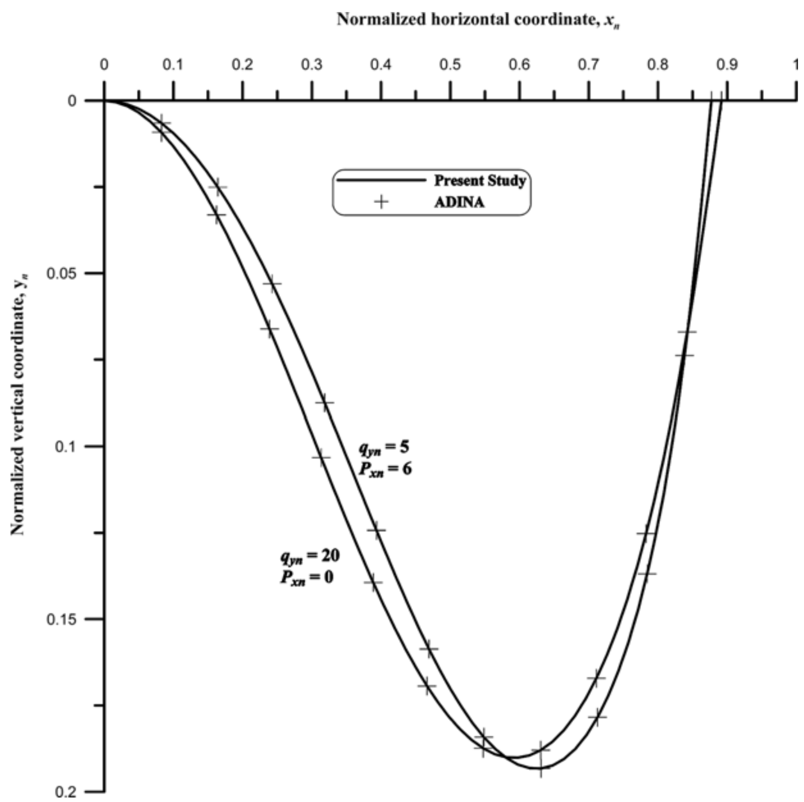



Fig. 8 Deformed configurations for Fig. 7

### 10.5 Example 5: Fixed-fixed beams subjected to linearly distributed load and uniformly axial distributed compressive load

Fig. 9 shows a prismatic Fixed-fixed beams subjected to linearly increasing distributed load $q_{yn}$ and uniformly axial distributed compressive load $q_{xn}$. Two cases are considered for the combined load. One with $q_{xn}$ = 90 and $q_{yn}$ = 0 and the other with $q_{xn}$ = 45 and $q_{yn}$ = 95. Fig. 10 shows the

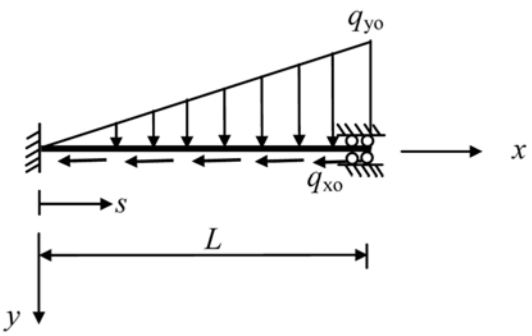



Fig. 9 Fixed-fixed beams subjected to linearly distributed load and uniformly axial distributed compressive load

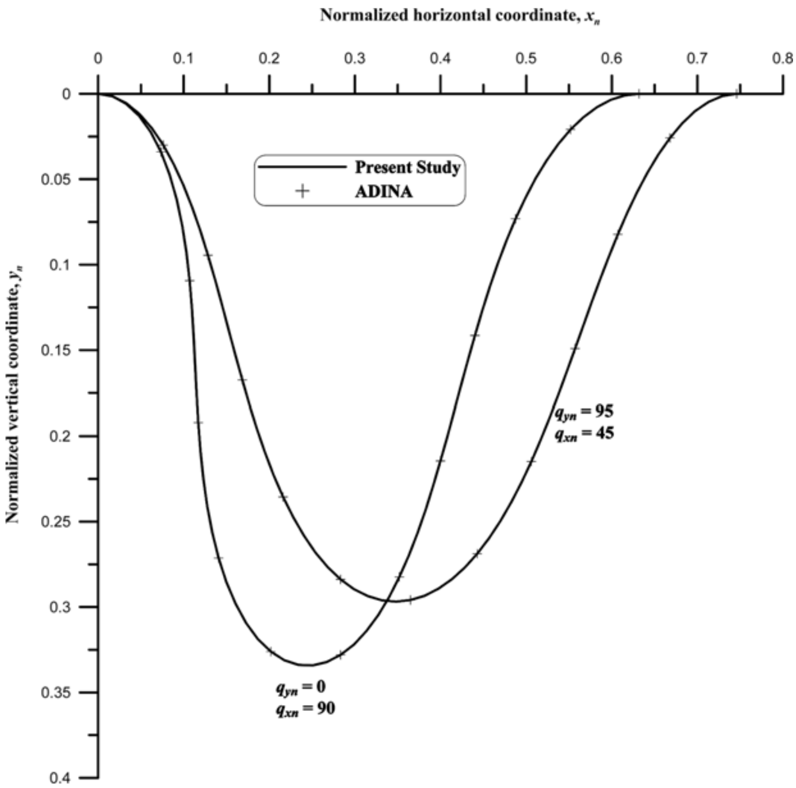



Fig. 10 Deformed configurations for Fig. 9

deformed configurations for the two cases. Fig. 10 also shows results obtained via ADINA [44]. A comparison between the present scheme and ADINA reveals excellent agreement. The number of iterations required to achieve good accuracy were equal to 3.

## 11. Conclusions

A new, simple, robust and accurate finite element scheme for large deflection analysis of inextensible slender prismatic and nonprismatic beams with different loading and boundary conditions is presented and verified through comparison with available analytic solutions and numerical results by available references and results by ADINA. An evaluation of the results leads to a number of conclusions. These are listed herein.

1. The present formulation and solution procedure is both simple and practical in studying different large deflection problems of prismatic and nonprismatic slender beams subjected to continuous and discontinuous loadings and with different boundary conditions.
2. Convergence with good accuracy is normally achieved with a small number of elements ($n = 10$).
3. The total number of iterations required for solving the nonlinear equations ranged between (3 and 5).

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