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A multiple scales method solution for the free and forced nonlinear transverse vibrations of rectangular plates

A. Shooshtari† and S. E. Khadem‡

Mechanical Engineering Department, Tarbiat Modares University, P.O. Box 14115-177, Tehran, Iran

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Abstract. In this paper, first, the equations of motion for a rectangular isotropic plate have been derived. This derivation is based on the Von Karmann theory and the effects of shear deformation have been considered. Introducing an Airy stress function, the equations of motion have been transformed to a nonlinear coupled equation. Using Galerkin method, this equation has been separated into position and time functions. By means of the dimensional analysis, it is shown that the orders of magnitude for nonlinear terms are small with respect to linear terms. The Multiple Scales Method has been applied to the equation of motion in the forced vibration and free vibration cases and closed-form relations for the nonlinear natural frequencies, displacement and frequency response of the plate have been derived. The obtained results in comparison with numerical methods are in good agreements. Using the obtained relation, the effects of initial displacement, thickness and dimensions of the plate on the nonlinear natural frequencies and displacements have been investigated. These results are valid for a special range of the ratio of thickness to dimensions of the plate, which is a characteristic of the Multiple Scales Method. In the forced vibration case, the frequency response equation for the primary resonance condition is calculated and the effects of various parameters on the frequency response of system have been studied.

Keywords: multiple scales method; plate; nonlinear vibration; free vibration; force vibration; frequency response.

1. Introduction

Large amplitude vibrations of rectangular plates have been investigated rigorously by many authors. A good review has been carried out by Sathyamoorphy (1996). Benammar *et al.* (1990, 1991a,b, 1993) have developed one model for large displacement and nonlinear vibrations of plates with different boundary conditions. This model is based on the Hamilton principle and the spectral analysis. In this method, one assumes that for a system which has weak nonlinear terms, the response is assumed:

$$w(x, y, t) = \phi(x, y)q(t)$$

† PhD Student of Mechanical and Aerospace Engineering Department
‡ Professor of Mechanical and Aerospace Engineering Department, Corresponding author,
E-mail: Khadem@modares.ac.ir

Assuming $q(t)$ as a harmonic function, the $\phi(x, y)$ will be found by the harmonic balance method.

Also Bikri *et al.* (2003) investigated the effects of geometrically nonlinear free vibration of thin isotropic laminated rectangular composite plates with clamped simply supported rectangular plates, in order to determine their fundamental nonlinear mode shapes, and other dynamical properties of the plate with a numerical method.

Ribeiro and Petyt (2000) used the hierarchical finite element method for the free vibration analysis and discovered the internal resonance of the system by this method. In this work, they considered the geometrical nonlinearity too, and demonstrated the plate mode shapes with the amplitude of the vibration.

Amabili (2004) investigated theoretically and experimentally large amplitude (geometrically nonlinear) vibrations of rectangular plates subjected to radial harmonic excitation in the spectral neighborhood of the lowest resonances with different boundary conditions.

In all the above works, the researchers have considered the effects of geometrical nonlinearity which is a kind of nonlinearity in the stiffness. But in a nonlinear system, there may be some other kinds of nonlinearities such as nonlinear inertia and nonlinear damping.

Shaw and Pierre (1993) defined a new method which is named invariable manifold for nonlinear continuous systems and detected the kind of nonlinearity in the nonlinear equations. This method can be used for systems with weak nonlinearities. Based on this method, Nayfeh and Nayfeh (1994) obtained the nonlinear frequencies and mode shapes for one-dimensional continuous systems which have nonlinearities in stiffness and inertia. Then, Nayfeh and Chin (1995) used a model for analysis of a cantilever beam. Also, in another paper he investigated internal resonances for the one dimensional continuous nonlinear system with this method (Nayfeh *et al*. 1996). Mahmoodi *et al.* (2004) analyzed the nonlinear free vibration of continuous damped system which contains nonlinearity terms in inertia, stiffness and damping.

In the above works, the method of invariant manifold was applied to one dimensional continuous problem such as beams. In this paper, the results of invariant manifold theory (Nayfeh and Nayfeh 1994) are used to show that the system has nonlinearity in inertia and stiffness.

Then the Multiple Scale Method is applied on the free and forced vibration analysis. The advantage of this method is that it yields closed form relations for the nonlinear natural frequencies and displacements in free vibration case and frequency response in the forced vibration case. Also the effect of system parameters on the natural frequencies and nonlinear mode shapes can be determined accurately. Obtained results have good agreement with respect to the available numerical results.

In the free vibration case the effects of parameters such as thickness and initial amplitude on the dynamical responses of the plate have been investigated. Also it is shown that by increasing the ratio of thickness to dimension of the plate, the nonlinear frequency of the plate will increase.

Here, also the forced vibration of the plate has been investigated and the frequency response of the plate has been calculated with the Multiple Scales Method. The effects of parameters such as thickness, Poisson's ratio and force amplitude excitation on the frequency response have been studied.

It is shown that by increasing the thickness of the plate, nonlinearity effects will increase, due to increase of the effects of rotary inertia and shear deformation. Also, an increase of Poisson's ratio of the material will increase the nonlinearity of the plate, but it does not have any effect on the sharpness of the response curve. Also, it is shown that by increasing the force amplitude of the excitation, the nonlinearity and deviation of frequency response of system, will not change. But the sharpness of the response curve is decreased.

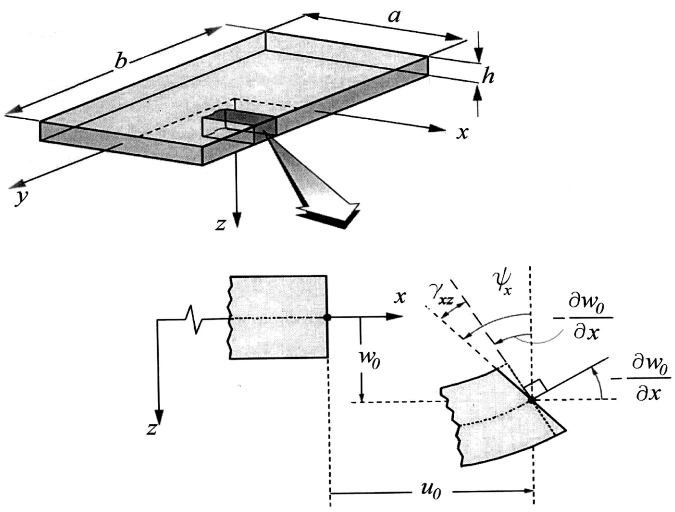

Fig. 1 Rectangular isotropic plate and the displacement after deformation

2. Description of the motion

Considering an isotropic elastic rectangular plate with dimensions a and b and thickness h as shown in Fig. 1. This plate is under large deflection and the displacements of the plate are unknown.

In the analysis, the effect of shear deformations and rotary inertia are considered. So, first order shear deformation theory is used. Based on this theory, displacement relations are:

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) + z\,\psi_x(x,y,t) \\ v(x,y,z,t) &= v_0(x,y,t) + z\,\psi_y(x,y,t) \\ w(x,y,z,t) &= w_0(x,y,t) \end{aligned} \tag{1}$$

where u, v and w are the displacement in directions of x, y and z respectively, and u_0, v_0, w_0 are the displacements of the mid-plane. Also ψ_x and ψ_y are the angles between the normal to mid-plane before and after deformation and t is the time. The components of stresses are:

$$\begin{aligned} \sigma_x &= \frac{N_x}{h} + \frac{12M_x}{h^3} = \frac{N_x}{h} + \frac{M_x}{I} \\ \sigma_y &= \frac{N_y}{h} + \frac{12M_y}{h^3} = \frac{N_x}{h} + \frac{M_y}{I} \\ \sigma_{xy} &= \frac{N_{xy}}{h} + \frac{12M_{xy}}{h^3} = \frac{N_{xy}}{h} + \frac{M_{xy}}{I} \\ \tau_{xz} &= \frac{3}{2}\left(\frac{Q_x}{h}\right)\left(1 - 4\frac{z^2}{h^2}\right) \end{aligned} \tag{2}$$

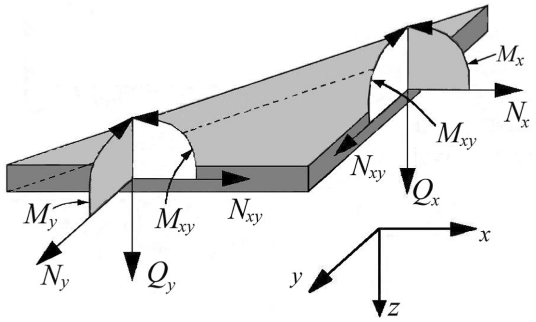

Fig. 2 Internal forces and bending moment in the plate

$$\tau_{yz} = \frac{3}{2}\left(\frac{Q_y}{h}\right)\left(1 - 4\frac{z^2}{h^2}\right)$$

where in the above equations N_x, N_y and N_{xy} are the in-plane forces, M_x, M_y and M_{xy} are bending moments, Q_x and Q_y are shear forces and I is the moment of inertia. The internal forces and bending moments are shown in Fig. 2.

Considering the first order shear deformation theory and by using the variation calculus, the force-displacement relations for an elastic isotropic plate will become as:

$$\begin{aligned}
&\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 - \frac{N_x}{Eh} + \nu\frac{N_y}{Eh} = 0 \\
&\frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 - \frac{N_y}{Eh} + \nu\frac{N_x}{Eh} = 0 \\
&\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\cdot\frac{\partial w_0}{\partial y} - 2\frac{(1+\nu)}{Eh}N_{xy} = 0 \\
&\frac{\partial \psi_x}{\partial x} - \frac{12 M_x}{Eh^3} + \frac{12\,\nu M_y}{Eh^3} = 0 \\
&\frac{\partial \psi_y}{\partial y} - \frac{12 M_y}{Eh^3} + \frac{12\,\nu M_x}{Eh^3} = 0 \\
&\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} - \frac{24(1+\nu)}{Eh^3}M_{xy} = 0 \\
&\alpha + \frac{\partial w}{\partial x} - \frac{12(1+\nu)}{5Eh}Q_x = 0 \\
&\beta + \frac{\partial w}{\partial y} - \frac{12(1+\nu)}{5Eh}Q_y = 0
\end{aligned} \tag{3}$$

Computing the kinetic and potential energy and using the Hamilton principle for conservative systems, the equations of motion are found as the following:

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = \frac{\rho h^3}{12}\psi_{x,tt} \tag{4}$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = \frac{\rho h^3}{12}\psi_{y,tt} \tag{5}$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \rho h u_{,tt} \tag{6}$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = \rho h v_{,tt} \tag{7}$$

$$\frac{\partial N_x}{\partial x}w_{,x} + N_x\frac{\partial^2 w}{\partial x^2} + \frac{\partial N_y}{\partial y}w_{,y} + N_y\frac{\partial^2 w}{\partial y^2} + \frac{\partial N_{xy}}{\partial x}w_{,y} + \frac{\partial N_{xy}}{\partial y}w_{,x}$$

$$+ 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = \rho h w_{,tt} + F \tag{8}$$

Where ρ is the density, ν is the Poisson's ratio, E is the Young modulus and F is the excitation force which is effect on the lateral direction of plate and can be expressed as:

$F = F_0\cos(\Omega t)$ where F_0 is the amplitude of force and Ω is the frequency of excitation.

Considering simply supported boundary conditions it can be written as:

at $x = 0$ and $x = a$:

$$u = 0 \quad \text{or} \quad N_x = 0$$

$$v = 0 \quad \text{or} \quad N_{xy} = 0$$

$$w = 0 \quad \text{or} \quad Q_x + N_x\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y} = 0 \tag{9}$$

$$\psi_x = 0 \quad \text{or} \quad M_x = 0$$

$$\psi_y = 0 \quad \text{or} \quad M_{xy} = 0$$

and at $y = 0$ and $y = b$:

$$u = 0 \quad \text{or} \quad N_{xy} = 0$$

$$v = 0 \quad \text{or} \quad N_y = 0$$

$$w = 0 \quad \text{or} \quad Q_y + N_{xy}\frac{\partial w}{\partial x} + N_y\frac{\partial w}{\partial y} = 0 \tag{10}$$

$$\psi_x = 0 \quad \text{or} \quad M_{xy} = 0$$

$$\psi_y = 0 \quad \text{or} \quad M_y = 0$$

The Eqs. (4) through (8) are the equations of motion for a rectangular isotropic elastic plate under large amplitude vibration with considering the shear deformation and rotary inertia phenomena.

Assuming principally transverse motion which is based on the results of previous investigations that the in-plane natural frequencies are very far away from transverse natural frequencies (Nayfeh and Mook 1979), it can be considered that:

$$\rho h u_{,tt} \approx 0 \qquad \rho h v_{,tt} \approx 0$$

So that an Airy stress function φ is introduced as:

$$N_x = \frac{\partial^2 \phi}{\partial y^2} \qquad N_y = \frac{\partial^2 \phi}{\partial x^2} \qquad N_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \tag{11}$$

Substituting the above function and Eqs. (3) in the equations of motion, the Eqs. (6) and (7), are satisfied automatically in the above conditions. The other equations become as:

$$D\frac{\partial^2 \psi_x}{\partial x^2} + \frac{D(1-\nu)}{2}\frac{\partial^2 \psi_x}{\partial y^2} + \frac{D(1+\nu)}{2}\frac{\partial^2 \psi_y}{\partial x \partial y} - \frac{5Eh}{12(1+\nu)}\left(\psi_x + \frac{\partial w}{\partial x}\right) - \frac{\rho h^3}{12}\psi_{x,tt} = 0 \tag{12}$$

$$D\frac{\partial^2 \psi_y}{\partial y^2} + \frac{D(1-\nu)}{2}\frac{\partial^2 \psi_y}{\partial x^2} + \frac{D(1+\nu)}{2}\frac{\partial^2 \psi_x}{\partial x \partial y} - 5\frac{Eh}{12(1+\nu)}\left(\psi_y + \frac{\partial w}{\partial y}\right) - \frac{\rho h^3}{12}\psi_{y,tt} = 0 \tag{13}$$

$$\frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 \phi}{\partial x \partial y} + \frac{5Eh}{12(1+\nu)}\left(\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} + \nabla^2 w\right) - \rho h w_{,tt} - F = 0 \tag{14}$$

The proper compatibility equation must be considered for the middle surface strains, which states as:

$$\nabla^4 \phi = Eh\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \tag{15}$$

Eliminating ψ_x and ψ_y terms from Eqs. (12) through (14), it will become as:

$$\left[\frac{D^2(1-\nu)}{2}\nabla^4 - \frac{D\rho h^3(3-\nu)}{24}\nabla^2\frac{\partial^2}{\partial t^2} - \frac{5DEh(3-\nu)}{24(1+\nu)}\nabla^2 + \frac{25E^2h^2}{144(1+\nu)^2} + \frac{5\rho E h^4}{72(1+\nu)}\frac{\partial^2}{\partial t^2}\right] \times$$
$$\left[\frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{5Eh}{12(1+\nu)}\nabla^2 w - \rho h \ddot{w}\right]$$
$$+ \frac{5Eh}{12(1+\nu)}\left[\frac{5DEh}{24}\frac{(1-\nu)}{(1+\nu)}\nabla^4 - \frac{25E^2h^2}{144(1+\nu)^2}\nabla^2 - \frac{5\rho E h^4}{144(1+\nu)}\frac{\partial^2}{\partial t^2}\right]w + F = 0 \tag{16}$$

The above equation is the equation of motion in transverse direction for a rectangular isotropic elastic plate where $D = Eh^3/12(1-\nu^2)$.

3. Forced vibration case

Now in this section the forced vibration analysis in the case of primary resonance is investigated. In this analysis, the symmetric mode shapes ($m = n$) are considered and it is assumed that only one mode is excited. So the effects of internal resonances are neglected. For this purpose, to describe the nearness of the primary resonance to natural frequency, the detuning parameter δ is defined as:

$$\Omega = \omega + \varepsilon\delta \tag{17}$$

In order to obtain the non-linear natural frequencies and mode shapes, there are many different methods i.e., The Numerical methods, the Finite elements methods and the Analytical analysis such as perturbation methods.

The method that will be used in this paper is the multiples scales method. The most important advantage of this method is that by identification of a non-dimensional small parameter which has a physical interpretation and using several time scales, one can obtain a complete physical understanding about the behavior of the system and the influence of different parameters and terms on the final response of system.

To solve the above nonlinear Eq. (16), first by using the Galerkin method, one discretizes the equation by writing the solution as:

$$w(x, y, t) = \sum_{i=1}^{N} \psi_{ij}(x, y)f(t) \tag{18}$$

Then, by selecting the $\psi_{ij}(x, y)$ so that they satisfies the boundary conditions, we use the multiples scales method to find the time function $f(t)$. For this purpose without any lack of generality, the case of a square plate with all sides hinged will be considered in the first mode. Therefore

$$\psi_{11}(x, y) = h\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right) \tag{19}$$

So

$$w_{11}(x, y, t) = hf(t)\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right) \tag{20}$$

Also the function ϕ will be selected as:

$$\phi = f^2(t)\frac{Eh^3}{32}\left[\frac{2\pi^2}{a^2(1-\nu^2)}(x^2 + y^2) + \cos^2\frac{\pi x}{a} + \cos^2\frac{\pi y}{a}\right] \tag{21}$$

To satisfy the compatibility Eq. (15)

The boundary condition of the problem is considered as simply support. So:
at $x = 0$ and $x = a$:

$$w(x, y) = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \tag{22}$$

and at $y = 0$ and $y = a$:

$$w(x, y) = 0, \quad \frac{\partial^2 w}{\partial y^2} = 0 \tag{23}$$

Defining the non-dimensional parameters as:

$$\tau = t\sqrt{\frac{E}{\rho a^2}}, \quad x^* = \frac{x}{a}, \quad y^* = \frac{y}{a} \tag{24}$$

and substituting Eq. (20) and Eq. (21) in to Eq. (16) and multiplying the left-hand side by $\sin(\pi x/a)$ $\sin(\pi y/a)$, and integrating over the plate area, one obtains non-dimensional equation by using the non-dimensional parameters (24) as:

$$a_1 f^3 + a_2 f + a_3 f'' + a_4 f^2 f'' + a_5 f (f')^2 = F \tag{25}$$

where:

$$a_1 = \frac{\pi^8 r^6}{2304}\frac{(3-\nu)}{(1-\nu^2)^2} + \frac{5\pi^6(3-\nu)^2 r^4}{4608(1-\nu^2)^2} + \frac{25\pi^4 r^2(3-\nu)}{4608(1+\nu^2)(1-\nu)}$$

$$a_2 = \frac{5\pi^6 r^4}{1728(1+\nu)^2(1-\nu^2)} + \frac{25\pi^4 r^2}{1728(1+\nu)^2(1-\nu^2)}$$

$$a_3 = \frac{\pi^4 r^4}{1728(1+\nu)^2(1-\nu^2)} + \frac{5\pi^2 r^2(23-11\nu)}{3456(1+\nu)(1-\nu^2)} + \frac{25}{576(1+\nu)^2} \tag{25a}$$

$$a_4 = \frac{\pi^6 r^6(3-\nu)^2}{1536(1-\nu)(1-\nu^2)} + \frac{5\pi^4 r^4(3-\nu)}{768(1-\nu^2)}$$

$$a_5 = \frac{\pi^6 r^6(3-\nu)^2}{768(1-\nu)(1-\nu^2)} + \frac{5\pi^4 r^4(3-\nu)}{384(1-\nu^2)}$$

and $r = h/a$. Also the prime means derivatives about the non dimensional time τ.

Comparing the Eq. (25) with the equation which is derived by Nayfeh and Nayfeh (1994), it is found that the first term of this equation presents the nonlinearity in the stiffness and two last terms are the nonlinear terms in inertia. So, this system has nonlinearity in stiffness and inertia.

Now Eq. (25) is rewritten as:

$$f'' + b_1 f + b_2 f^3 + b_3 f^2 f'' + b_4 f (f')^2 = F \tag{26}$$

where

$$b_1 = \frac{a_2}{a_3} = O(1) \qquad b_2 = \frac{a_1}{a_3} = O(r^2)$$

$$b_3 = \frac{a_4}{a_3} = O(r^2) \qquad b_4 = \frac{a_5}{a_3} = O(r^2) \tag{27}$$

where the O means the order of magnitude of the coefficients. With definition of the parameter $r = h/a$ as a small parameter:

$$\sqrt{\varepsilon} = r = h/a \tag{28}$$

the Eq. (26) will be written as:

$$f'' + \omega^2 f + \varepsilon G(f, f', f'') = F \tag{29}$$

where ω is the linear natural frequency and

$$G(f, f', f'') = b_2 f^3 + b_3 f^2 f'' + b_4 f(f')^2 \tag{30}$$

Therefore due to the presence of ε, the Eq. (29) has weakly nonlinear terms. Assuming the primary resonance condition, force function F has weak amplitude of excitation and so, one can write:

$$F = \varepsilon F_0 \cos(\Omega t) \tag{31}$$

And the Eq. (29) can be written as:

$$f'' + \omega^2 f + b_2 f^3 + b_3 f^2 f'' + b_4 f(f')^2 - \varepsilon F_0 \cos\Omega t = 0 \tag{32}$$

Two time scales are defined as

$$T_0 = t \qquad T_1 = \varepsilon t \tag{33}$$

Thus the first and second derivatives of f are written as:

$$\frac{df}{dt} = \frac{df}{dT_0} + \varepsilon\frac{df}{dT_1}$$

$$\frac{d^2f}{dt^2} = \frac{d^2f}{dT_0^2} + 2\varepsilon\frac{d^2f}{dT_0 dT_1} + \varepsilon^2\frac{d^2f}{dT_1^2} \tag{34}$$

Considering the above relations and expanding the function f as:

$$f = f_0 + \varepsilon f_1 + \dots \tag{35}$$

and substituting in the Eq. (32) and equating the coefficients of ε^0 and ε^1 equal to zero:

$$0(\varepsilon^0) = 0(1): \quad \frac{d^2 f_0}{dT_0^2} + \omega^2 f_0 = 0 \tag{36}$$

$$0(\varepsilon^1): \quad \frac{d^2 f_1}{dT_0^2} + \omega^2 f_1 = -2\frac{d^2 f_0}{dT_0 dT_1} - b_2 f_0^3 - b_3 f_0^2 \frac{d^2 f_0}{dT_0^2} - b_4 f_0\left(\frac{df_1}{dT_0}\right)^2 - F_0\cos(\Omega t) \tag{37}$$

a general solution for Eq. (36) can be written as:

$$f_0(T_0, T_1) = A(T_1)e^{i\omega T_0} + \bar{A}(T_1)e^{-i\omega T_0} \tag{38}$$

Substituting f_0 from Eq. (38) in Eq. (37) and using Eq. (17), it will become:

$$\frac{d^2 f_1}{dT_0^2} + \omega^2 f_1 = -2iA'\omega e^{i\omega T_0} + 2i\bar{A}' e^{-i\omega T_0} - b_2(Ae^{i\omega T_0} + \bar{A}e^{-i\omega T_0})^3 -$$
$$- b_3(Ae^{i\omega T_0} + \bar{A}e^{-i\omega T_0})^2(-A\omega^2 e^{i\omega T_0} - \bar{A}\omega^2 e^{-i\omega T_\cdot}) - b_4(Ae^{i\omega T_0} + \bar{A}e^{-i\omega T_\cdot})$$
$$(Ai\omega e^{i\omega T_0} - \bar{A}i\omega e^{-i\omega T_0})^2 + \left(\frac{F_0}{2}e^{i\delta T_1}\right)e^{i\omega T_0} + cc \tag{39}$$

Secular terms which are the coefficients of $\exp(i\omega T_0)$, will produce non periodic solution and must be eliminated from the Eq. (38) (Neyfeh and Mook 1979). So the solvability condition for Eq. (39) is:

$$2A'i\omega + [3b_2^* + \omega^2(b_4^* - 3b_3^*)]A^2\bar{A} + \frac{F_0}{2}e^{i\delta T_1} = 0 \tag{40}$$

Now for determining $\bar{A}(T_1)$, it is stated in the polar form as:

$$A(T_1) = \frac{1}{2}\zeta(T_1)e^{i\beta(T_1)} \tag{41}$$

Substituting Eq. (41) in Eq. (40) and separating the real and imaginary components equal to zero, it is obtained:

$$\zeta' = \frac{F_0}{2\omega}\sin(\delta T_1 - \beta) \tag{42}$$

$$\zeta\beta' = -\frac{[3b_2 + \omega^2(b_4 - 3b_3)]\lambda^3}{8} + \frac{F_0}{2\omega}\cos(\delta T_1 - \beta) \tag{43}$$

Eqs. (42) and (43) are transformed to an autonomous system (i.e., one in which T_1 does not appear explicitly) by introducing λ as:

$$\lambda = \delta T_1 - \beta \tag{44}$$

So, the Eqs. (42) and (43) will be rewritten as:

$$\zeta' = \frac{F_0}{2\omega}\sin\lambda \tag{45}$$

$$\zeta\lambda' = \gamma\sigma + \frac{[3b_2 + \omega^2(b_4 - 3b_3)]\gamma^3}{8} - \frac{F_0}{2\omega}\cos\lambda \tag{46}$$

These equations are the modulation equations of the system. To determine the character of the solutions, first one must locate the singular points and then examine the motion in their neighborhoods. Because the amplitude and phase at a singular point don't change, so the steady-state motion is considered to determine the singular points.

Steady-state motions occur when $d\zeta/dT_1 = d\lambda/dT_1 = 0$. So from Eqs. (45) and (46) in steady-state motions we have:

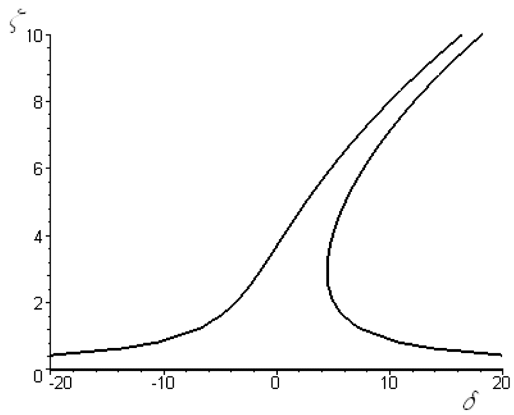

Fig. 3 Variation of non-dimensional amplitude of the middle point of plate with respect to δ, $r = h/a = 0.1$, $\nu = 0.3$, $F_0 = 10$

$$\left[\zeta\delta + \frac{[3b_2 + \omega^2(b_4 - 3b_3)]\zeta^3}{8}\right]^2 = \frac{F_0^2}{4\omega^2} \tag{47}$$

With this equation the frequency response of the system can be obtained. The frequency response is the solution of Eq. (47) with respect to δ.

Now the frequency response of the plate at the middle point of plate $(x = a/2, y = b/2)$, and the effects of thickness, Poisson's ratio and force amplitude excitation on the frequency response will be investigated.

Fig. 3 shows the frequency response of the plate (backbone curve) at the middle point of plate.

It shows that the frequency response curve has a deviation to the right. So the system has positive nonlinearity and behaves as a hardening nonlinear system (Nayfeh and Mook 1979). Also it is shown that when δ goes to zero, the excitation frequency will go to natural frequency and therefore the resonance will be occurred. Because there is not considered any damping in the system and the material of plate is elastic, so the amplitude will go to infinity in the resonance case and the two branches of the curve will not have any intersection with together.

Fig. 4 shows the effect of thickness on the frequency response of the system.

As it is shown, an increase in the ratio of thickness to dimension of the plate, the nonlinearity of system will increase and become hardener.

Fig. 5 shows the effect of Poisson's ratio on the frequency response of the plate.

In this figure, one can see that an increase in the Poisson's ratio of the material, will increase the nonlinearity of the plate, but it does not have any effect on the sharpness of response curve.

Also Fig. 6 shows the effects of force amplitude F_0 on the frequency response of the system.

It is shown that an increase in the force amplitude of the excitation, will not change the nonlinearity of system. But, it decreases the sharpness of the response curve.

4. Free vibration case

In order to calculate the natural frequencies and transverse mode shapes of the system, Eq. (16) must be solved in the free vibration case: $F = 0$.

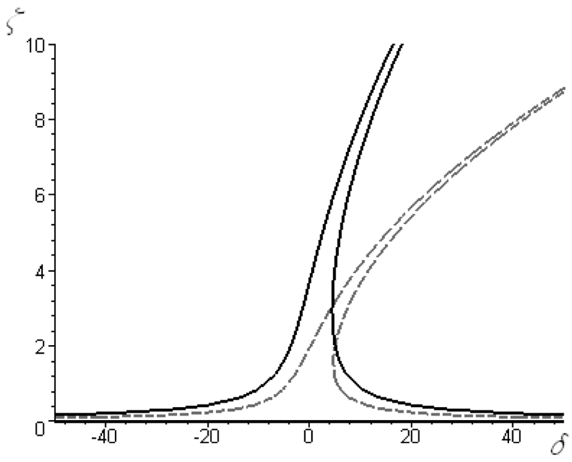

Fig. 4 The effects of thickness on the frequency response curve ______ $r = h/a = 0.1$, _ _ _ _ _ $r = h/a = 0.2$

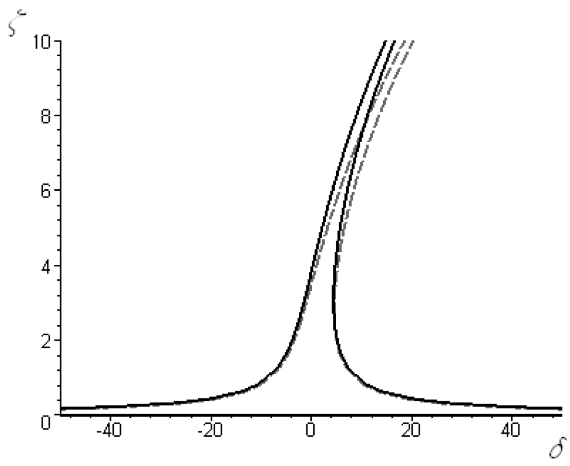

Fig. 5 The effect of Poisson's ratio on the frequency response curve ______ $\nu = 0.2$, _ _ _ _ _ $\nu = 0.4$

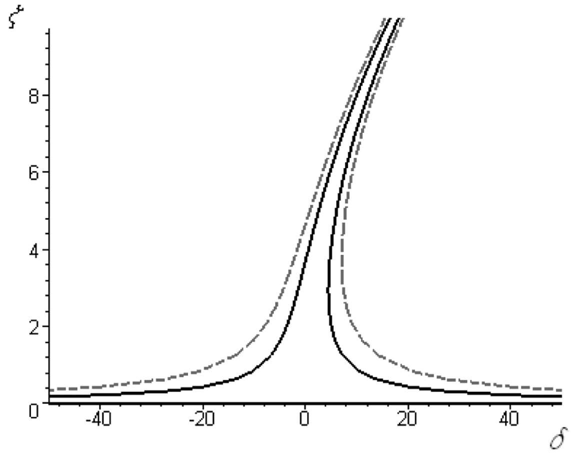

Fig. 6 Effect of force amplitude excitation on the frequency response of the plate ______ $F_0 = 10$, _ _ _ _ _ $F_0 = 20$

So that the discreatize Eq. (25) in the case of free vibration case will become as:

$$a_1 f^3 + a_2 f + a_3 f'' + a_4 f^2 f'' + a_5 f (f')^2 = 0 \tag{48}$$

where ω is the linear natural frequency.

Now, the Eq. (48) will be solved by the method of multiple scales. First this equation is rewritten as

$$f'' + \omega^2 f + \varepsilon (b_2 f^3 + b_3 f^2 f'' + b_4 f (f')^2) = 0 \tag{49}$$

Which b_1 to b_4 are defined in Eq. (27). Using the Eqs. (32) to (35) and equating the different order of ε equal to zero, it is obtained:

$$0(\varepsilon^0) = 0(1): \quad \frac{d^2 f_0}{dT_0^2} + \omega^2 f_0 = 0 \tag{50}$$

$$0(\varepsilon^1): \quad \frac{d^2 f_1}{dT_0^2} + \omega^2 f_1 = -2\frac{d^2 f_0}{dT_0 dT_1} - b_2 f_0^3 - b_3 f_0^2 \frac{d^2 f_0}{dT_0^2} - b_4 f_0 \left(\frac{df_1}{dT_0}\right)^2 \tag{51}$$

Solving Eq. (50) as described in Eq. (38) and substituting in Eq. (51), by eliminating the secular terms which is described in Eq. (39) for the free vibration case, one can obtain the solvability condition as:

$$2A' i\omega + [3b_2 + \omega^2(b_4 - 3b_3)]A^2\overline{A} = 0 \tag{52}$$

This is useful for calculating $A(T_1)$ in this case. As described in the forced vibration case, $A(T_1)$ is stated in the polar coordinate as:

$$A(T_1) = \frac{1}{2}\gamma(T_1)e^{i\beta(T_1)} \tag{53}$$

where $\gamma(T_1)$ is the amplitude and $\beta(T_1)$ is the phase of the oscillation of the plate. Substituting Eq. (53) in Eq. (52) and separating the real and imaginary components equal to zero, it is obtained:

$$\gamma' = 0 \tag{54}$$

$$\beta' = \frac{3}{8}b_2\omega\gamma^2(b_4 - 3b_3) \tag{55}$$

Eq. (54) means that the amplitude of the mode shape is constant and it is not a function of any time scales. This is because that the system does not posses any kind of damping. Also from Eq. (55) it is obtained:

$$\beta = \frac{3}{8}b_2\omega\gamma^2(b_4 - 3b_3)T_1 + \beta_0 \tag{56}$$

where β_0 is constant. Therefore, substituting from Eqs. (53) and (54) in Eq. (38) which can be used for free vibration case, it is obtained:

$$f_0(T_0, T_1) = \frac{1}{2}\gamma e^{i\left[\omega T_0 + \frac{3}{8}b_2\omega\gamma^2(b_4 - 3b_3)T_1 + \beta_0\right]} + cc \tag{57}$$

So by eliminating the secular terms, the Eq. (51) will become:

$$\frac{d^2 f_1}{dT_0^2} + \omega^2 f_1 = [-b_2 + \omega^2(b_3 + b_4)]A^3 e^{3i\omega T_0} + cc \tag{58}$$

The solution of this equation will be:

$$f_1(T_0, T_1) = \frac{\gamma^3}{64\omega^2}[b_2 - \omega^2(b_3 + b_4)]e^{i\left[\frac{9}{8}b_2\omega\gamma^2(b_4 - 3b_3)T_1 + 3\beta_0\right] + 3i\omega T_0} \tag{59}$$

Substituting Eqs. (57) and (59) in to Eq. (35) and converting it to triangular form, the time function will be:

$$f = \gamma\cos\theta + \varepsilon\frac{\gamma^3}{64\omega^2}[b_2 - \omega^2(b_3 + b_4)]\cos(3\theta) + \ldots \tag{60}$$

where θ and ω_N (Nonlinear natural frequency) are introduced as:

$$\theta = \omega_N t + \beta_0 \tag{61}$$

and

$$\omega_N = \omega + \frac{3b_2\omega^2(b_4 - 3b_3)}{8\omega}\varepsilon\gamma^2 + \ldots \tag{62}$$

The above equations are in the good agreement with the relation which is obtained by Nayfeh and Nayfeh (1994) which has been derived closed-form relation for the continuous systems. Eq. (62) means that the frequency of nonlinear oscillation is dependent on the parameters of system, the small parameter (relative thickness) ε and the amplitude of oscillation, which are characteristics of nonlinear systems.

The dependence of nonlinear frequencies with respect to amplitude (initial displacement of mid-point of plate) is illustrated in Fig. 7.

It is shown that for an elastic plate, by increasing the initial displacement, the nonlinear natural frequency is decreased quadratically.

The effect of small parameter on the amplitude and first frequency of plate is shown in Fig. 8.

It is shown that because there is no any damping in this system, the amplitude is constant and does not vary with respect to time, but the period of harmonic oscillation is decreased by increasing the small parameter.

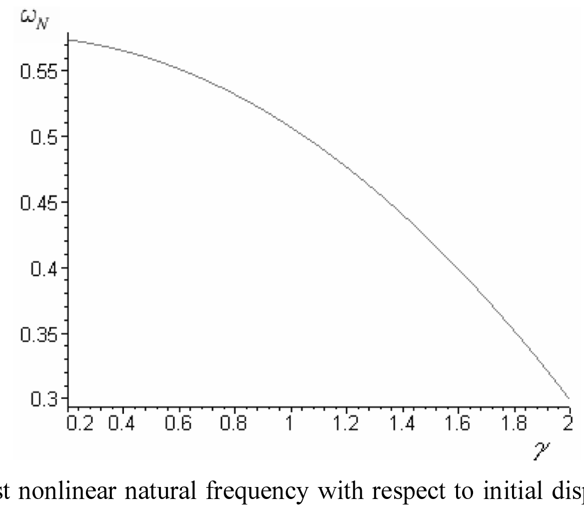

Fig. 7 Variation of first nonlinear natural frequency with respect to initial displacement of mid-point

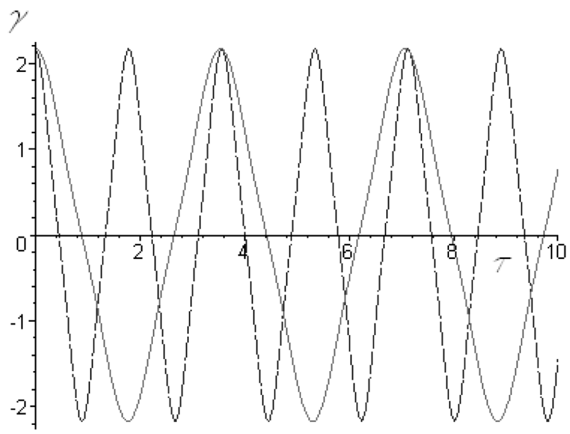

Fig. 8 Effect of small parameter on the displacement of the mid-point ($x = a/2$, $y = a/2$) (_____ $\varepsilon = 0.04$, ------ $\varepsilon = 0.01$)

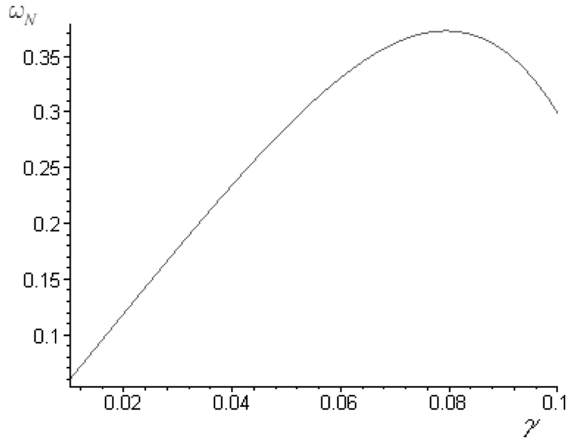

Fig. 9 Variation of first nonlinear frequency with respect to the ratio of thickness to dimension In amplitude $a = 2$

Fig. 9 shows the variation of the first nonlinear frequency of the plate with respect to the ratio of thickness to dimension, i.e., $r = h/a$.

It is shown that the frequency is increased with respect to the ratio of thickness to dimension and after a critical value of this ratio, the nonlinear frequency is decreased. For some values above the critical value of this ratio, the frequency will become negative. This means that the method which is describe here is valid when the ratio of thickness to dimension is small and therefore the Multiple Scales Method can be used.

To compare the above solution with respect to other solutions, the Eq. (38) has been solved by Runge-Kutta numerical method. This solution has been obtained using MAPLE 9 software and the following values for numerical parameters as $\nu = 0.3$, $h = 2$, $a = 50$. The numerical method and Multiple Scale Method results have been compared in the Fig. 10.

One can see from Fig. 6 that the numerical results have good agreements with the Multiple Scale

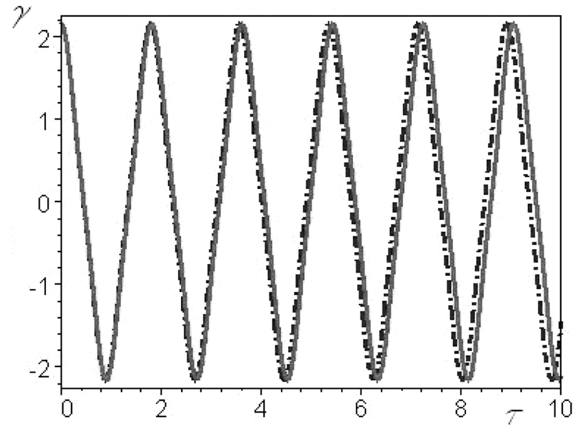

Fig. 10 Comparison the Numerical and Multiple scales method (Numerical method _____, Multiple Scale Methods -------) $\nu = 0.3$, $h = 2$, $a = 50$

method. But the Multiple Scale Method Solution has this advantage that it provides a closed-form solution with a good physical insight, whereas the numerical methods do not provide a closed form solution and lack this type of physical insights.

5. Conclusions

In this paper, first, the equations of motion for an isotropic elastic plate were derived. In this derivation, the effects of shear deformation and rotary inertia were considered. Then, by using an Airy stress function, these equations were converted to one coupled equation and a compatibility equation. Using the Galerkin method, a nonlinear differential equation with respect to time was obtained. This equation has nonlinearities in stiffness and inertia.

Using the method of multiple scales, this equation was solved for the forced vibration and free vibration cases. The advantage of the present solution is that the effects of nonlinearities can be determined accurately. Using the dimensional analysis, it is shown that the order of magnitude of nonlinear terms is smaller than the linear terms.

In the forced vibration case the frequency response of the plate has been calculated with the Multiple Scales Method. The effects of parameters such as thickness, Poisson's ratio and force amplitude excitation on the frequency response have been studied.

It is shown that by increasing the thickness of the plate, the nonlinearity effect of the system will increase due to an increase of the effect of rotary inertia and shear deformation. Also, an increase in the Poisson's ratio of the material will increase the nonlinearity of the plate, but it does not have any effect on the sharpness of the response curve. Also, it is shown that an increasing in the force amplitude of the excitation, the nonlinearity and deviation of frequency response of system, does not change. But the sharpness of the response curve will decrease.

In the free vibration case, the closed form solution for the amplitude and nonlinear natural frequency have been obtained and the effect of several system parameter have been investigated. It is shown that by increasing the initial displacement of the mid-point of the plate, the first nonlinear frequency is decreased quadratically. Also, it is shown that by increasing the ratio of thickness to dimension of the plate, the nonlinear frequency of plate will increase, but this result is valid for a special range of this ratio, which is the characteristic of Multiple Scales Method. In the end, by comparing the obtained results and numerical results, it is shown that the analytical results have good agreement with the numerical results.

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Notation

a, b	: dimensions of plate
E	: Young modulus
f	: time function
h	: thickness of pate
I	: moment of Inertia
N_x, N_y, N_{xy}	: in-plane internal forces

M_x, M_y, M_{xy}	: internal bending moments
Q_x, Q_y	: internal shear forces
T_0, T_1	: time scales
u, v	: in-plane displacement
w	: lateral displacement

Greek

ψ_x, ψ_y	: angle of rotation
ε	: non-dimensional small parameter
ϕ	: Airy stress function
ν	: Poisson's ratio
ρ	: density
ω	: linear natural frequency
ω_N	: nonlinear natural frequency
τ	: non-dimensional time parameter
ψ	: position function