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Vibrations and thermal stability of functionally graded spherical caps

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Abstract. Here, the axisymmetric free flexural vibrations and thermal stability behaviors of functionally graded spherical caps are investigated employing a three-noded axisymmetric curved shell element based on field consistency approach. The formulation is based on first-order shear deformation theory and it includes the in-plane and rotary inertia effects. The material properties are graded in the thickness direction according to the power-law distribution in terms of volume fractions of the constituents of the material. The effective material properties are evaluated using homogenization method. A detailed numerical study is carried out to bring out the effects of shell geometries, power law index of functionally graded material and base radius-to-thickness on the vibrations and buckling characteristics of spherical shells.

Keywords: functionally graded; vibration; spherical shell; thermal buckling; power law index.

1. Introduction

The demand for improved structural efficiency in space structures and nuclear reactors has resulted in development of a new class of materials, called functionally graded materials (FGMs) (Koizumi 1997, Suresh and Mortensen 1998, Fukui 1991). FGMs are microscopically inhomogeneous, in which the material properties vary smoothly and continuously from one surface of the material to the other surface and thus, distinguish FGMs from conventional composite materials. Typically, these materials are made from a mixture of ceramic and metal, or a combination of different materials. Further, varying the properties in FGMs in a continuous manner is achieved by gradually changing the volume fraction of the constituent materials. The advantages of using these materials are that they are able to withstand high-temperature gradient environment while maintaining their structural integrity, and they avoid the interface problem that exists in homogeneous composites. Furthermore, a mixture of ceramic and metal with a continuously varying

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volume fraction can be easily manufactured (Koizumi 1993, Yamaoka *et al*. 1993, Wetherhold *et al*. 1996). Although these materials are initially designed as thermal barrier materials for aerospace structural applications and fusion reactors, they are now employed for general use as structural elements for different applications. For example, a common structural element for such applications is the rectangular plate, for which several recent studies on static buckling, vibration and dynamic behaviors have been performed (Praveen and Reddy 1998, Lanhe 2004, Tauchert 1991, Ma and Wang 2003, Yang *et al*. 2003).

Among various structural elements, shell elements form an important class of structural components with many significant engineering applications such as vessels or vessel's enclosures. Studies pertaining to FGMs shell structures are mainly limited to thermal stress, deformation, and fracture analysis in the literature (Makino *et al.* 1994, Obata and Noda 1994, Takezono *et al.* 1994, Durodola and Adlington 1996, Oh *et al.* 2003, Dao *et al.* 1997, Weisenbek *et al.* 1997). Makino *et al.* (1994), Obata and Noda (1994), and Takezono *et al.* (1994) have investigated thermal stress of FGM shells whereas the discs and rotors have been examined based on analytical approach by Durodola and Adlington (1996), and Oh *et al.* (2003). The elasto-plastics deformation of FGM shell is studied in the work of Dao *et al*. (1997), and Weisenbek *et al*. (1997). Few transient dynamic analyses of cracked FGM structural components are also reported in the literature (Li *et al.* 2001, Zhang *et al.* 2003). Li *et al*. have analyzed the stress intensity factor of FGMs under dynamic situation whereas Zhang *et al.* studied the dynamic responses of cracked FGM structural components. The parametric instability analysis of functionally graded cylindrical shells under harmonic axial loading has been carried out by Loy *et al*. (1999) and Ng *et al*. (2001) whereas the effect of FGM materials on the parametric response of plate structures has been reported by Ng *et al*. (2000). He *et al.* (2001) and Ng *et al*. (2002) have studied the active control of FGM structures integrated with piezoelectric sensors and actuators. However, to the authors' knowledge, it must be stressed that work on the vibrations and stability behavior of functionally graded material spherical shells is not commonly yet available in the literature, and such study is immensely useful to the designers while optimizing the designs of FGMs spherical shell structures.

In the present work, a three-noded shear flexible axisymmetric curved shell element developed based on the field-consistency principle (Prathap and Ramesh Babu 1986, Ganapathi *et al*. 2003) is employed to analyze the axisymmetric vibration, and thermal stability of functionally graded material spherical caps. The material properties are graded in the thickness direction according to the power-law distribution in terms of volume fractions of the material constituents. The present formulation is validated considering isotropic case for which solutions are available. Numerical results are presented considering different values for shell geometrical parameter, power law index, and boundary conditions on the axisymmetric vibration and thermal stability behavior of functionally graded spherical caps.

2. Formulation

An axisymmetric functionally graded shell of revolution (radius a, thickness h) made of a mixture of ceramics and metals is considered with the coordinates s, θ and z along the meridional, circumferential and radial/thickness directions, respectively as shown in Fig. 1(a). The materials in outer ($z = h/2$) and inner ($z = -h/2$) surfaces of the spherical shell are ceramic and metal, respectively. The locally effective material properties are evaluated using homogenization method

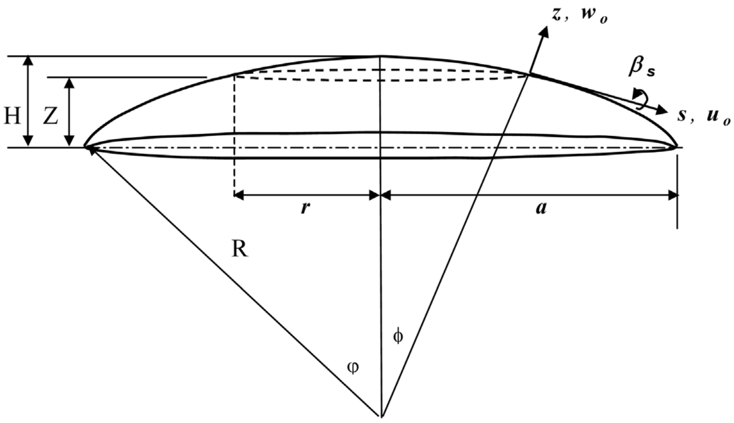

Fig. 1(a) Geometry and the coordinate system of a spherical cap

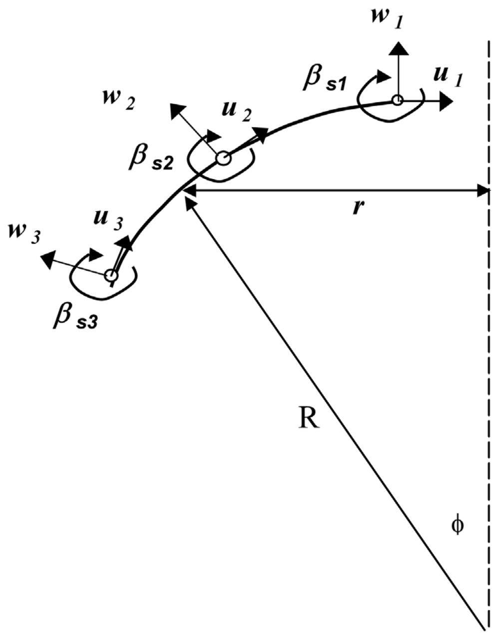

Fig. 1(b) Curved axisymmetric quadratic shell element

that is based on the Mori-Tanaka scheme. The effective bulk modulus K and shear modulus G of the functionally gradient material evaluated using the Mori-Tanaka estimates (Mori and Tanaka 1973, Benveniste 1987, Qian *et al*. 2004) are as

$$\frac{K-K_m}{K_c-K_m} = V_c/\left[1+(1-V_c)\frac{3(K_c-K_m)}{3K_m+4G_m}\right] \tag{1}$$

$$\frac{G-G_m}{G_c-G_m} = V_c/\left[1+(1-V_c)\frac{(G_c-G_m)}{G_m+f_1}\right] \tag{2}$$

where $f_1 = \dfrac{G_m(9K_m + 8G_m)}{6(K_m + 2G_m)}$.

Here, V is volume fraction of phase material. The subscripts m and c refer to the ceramic and metal phases, respectively. The volume-fractions of ceramic and metal phases are related by $V_c + V_m = 1$, and V_c is expressed as

$$V_c(z) = \left(\frac{2z + h}{2h}\right)^k \tag{3}$$

where k is the volume fraction exponent $(k \geq 0)$.

The effective values of Young's modulus E and Poisson's ratio υ can be found as

$$E(z) = \frac{9KG}{3K + G} \quad \text{and} \quad \upsilon(z) = \frac{3K - 2G}{2(3K + G)} \tag{4}$$

The locally effective heat conductivity coefficient κ is given as (Hatta and Taya 1985)

$$\frac{\kappa - \kappa_m}{\kappa_c - \kappa_m} = V_c / \left[1 + (1 - V_c)\frac{(\kappa_c - \kappa_m)}{3\kappa_m}\right] \tag{5}$$

The coefficient of thermal expansion α is determined in terms of the correspondence relation (Rosen and Hashin 1970)

$$\frac{\alpha - \alpha_m}{\alpha_c - \alpha_m} = \left(\frac{1}{K} - \frac{1}{K_m}\right) / \left(\frac{1}{K_c} - \frac{1}{K_m}\right) \tag{6}$$

The effective mass density ρ can be given by rule of mixture as (Senthil and Batra 2004)

$$\rho(z) = \rho_c V_c + \rho_m V_m \tag{7}$$

The temperature variation is assumed to occur in the thickness direction only and the temperature field is considered constant in the xy plane. In such a case, the temperature distribution along the thickness can be obtained by solving a steady-state heat transfer equation

$$-\frac{d}{dz}\left[\kappa(z)\frac{dT}{dz}\right] = 0, \quad T = T_c \text{ at } z = h/2; \quad T = T_m \text{ at } z = -h/2 \tag{8}$$

The solution of this boundary value problem provides the temperature distribution through the thickness of the plate (Cheng and Batra 2000).

By using the Mindlin formulation for an axi-symmetric spherical shell, the displacements at a point (s, z) are expressed as functions of the mid-plane displacements u_o, and w, and independent rotation β_s of the radial section, as

$$\begin{aligned} u(s, z, t) &= u_o(s, t) + z\,\beta_s(s, t) \\ w(s, z, t) &= w(s, t) \end{aligned} \tag{9}$$

where t is the time. The various strain components such as the membrane strains $\{\varepsilon_p\}$, bending strains $\{\varepsilon_b\}$, shear strains $\{\varepsilon_s\}$ are written as (Kraus 1967)

$$\{\varepsilon_p\} = \begin{Bmatrix} \dfrac{\partial u_o}{\partial s} + \dfrac{w}{R} \\ \dfrac{u_o \cos\phi}{r} + \dfrac{w \sin\phi}{r} \\ 0 \end{Bmatrix}$$

$$\{\varepsilon_b\} = \begin{Bmatrix} \dfrac{\partial \beta_s}{\partial s} \\ \dfrac{\beta_s \cos\phi}{r} \\ 0 \end{Bmatrix}$$

$$\{\varepsilon_s\} = \begin{Bmatrix} \beta_s + \dfrac{\partial w}{\partial s} - \dfrac{u_0}{R} \\ 0 \end{Bmatrix} \tag{10}$$

where $r = R\sin\phi$, R and ϕ are the radius of the parallel circle, radius of the sphere and meridional angle (Fig. 1a).

If $\{N\}$ represents the stress resultants (N_{ss}, $N_{\theta\theta}$, $N_{s\theta}$) and $\{M\}$ the moment resultants (M_{ss}, $M_{\theta\theta}$, $M_{s\theta}$), one can relate these to membrane strains $\{\varepsilon_p\}$ and bending strains $\{\varepsilon_b\}$ through the constitutive relations as

$$\{N\} = \begin{Bmatrix} N_{ss} \\ N_{\theta\theta} \\ N_{s\theta} \end{Bmatrix} = [A_{ij}]\{\varepsilon_p\} + [B_{ij}]\{\varepsilon_b\} - \{N^T\} \tag{11}$$

$$\{M\} = \begin{Bmatrix} M_{ss} \\ M_{\theta\theta} \\ M_{s\theta} \end{Bmatrix} = [B_{ij}]\{\varepsilon_p\} + [D_{ij}]\{\varepsilon_b\} - \{M^T\} \tag{12}$$

where the matrices $[A_{ij}]$, $[B_{ij}]$ and $[D_{ij}]$ $(i, j = 1, 2, 6)$ are the extensional, bending-extensional coupling and bending stiffness coefficients and are defined as $[A_{ij}, B_{ij}, D_{ij}] = \int_{-h/2}^{h/2} [\overline{Q}_{ij}](1, z, z^2)dz$. The thermal stress resultant $\{N^T\}$ and moment resultant $\{M^T\}$ are

$$\{N^T\} = \begin{Bmatrix} N^T_{ss} \\ N^T_{\theta\theta} \\ N^T_{s\theta} \end{Bmatrix} = \int_{-h/2}^{h/2} [\overline{Q}_{ij}] \begin{Bmatrix} \alpha(z) \\ \alpha(z) \\ 0 \end{Bmatrix} \Delta T(z)dz \tag{13}$$

$$\{M^T\} = \begin{Bmatrix} M_{ss}^T \\ M_{\theta\theta}^T \\ M_{s\theta}^T \end{Bmatrix} = \int_{-h/2}^{h/2} [\overline{Q}_{ij}] \begin{Bmatrix} \alpha(z) \\ \alpha(z) \\ 0 \end{Bmatrix} \Delta T(z) z dz \tag{14}$$

where the thermal coefficient of expansion $\alpha(z)$ is given by Eq. (6), and $\Delta T(z) = T(z) - T_0$ is temperature rise from the reference temperature T_0 at which there are no thermal strains.

Similarly the transverse shear force $\{Q\}$ representing the quantities $\{Q_{sz}, Q_{\theta z}\}$ is related to the transverse shear strains $\{\varepsilon_s\}$ through the constitutive relations as

$$\{Q\} = [E_{ij}]\{\varepsilon_s\}, \quad \text{where} \quad E_{ij} = \int_{-h/2}^{h/2} [\overline{Q}_{ij}] \kappa dz \tag{15}$$

Here $[E_{ij}]$ $(i, j = 4, 5)$ are the transverse shear stiffness coefficients, κ is the transverse shear correction factor for non-uniform shear strain distribution through the shell thickness, taken as 5/6. $\overline{Q}_{ij}$ are the stiffness coefficients and are defined as

$$\overline{Q}_{11} = \overline{Q}_{22} = \frac{E(z)}{1-\nu^2}; \quad \overline{Q}_{12} = \frac{\nu E(z)}{1-\nu^2}; \quad \overline{Q}_{16} = \overline{Q}_{26} = 0; \quad \overline{Q}_{44} = \overline{Q}_{55} = \overline{Q}_{66} = \frac{E(z)}{2(1+\nu)} \tag{16}$$

where the modulus of elasticity $E(z)$ is given by Eq. (4).

The strain energy functional U is given as

$$U(\delta) = (1/2) \int_A [\{\varepsilon_p\}^T [A_{ij}] \{\varepsilon_p\} + \{\varepsilon_p\}^T [B_{ij}] \{\varepsilon_b\} + \{\varepsilon_b\}^T [B_{ij}] \{\varepsilon_p\} + \{\varepsilon_b\}^T [D_{ij}] \{\varepsilon_b\} + \{\varepsilon_s\}^T [E_{ij}] \{\varepsilon_s\}] dA \tag{17}$$

In general, Eq. (17) can be rewritten as

$$U(\delta) = (1/2)\{\delta\}^T [\mathbf{K}] \{\delta\} \tag{18}$$

where $[\mathbf{K}]$ is the elastic stiffness matrix and δ is the vector of the degree of freedom associated to the displacement field in a finite element discretisation.

The kinetic energy (T) of the shell is given by

$$T(\delta) = 1/2 \int_A [p(\dot{u}_o^2 + \dot{w}_o^2) + I\dot{\beta}_s^2] dA = (1/2)\{\dot{\delta}\}^T [\mathbf{M}] \{\dot{\delta}\} \tag{19}$$

where $p = \int_{-h/2}^{h/2} \rho(z) dz$, $I = \int_{-h/2}^{h/2} z^2 \rho(z) dz$ and $\rho(z)$ is mass density which varies through the thickness of the spherical shell and is given by Eq. (7). $[\mathbf{M}]$ is the consistent mass matrix. The dot over the variable denotes derivative with respect to time.

The shell is subjected to temperature field and this, in turn, results in in-plane stress resultants $(N_{ss}^{th}, N_{s\theta}^{th})$. Thus, the potential energy (V) due to unit thermal pre-buckling stresses $(N_{ss}^{th}, N_{s\theta}^{th})$ developed during buckling can be written as

$$V(\delta) = \frac{1}{2}\int_A \left\{ N_{ss}^{th}\left(\frac{\partial w}{\partial s} - \frac{u_s}{R}\right)^2 + \frac{h^2}{12}\left[N_{ss}^{th}\left(\frac{\partial \beta_s}{\partial s}\right)^2 + N_{\theta\theta}^{th}\left(\frac{\beta_s \cos\phi}{r}\right)^2 \right] \right\} dA$$

$$= (1/2)\{\delta\}^T[\mathbf{K_G}]\{\delta\} \tag{20}$$

where $[\mathbf{K_G}]$ represents the geometric stiffness matrix due to unit thermal loads.

By minimization of total potential energy obtained $U_T (= U - V)$ contributed by Eqs. (18) and (20), the governing equations are derived for thermal stability case as (Zienkiewicz and Taylor 1989)

$$([\mathbf{K}] + \Delta T[\mathbf{K_G}])\{\delta\} = \{0\} \tag{21}$$

where ΔT $(= T_c - T_m)$ is the critical temperature difference.

Similarly, substituting Eqs. (18) and (19) in Lagrange's equation of motion of the form

$$d/dt[\partial(T-U)/\partial\dot{\delta}] - [\partial(T-U)/\partial\delta] = 0 \tag{22}$$

the governing equation for the free vibration case is obtained as (Zienkiewicz and Taylor 1989)

$$[\mathbf{M}]\{\ddot{\delta}\} + [\mathbf{K}]\{\delta\} = \{0\} \tag{23}$$

where $\{\ddot{\delta}\}$ is the acceleration vector. For free vibration case, assuming harmonic vibration, $\{\ddot{\delta}\} = -\omega^2\{\delta\}$, Eq. (23) leads to

$$([\mathbf{K}])\{\delta\} - \omega^2[\mathbf{M}]\{\delta\} = \{0\} \tag{24}$$

where ω is the natural frequency.

The frequency and the critical temperature difference can be calculated using standard eigenvalue extraction algorithm.

3. Element description

The axisymmetric three-noded curved shell element used here is a C^0 continuous shear flexible one and has 3 nodal degrees of freedoms (Fig. 1b). If the interpolation functions for three-noded element are used directly to interpolate the three field variables u_o, w and β_s in deriving the transverse shear and membrane strains, the element will lock and show oscillations in the shear and membrane stresses. Field consistency requires that the membrane and transverse shear strains must be interpolated in a consistent manner. Thus, β_s and u_o terms in the expression for $\{\varepsilon_s\}$ given in Eq. (10) have to be consistent with field function $\partial w/\partial s$ as shown in the works of Prathap and Ramesh Babu (1986). Similarly w and u_o terms in the expression of $\{\varepsilon_p\}$ given in Eq. (10) have to be consistent with the field function $\partial u_o/\partial s$ respectively. This is achieved by using the field redistributed substitute shape functions to interpolate those specific terms that must be consistent as described by Prathap and Ramesh Babu (1986) and Ganapathi *et al.* (2003). The element derived in this fashion behaves very well for both thick and thin situations, and permits the greater flexibility in the choice of integration order for the energy terms. It has good convergence and has no spurious rigid modes. The original shape functions used for the present three-noded quadratic element are

$$N_1 = (\xi^2 - \xi)/2, \quad N_2 = (1 - \xi^2), \quad N_3 = (\xi^2 + \xi)/2 \tag{25}$$

whereas the redistributed substitute shape functions used for avoiding locking syndrome here are

$$\overline{N}_1 = (1/3 - \xi)/2, \quad \overline{N}_2 = 2/3, \quad \overline{N}_3 = (1/3 + \xi)/2 \tag{26}$$

4. Results and discussion

In this section, we use the above formulation to investigate the effect of parameters like gradient index, shell geometrical parameter on the axisymmetric free flexural vibration characteristics and thermal buckling of functionally graded material spherical caps. Since the finite element used in this study is based on field consistency approach, an exact integration is employed to evaluate all the strain energy terms. The shear correction factor, which is required in a first order theory to account for the variation of transverse shear stresses, is taken as 5/6. For the present study based on progressive mesh refinement, a 15-element idealization is found to be adequate in modeling the spherical caps.

Fig. 2(a) shows the typical variation of the volume fractions of ceramic in the thickness direction z for the FGM spherical cap. The outer surface is ceramic rich and the inner surface is metal rich. The typical temperature variation through the thickness direction is presented in Fig. 2(b) and it can be noted that the temperature variation in the thickness of functionally graded shell is nonlinear compared to those of pure ceramic and metal cases ($k = 0$ and $k = 100$). The FGM spherical shell considered here consists of aluminum and alumina (Lanhe 2004). The Young's modulus, conductivity and the coefficient of thermal expansion for alumina is $E_c = 380$ GPa, $\kappa_c = 10.4$ W/mK, $\alpha_c = 7.4 \times 10^{-6}$ 1/°C, and for aluminum is $E_m = 70$ GPa, $\kappa_m = 204$ W/mK, $\alpha_m = 23 \times 10^{-6}$ 1/°C, respectively. The shell is of uniform thickness and boundary conditions considered here are:

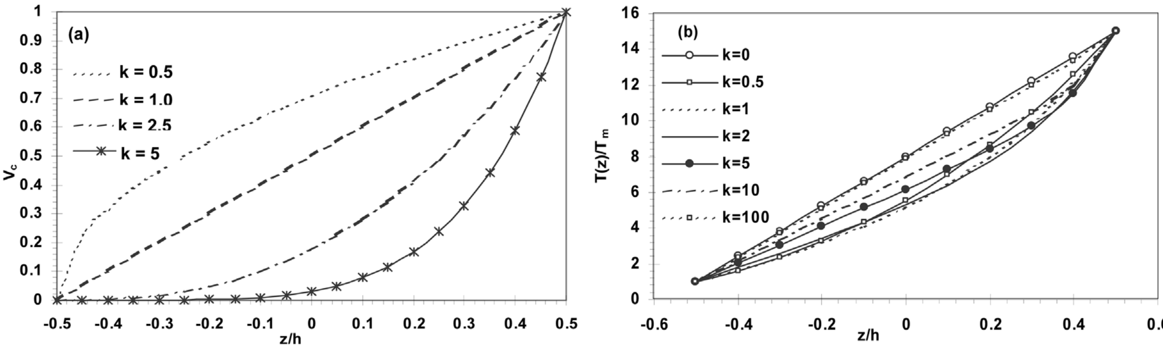

Fig. 2 Variation of volume fraction of ceramic and temperature through thickness; (a) Volume fraction of ceramic, (b) Temperature

Table 1(a) Comparison of fundamental frequency, Ω for isotropic shallow spherical shell

H/h	Fundamental frequency	
	Sathyamoorthy (1994)	Present
2	6.14	6.42
5	13.08	13.37

Table 1(b) Comparison of critical thermal buckling strain ε_T for isotropic hemi spherical shell

h/R	Ganesan and Ravikiran (2005)	Present
0.01	0.00407	0.00424
0.02	0.00844	0.00840
0.03	0.01204	0.01251
0.04	0.01624	0.01657
0.05	0.02034	0.02052
0.06	0.02379	0.02446
0.07	0.02764	0.02844
0.08	0.03164	0.03222
0.09	0.03530	0.03584
0.10	0.03927	0.03944

simply supported:

$$u_0 = w = 0 \text{ and } M_s = 0 \text{ (Natural BC) on } r = a$$

clamped support:

$$u_0 = w = \beta_s \quad \text{on} \quad r = a$$

Before proceeding for the free flexural vibration characteristic study of FG spherical cap, the formulation developed herein is validated against the available clamped isotropic spherical shells results pertaining to the free vibrations and thermal buckling cases in Tables 1(a) and (b), respectively. Here, the nondimensional frequency, Ω is defined as $\Omega = \omega(a/h)(\rho_c a^2/E_c)^{1/2}$, where ρ_c and E_c are the mass density and Young's modulus of ceramic, respectively. The results are found to be in good agreement with the existing solutions (Sathyamoorthy 1994, Ganesan and Ravikiran 2005).

Next, the detailed investigations for free flexural vibrations of spherical caps are carried out for different geometrical parameters and material power law index, k. Fig. 3 highlights the non-dimensional fundamental frequencies of simply supported FGM spherical caps for different values of thickness-to-radius ratio, material power law index and different spherical angle. It is observed that the increase in material power law index value results in decrease in non-dimensional frequency

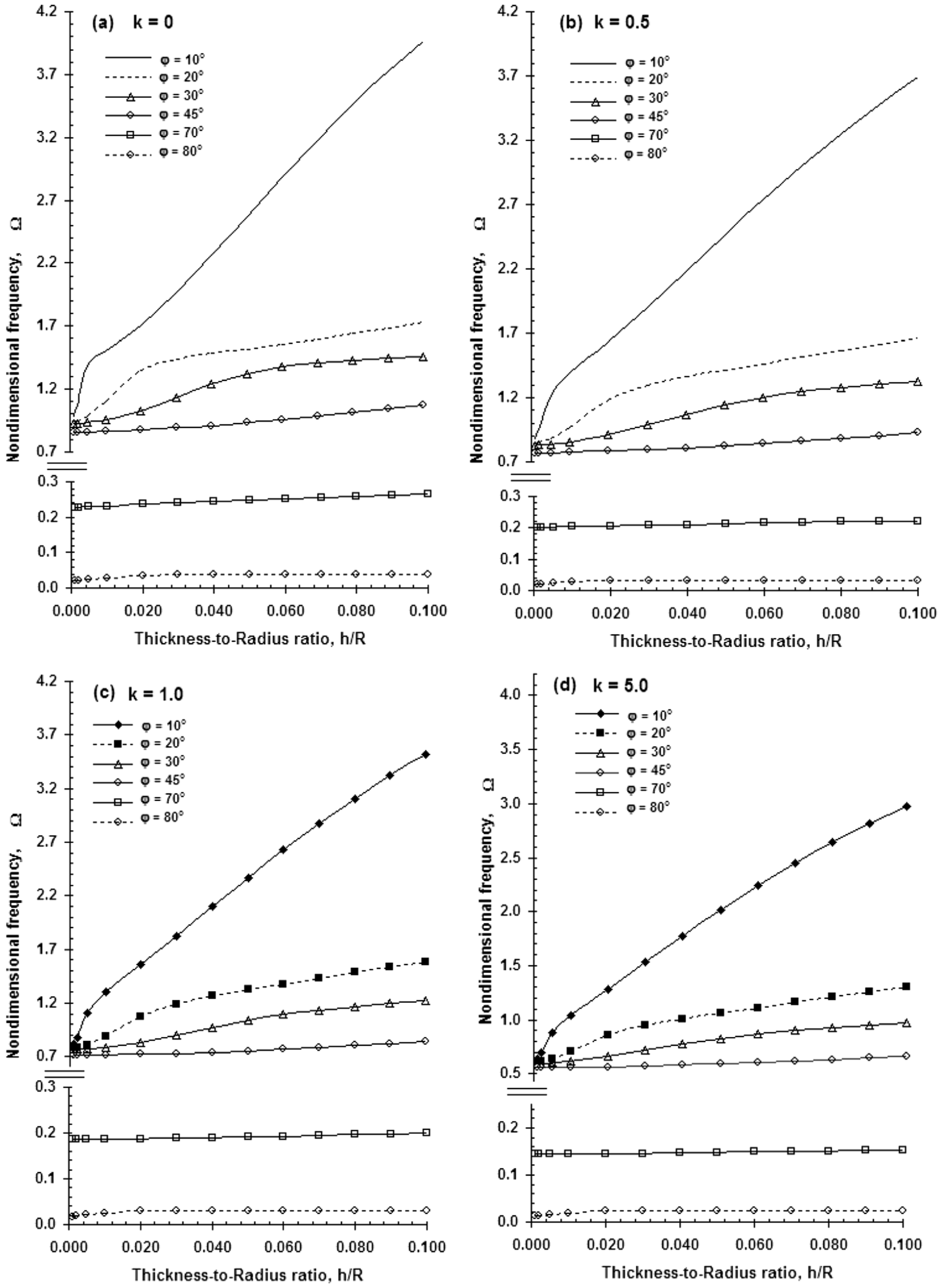

Fig. 3 Variation of nondimensional frequency Ω for a simply supported FG spherical shell for different gradient index

value. This is attributed due to the stiffness reduction because of the increase in the metallic volumetric fraction and the introduction of different stiffness couplings due to elastic properties variation through the thickness of FGM shell. It can also be opined that the frequency value

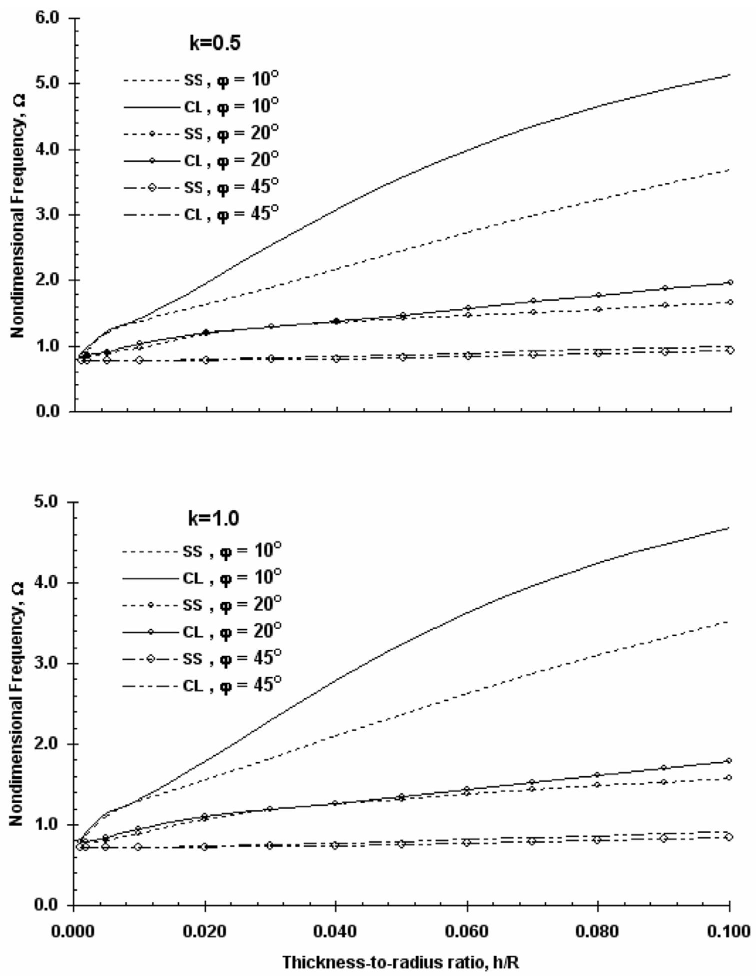

Fig. 4 Comparison of nondimensional frequency, Ω for simply supported and clamped FG spherical shells

increases with the increase in thickness-to-radius ratio. However, the rate of increase in non-dimensional frequency value is high for shallow spherical cases compared to deep shells. The effect of boundary conditions on frequency can be further viewed from Fig. 4 for two values of material index. It is noticed from Fig. 4 that the frequency values for clamped case is higher than those of simply supported shell, as expected and the difference in frequency values is with respect to thickness-to-radius however less for deep shells.

Similar analysis for the thermal buckling behavior of FGM spherical caps with simply supported boundary condition has been done considering different values of thickness-to-radius ratio and geometrical parameter. The results are plotted in Fig. 5. It can be concluded that the influence of material power law index and spherical included angle (φ) on critical values is qualitatively similar to those of vibration case, i.e., reduction in thermal buckling temperature difference with increase in the values of material index and deepness of the shell. However, the critical buckling temperature difference is quite high for very shallow shells compared to those of deep cases. For moderately

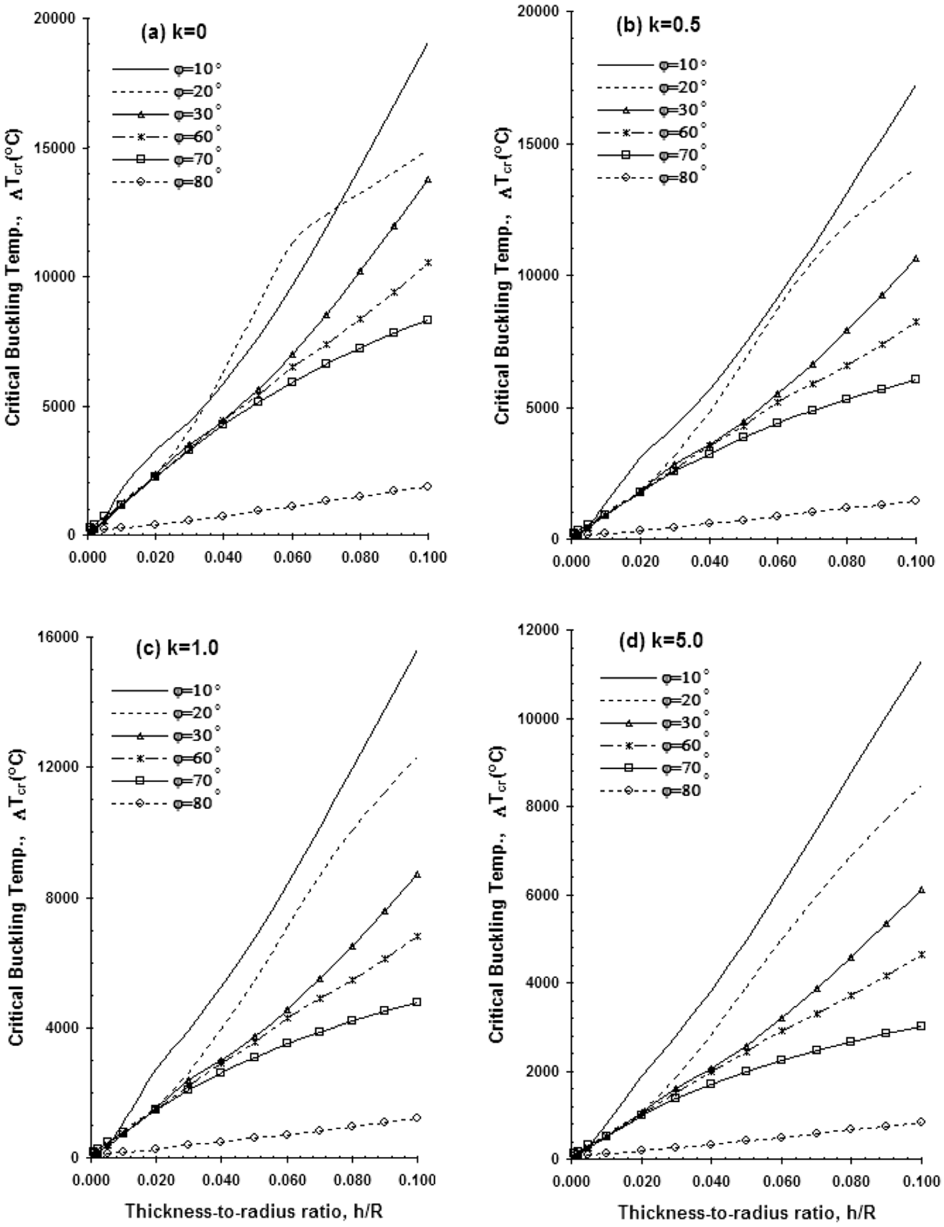

Fig. 5 Variation of axisymmetric critical buckling temperature difference, ΔT_{cr} (°C) for a simply supported FG spherical shell

deep shell structures, the change in the buckling values is noticeable for higher values of thickness-to-radius ratio. The critical buckling temperature for ceramic is higher than the microscopically heterogeneous mixture of ceramic and metal, as expected and this is mainly due to the increase in metallic volumetric fraction. Furthermore, the results pertaining to FGM hemi-spherical shells are highlighted in Fig. 6 and the observation is similar to those of other spherical caps.

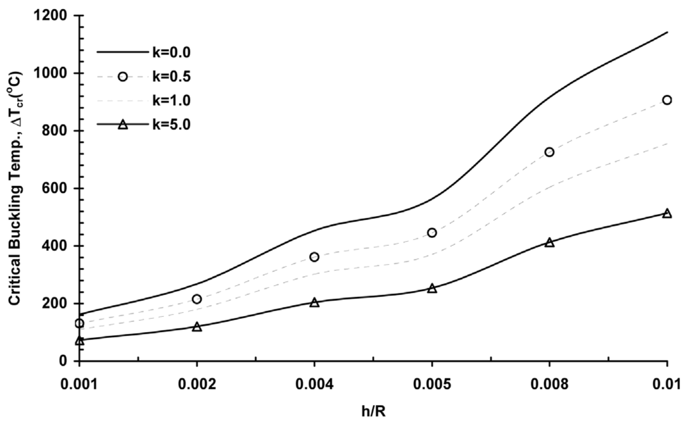

Fig. 6 Variation of axisymmetric critical buckling temperature difference, ΔT_{cr} (°C) for a simply supported FG hemi-spherical shell

5. Conclusions

Axisymmetric free flexural vibrations and thermal stability of FGM spherical caps have been investigated using a three-noded axisymmetric curved shell element employing a field consistency approach. Numerical results obtained here for an isotropic case are found to be in good agreement with the previous findings. From the detailed parametric study, it is observed that the frequency and critical buckling temperature decrease with the increase in metallic volume fraction and spherical angle whereas they increase with the increase in thickness-to-radius ratio of the shells. It is hoped that this study will be useful for the designers while optimizing the FGM based spherical shell structures.

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