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Propagation of non-uniformly modulated evolutionary random waves in a stratified viscoelastic solid

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Abstract. The propagation of non-uniformly modulated, evolutionary random waves in viscoelastic, transversely isotropic, stratified materials is investigated. The theory is developed in the context of a multi-layered soil medium overlying bedrock, where the material properties of the bedrock are considered to be much stiffer than those of the soil and the power spectral density of the random excitation is assumed to be known at the bedrock. The governing differential equations are first derived in the frequency/wave-number domain so that the displacement response of the ground may be computed. The eigen-solution expansion method is then used to solve for the responses of the layers. This utilizes the precise integration method, in combination with the extended Wittrick-Williams algorithm, to obtain all the eigen-solutions of the ordinary differential equation. The recently developed pseudo-excitation method for structural random vibration is then used to determine the solution of the layered soil responses.

Keywords: layered material; precise integration; pseudo-excitation method; extended Wittrick-Williams algorithm; wave propagation; random vibration.

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1. Introduction

Wave propagation in stratified or layered material is an important issue for many physical and industrial problems, ranging from seismic wave propagation along layers of earth to ultrasonic wave propagation in fiber-reinforced composite materials. The inverse problem of determining the material properties of stratified layers is also important in the exploration of deep-underground water, petroleum or mineral resources by means of surface explosion or remote sensing techniques.

Many different models and approaches to wave propagation in a stratified medium can be found in the literature. The medium can be solid, porous or fluid, while the properties of the layers may be transversely isotropic or anisotropic, elastic or viscoelastic, linear or nonlinear, homogeneous or inhomogeneous. In general there are two kinds of problem: the first is the forward problem, in which the main purpose is to determine the wave field or the reflection and transmission at the interface; while the second is its inverse problem, in which the main purpose is to determine the material parameters of the medium. Early treatments of such problems were analytically-based (Timoshenko and Goodier 1951, Ewing *et al.* 1957, Achenback 1973, Graff 1975, Aki and Richards 1980, Brekhovskikh 1980, Doyle 1989). More recently Rizzi and Doyle (1992a,b) developed a spectral element approach based on the Fast Fourier Transform technique (FFT) and applied it to the study of transient waves in elastic layered solids. Similar techniques were proposed by Khoury *et al.* (2002a,b) for determining the parameters of the layers.

Transient wave propagation has also been investigated when considering viscoelastic media (Alshaikh *et al.* 2001) as well as anisotropic stratified media (Thomson 1997, Caviglia and Morro 2000a,b, Caviglia and Morro 2002, Verma 2002, Gulyayev *et al.* 2003). A method of simulating the propagation of elastic waves in stratified transversely inhomogeneous media using a generalized Thomson-Haskell matrix method was proposed by Zhang and Li (1997), while the propagation of waves in stratified transversely orthogonal porous media has been investigated by Vashishth and Khurana (2002).

The conventional way of dealing with the problem of wave propagation is to transform the partial differential equation (PDE) into the frequency/wave-number domain. However, deriving an analytical solution is generally very difficult mathematically, thus matrix methods may be more effective (Kennett 1983). Nevertheless, it is still quite difficult to obtain the numerical solution with sufficient accuracy, although the precise integration method (Zhong 1994) is quite effective for such elastic wave propagation problems.

The randomness in wave propagation in layered material comes mainly from random waves, random parameters of materials and random interfaces or boundaries (Mamolis 2002). There are many publications on random material parameters and random interfaces (Mamolis and Shaw 1996, 1997, Zhang and Shinozuka 1996, Zhang and Lou 2001, Mamolis 2002). However the random property caused by the input waves has rarely been studied. In this paper, the propagation of non-uniformly modulated evolutionary random waves in transversely isotropic viscoelastic media will be solved using the precise integration method (PIM) (Zhong 1994) combined with the extended Wittrick-Williams (W-W) algorithm (Zhong *et al.* 1997) and the pseudo excitation method (PEM) (Lin 1992, Lin *et al.* 1993, 1994, 1995a,b,c).

In the frequency domain, random excitations and responses are usually described in terms of the power spectral density (PSD). As the bedrock is a continuum, it will generate an infinite number of plane incident waves. If these waves are assumed to be plane waves travelling in the x-y plane, then a spectrum associated with the wave-number can be obtained through use of a Fourier transform.

Priestley (1967) suggested a kind of non-uniformly modulated evolutionary random excitation that can be described by means of the Riemann-Stieltjes integration. However, the relative computation is rather difficult. In this paper, by using PEM, the non-uniformly modulated evolutionary random excitations are transformed into deterministic pseudo excitations, so that the problem can be solved by means of a transient direct integration.

In the work that follows, the non-uniformly modulated evolutionary random excitations of the bedrock are given in the frequency/wave-number domain, from which the time dependent PDE system is developed. The parameters of this system are deterministic, whereas the boundary motion in the bedrock is random. This is similar to the random vibration of a structure subjected to excitation by the ground. Clearly, PEM is applicable to this system. The key is to compute the response of the ground under the pseudo bedrock excitations for any given frequency and wave-number. In addition, by virtue of the extended W-W algorithm, the participating eigen-pairs may be obtained and used in the mode superposition prior to using PIM to solve the reduced partial differential equations. The PSD functions of the ground excitation $S_{uu}(\kappa_x, \kappa_y, \omega)$, $S_{vv}(\kappa_x, \kappa_y, \omega)$ and $S_{ww}(\kappa_x, \kappa_y, \omega)$ in the frequency/wave-number domain are then computed and correspond to the ground excitations that structures on the ground would be subjected to.

The W-W algorithm is highly accurate for eigenvalue extraction and accurate modes can be obtained by solving the dual equations. PEM is likewise an efficient and accurate method for computing the random responses of any linear system, while PIM can solve linear ordinary differential equations to the working accuracy of the host computer. Therefore, except for taking a limited number of modes, the proposed method is very accurate.

2. Fundamental equations

Consider the propagation of plane waves in the x-y plane in stratified transversely isotropic media. Let the z axis point downwards with $z = 0$ being at the free surface, see Fig. 1. The ith layer is separated by the horizontal planes $z = z_i (i = 1, 2, \ldots, l)$, where $z_i < z_j$ when $i < j$. The lowest boundary is at $z = z_l$, where enforced displacements (random excitations) $\hat{u}_g$, $\hat{v}_g$ and $\hat{w}_g(x, y, t)$ are

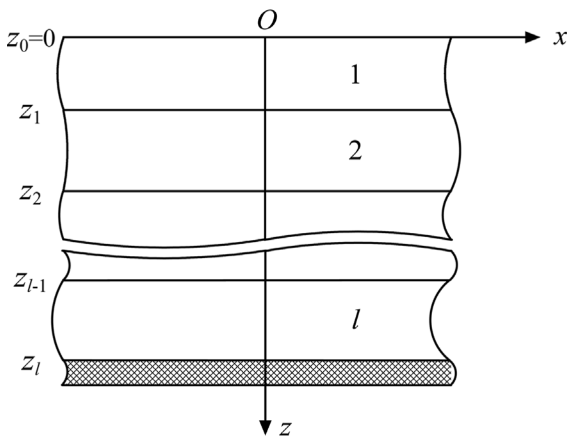

Fig. 1 Stratified material

given. Let $\hat{u}, \hat{v}$ and $\hat{w}$ be the displacements along the inertia coordinates, then the strain-displacement relations are

$$\hat{\varepsilon}_x = \partial\hat{u}/\partial x,\ \hat{\varepsilon}_y = \partial\hat{v}/\partial y,\ \hat{\varepsilon}_z = \partial\hat{w}/\partial z,\ \hat{\gamma}_{xy} = \partial\hat{v}/\partial x + \partial\hat{u}/\partial y,$$
$$\hat{\gamma}_{xz} = \partial\hat{w}/\partial x + \partial\hat{u}/\partial z,\ \hat{\gamma}_{yz} = \partial\hat{v}/\partial z + \partial\hat{w}/\partial y \tag{1}$$

The viscoelastic isotropic stress-strain relationships are

$$P\begin{Bmatrix}\hat{\sigma}_x\\ \hat{\sigma}_y\\ \hat{\sigma}_z\\ \hat{\tau}_{xy}\\ \hat{\tau}_{xz}\\ \hat{\tau}_{yz}\end{Bmatrix} = \mathbf{R}\times Q\begin{Bmatrix}\hat{\varepsilon}_x\\ \hat{\varepsilon}_y\\ \hat{\varepsilon}_z\\ \hat{\gamma}_{xy}\\ \hat{\gamma}_{xz}\\ \hat{\gamma}_{yz}\end{Bmatrix},\quad \mathbf{R} = \begin{bmatrix}\lambda+2G & \lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda+2G & \lambda & 0 & 0 & 0\\ \lambda & \lambda & \lambda+2G & 0 & 0 & 0\\ 0 & 0 & 0 & G & 0 & 0\\ 0 & 0 & 0 & 0 & G & 0\\ 0 & 0 & 0 & 0 & 0 & G\end{bmatrix},\quad \begin{aligned} P &= \sum_{k=0}^{m} p_k\frac{d^k}{dt^k}\\ Q &= \sum_{k=0}^{m} q_k\frac{d^k}{dt^k}\end{aligned} \tag{2}$$

Here: λ and G are the Lamé constants which may be different for different layers; P, Q are differential operators; p_k, q_k are the viscoelastic material constants; $p_0 = 1,\ p_1 \neq 0,\ q_1 \neq 0$ corresponds to the Maxwell fluid; $q_0 \neq 0,\ q_1 \neq 0$ corresponds to the Kelvin solid; and $q_0 \neq 0,\ p_1 \neq 0,\ q_1 \neq 0$ with $q_1 > p_1 \cdot q_0$ corresponds to the three-parameter solid.

The equations of motion can then be written as

$$\frac{\partial\hat{\sigma}_x}{\partial x} + \frac{\partial\hat{\tau}_{xy}}{\partial y} + \frac{\partial\hat{\tau}_{xz}}{\partial z} = \frac{\rho\partial^2\hat{u}}{\partial t^2},\quad \frac{\partial\hat{\tau}_{yx}}{\partial x} + \frac{\partial\hat{\sigma}_y}{\partial y} + \frac{\partial\hat{\tau}_{yz}}{\partial z} = \frac{\rho\partial^2\hat{v}}{\partial t^2},\quad \frac{\partial\hat{\tau}_{zx}}{\partial x} + \frac{\partial\hat{\tau}_{zy}}{\partial y} + \frac{\partial\hat{\sigma}_z}{\partial z} = \frac{\rho\partial^2\hat{w}}{\partial t^2} \tag{3}$$

in which ρ is the density, which may have different values for different layers.

3. The non-stationary modulated evolutionary random field

Priestley (1967) suggests that the non-uniformly modulated evolutionary random excitation $f(t)$ can be expressed as a Riemann-Stieltjes integration

$$f(t) = \int_{-\infty}^{+\infty} A(\omega, t)\exp(-i\omega t)d\alpha(\omega) \tag{4}$$

in which $\alpha(\omega)$ satisfies

$$f_g(t) = \int_{-\infty}^{+\infty}\exp(-i\omega t)d\alpha(\omega),\quad E[d\alpha^*(\omega_1)d\alpha(\omega_2)] = S(\omega_1)\delta(\omega_1 - \omega_2)d\omega_1 d\omega_2 \tag{5}$$

where $f_g(t)$ is a stationary random excitation with PSD $S(\omega)$ given, δ is the Dirac function and the superscript asterisk represents the complex conjugate.

In the present paper, the bedrock excitation $f(x, y, t)$ is a random field, which can also be written as a Riemann-Stieltjes integration

$$f(x, y, t) = \int_{-\infty}^{+\infty} A(\kappa_x, x, \kappa_y, y, \omega, t)\exp[i(\kappa_x x + \kappa_y y - \omega t)]d\alpha(\kappa_x, \kappa_y, \omega) \tag{6}$$

in which $\alpha(\kappa_x, \kappa_y, \omega)$ satisfies

$$f_g(x, y, t) = \int_{-\infty}^{+\infty} \exp[i(\kappa_x x + \kappa_y y - \omega t)] d\alpha(\kappa_x, \kappa_y, \omega) \tag{7}$$

$$\begin{aligned} &E[d\alpha^*(\kappa_{x1}, \kappa_{y1}, \omega_1) d\alpha(\kappa_{x2}, \kappa_{y2}, \omega_2)] \\ &= S(\kappa_{x1}, \kappa_{y1}, \omega_1)\delta(\kappa_{x1} - \kappa_{x2})\delta(\kappa_{y1} - \kappa_{y2})\delta(\omega_1 - \omega_2) d\kappa_{x1} d\kappa_{x2} d\kappa_{y1} d\kappa_{y2} d\omega_1 d\omega_2 \end{aligned} \tag{8}$$

where $f_g(x, y, t)$ is a stationary random field with PSD $S(\kappa_x, \kappa_y, \omega)$ given.

In this paper, it is assumed that A varies with ω and t, but not with x and y, that means the bedrock random excitation is non-uniformly modulated with the time coordinate t, but not with the space coordinates x and y. Thus, the random excitation of the bedrock can be written as

$$\{\hat{u}_g, \hat{v}_g, \hat{w}_g\}^T = \int_{-\infty}^{+\infty} A(\omega, t) \exp[i(\kappa_x x + \kappa_y y - \omega t)] \{d\alpha(\kappa_x, \kappa_y, \omega), d\beta(\kappa_x, \kappa_y, \omega), d\lambda(\kappa_x, \kappa_y, \omega)\}^T \tag{9}$$

$$\{x_g(x, y, t), y_g(x, y, t), z_g(x, y, t)\}^T = \int_{-\infty}^{+\infty} \exp[i(\kappa_x x + \kappa_y y - \omega t)] \{d\alpha, d\beta, d\gamma\}^T \tag{10}$$

where $x_g(x, y, t), y_g(x, y, t), z_g(x, y, t)$ are the stationary random components, their PSDs $S_{xg}(\kappa_x, \kappa_y, \omega)$, $S_{yg}(\kappa_x, \kappa_y, \omega)$, $S_{zg}(\kappa_x, \kappa_y, \omega)$ are known, and $\alpha(\kappa_x, \kappa_y, \omega), \beta(\kappa_x, \kappa_y, \omega)$, $\lambda(\kappa_x, \kappa_y, \omega)$ satisfy Eq. (8).

4. Pseudo Excitation Method (PEM)

The ground response PSDs $S_{uu}(\kappa_x, \kappa_y, \omega, t), S_{vv}(\kappa_x, \kappa_y, \omega, t), S_{ww}(\kappa_x, \kappa_y, \omega, t)$ are to be computed for a series of specified κ_x, κ_y, ω. To achieve this, the pseudo excitation method will be used (Lin 1992, Lin *et al.* 1993, 1994, 1995a,b,c, 1997, 2005). The pseudo excitations of the bedrock can be expressed as

$$\{\tilde{u}_g, \tilde{v}_g, \tilde{w}_g\}^T = \{\sqrt{S_{xg}}, \sqrt{S_{yg}}, \sqrt{S_{zg}}\}^T A(\omega, t) \exp[i(\kappa_x x + \kappa_y y - \omega t)] \tag{11}$$

If u_0, v_0, w_0 are the ground responses due to these pseudo excitations, then $S_{uu} = |u_0|^2$, $S_{vv} = |v_0|^2$, $S_{ww} = |w_0|^2$ will be the PSD functions of the corresponding ground displacements. Therefore the problem is reduced to the transient response analysis due to the deterministic pseudo excitations.

Assume the wave-numbers along the x, y directions are κ_x, κ_y. All unknowns can then be expressed as

$$\begin{aligned} &\hat{u} = u(z, t)\exp[i(\kappa_x x + \kappa_y y)], \quad \hat{v} = v(z, t)\exp[i(\kappa_x x + \kappa_y y)] \\ &\hat{\varepsilon}_x = \varepsilon_x(z, t)\exp[i(\kappa_x x + \kappa_y y)], \quad \hat{\sigma}_x = \sigma_x(z, t)\exp[i(\kappa_x x + \kappa_y y)] \end{aligned} \tag{12}$$

in which $u, v, \sigma_x, \varepsilon_x$ are all functions of z, t and κ_x, κ_y. Let

$$\mathbf{q} = \{u, v, w\}^T, \quad g(t) = A(\omega, t)\exp(-i\omega t) \tag{13}$$

Substituting Eq. (12) into Eqs. (1) and (2) and substituting the results into Eq. (3), gives the following equations in matrix form as

$$Q[\mathbf{K}_{22}\mathbf{q}'' + (\mathbf{K}_{21} - \mathbf{K}_{12})\mathbf{q}' - \mathbf{K}_{11}\mathbf{q}] = P\rho\ddot{\mathbf{q}} \tag{14}$$

in which $\dot{(\#)} = \partial(\#)/\partial t$, $(\#)' = \partial(\#)/\partial z$, and

$$\mathbf{K}_{22} = \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & \lambda+2G \end{bmatrix}, \quad \mathbf{K}_{21} = -\mathbf{K}_{12}^T = i\kappa_x \begin{bmatrix} 0 & 0 & G \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix} + i\kappa_y \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & G \\ 0 & \lambda & 0 \end{bmatrix}$$

$$\mathbf{K}_{11} = \kappa_x^2 \begin{bmatrix} \lambda+2G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} + \kappa_y^2 \begin{bmatrix} G & 0 & 0 \\ 0 & \lambda+2G & 0 \\ 0 & 0 & G \end{bmatrix} + \kappa_x\kappa_y \begin{bmatrix} 0 & \lambda+G & 0 \\ \lambda+G & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{15}$$

We can now constitute a dual vector

$$\mathbf{p} = \{\tau_{xz}, \tau_{yz}, \sigma_z\}^T \tag{16}$$

which satisfies

$$P\mathbf{p} = Q(\mathbf{K}_{22}\mathbf{q}' + \mathbf{K}_{21}\mathbf{q}) \tag{17}$$

The boundary and continuity conditions are

$$\mathbf{p}(0, t) = \mathbf{0} \text{ at } (z = 0); \ \mathbf{q}, \mathbf{p} \text{ continuous at } z_i(0 < i < l); \ \mathbf{q}(z_l, t) = \mathbf{s}g(t)$$

$$\mathbf{s} = \{\sqrt{S_{xg}}, \sqrt{S_{yg}}, \sqrt{S_{zg}}\}^T \tag{18}$$

Let $k = \max(m, n+2)$, then the initial conditions are

$$\frac{\partial^j \mathbf{q}}{\partial t^j}(z, 0) = \mathbf{0}, \quad (j = 0, 2, \ldots, k-1) \tag{19}$$

5. Processing of the inhomogeneous boundary conditions

In order to use the mode-superposition scheme to reduce the system, the inhomogeneous boundary conditions at $z = z_l$ must be transformed into a homogeneous form. Firstly, let us constitute a matrix $\mathbf{\Omega}_r$ for the r-th layer, which is continuous in $z_i(0 < i < l)$, and satisfies

$$\mathbf{K}_{22,r}\mathbf{\Omega}_r' + \mathbf{K}_{21,r}\mathbf{\Omega}_r = \mathbf{0}, \quad \mathbf{\Omega}_l(z_l) = \mathbf{I} \tag{20}$$

Thus the displacement field $\mathbf{\Omega}_r\mathbf{s}g(t)$ will be $\mathbf{s}g(t)$ at the bedrock ($z = z_l$), which is equal to the displacement $\mathbf{q}$ that takes place at the bedrock. Furthermore, $\mathbf{\Omega}_r\mathbf{s}g(t)$ is continuous at the interface. According to Eq. (17), the dual vector induced by the displacement $\mathbf{\Omega}_r\mathbf{s}g(t)$ is $\mathbf{0}$. If we let $\bar{\mathbf{q}} = \mathbf{q} - \mathbf{\Omega}\mathbf{s}g(t)$, then the value of the displacement $\bar{\mathbf{q}}$ at the bedrock is $\mathbf{0}$. As $\bar{\mathbf{q}}$ is continuous at the interfaces, the dual vector $\bar{\mathbf{p}}$ induced by $\bar{\mathbf{q}}$ is equal to that induced by $\mathbf{q}$. According to Eq. (18) and the analysis above, the boundary and the continuity conditions on the interface for displacement $\bar{\mathbf{q}}$ as well as the dual vector $\bar{\mathbf{p}}$ are: $\bar{\mathbf{p}}(0, t) = \mathbf{0}$, $\bar{\mathbf{q}}$, $\bar{\mathbf{p}}$ continuous at $z_i(0 < i < l)$ and

$\bar{\mathbf{q}}(z_l, t) = \mathbf{0}$. Therefore the inhomogeneous boundary conditions have been transformed into homogeneous ones. It is easy to prove that one solution of Eq. (20) is

$$\mathbf{\Omega}_r(z) = \exp[-\mathbf{K}_{22,r}^{-1}\mathbf{K}_{21,r}(z - z_r)]\mathbf{\Omega}_r(z_r), \quad z_{r-1} \le z \le z_r \tag{21}$$

Substituting $\mathbf{q} = \bar{\mathbf{q}} + \mathbf{\Omega}\mathbf{s}g(t)$ into Eq. (14) gives

$$Q[\mathbf{K}_{22}\bar{\mathbf{q}}'' + (\mathbf{K}_{21} - \mathbf{K}_{12})\bar{\mathbf{q}}' - \mathbf{K}_{11}\bar{\mathbf{q}}] = P\rho\ddot{\bar{\mathbf{q}}} + \bar{\mathbf{B}}\mathbf{\Omega}\mathbf{s}Qg + \rho\mathbf{\Omega}\mathbf{s}P\ddot{g}, \quad \bar{\mathbf{B}} = \mathbf{K}_{11} - \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21} \tag{22}$$

The initial conditions are

$$\frac{\partial^j \bar{\mathbf{q}}}{\partial t^j}(z, 0) = -\mathbf{\Omega}\mathbf{s}\frac{\partial^j g}{\partial t^j}(0), \quad (j = 0, 2, \ldots, k-1) \tag{23}$$

6. Mode superposition analysis

In order to solve Eq. (22), it is first necessary to solve the following eigen-equation without damping

$$\mathbf{K}_{22}\mathbf{q}_i'' + (\mathbf{K}_{21} - \mathbf{K}_{12})\mathbf{q}_i' - \mathbf{K}_{11}\mathbf{q}_i + \rho\omega_i^2\mathbf{q}_i = \mathbf{0} \tag{24}$$

Denoting the i-th eigenpair as $\mathbf{q}_i$, ω_i^2, it can readily be verified that

$$\sum_{r=1}^{l}\int_{z_{r-1}}^{z_r} \rho_r\mathbf{q}_{m,r}^H\mathbf{q}_{n,r}dz = \delta_{mn}, \quad \sum_{r=1}^{l}\int_{z_{r-1}}^{z_r} \mathbf{q}_{m,r}^H[\mathbf{K}_{22,r}\mathbf{q}_{n,r}'' + (\mathbf{K}_{21,r} - \mathbf{K}_{12,r})\mathbf{q}_{n,r}' - \mathbf{K}_{11,r}\mathbf{q}_{n,r}]dz = -\omega_m^2\delta_{mn} \tag{25}$$

where $\mathbf{q}_{m,r}$, $\mathbf{q}_{n,r}$ are the values of the modes $\mathbf{q}_m$, $\mathbf{q}_n$ at the r-th layer of the medium and ρ_r, $\mathbf{K}_{22,r}$ etc. are the density and $\mathbf{K}_{22}$ etc. of the r-th layer. The superscript H represents Hermitian transposition. It is assumed that the first p modes are used in the mode-superposition, then

$$\bar{\mathbf{q}} = \sum_{i=1}^{p}\mathbf{q}_i(z)\cdot T_i(t) \tag{26}$$

In terms of the orthogonality relation (25), the equation of motion (22) and the initial conditions (23) can be reduced to

$$P\ddot{T}_i + Q\omega_i^2T_i = \mathbf{N}_{1i}\mathbf{s}Qg + \mathbf{N}_{2i}\mathbf{s}P\ddot{g} \tag{27}$$

$$\frac{\partial^j T_i}{\partial t^j}(0) = \mathbf{N}_{2i}\mathbf{s}\frac{\partial^j g}{\partial t^j}(0), \quad (j = 0, 2, \ldots, k-1) \tag{28}$$

where

$$\mathbf{N}_{1i} = -\sum_{r=1}^{l}\int_{z_{r-1}}^{z_r} \mathbf{q}_{i,r}^H\bar{\mathbf{B}}_r\mathbf{\Omega}_r dz, \quad \mathbf{N}_{2i} = -\sum_{r=1}^{l}\int_{z_{r-1}}^{z_r} \rho_r\mathbf{q}_{i,r}^H\mathbf{\Omega}_r dz \tag{29}$$

Therefore, in order to solve the problem in the reduced way, it is necessary to: (1) Solve Eq. (24)

for eigenpairs using the precise integration method; (2) Compute $\mathbf{N}_{1i}$, $\mathbf{N}_{2i}$; (3) Solve Eq. (27). The precise integration method combined with the extended W-W algorithm is then used to solve Eq. (24) to obtain the required eigenvalues (Zhong 2002).

Once $\mathbf{q}_{i,1}(z_0)$ has been determined from Eq. (24), T_i can be obtained using the precise integration method (Zhong 2002). The responses u_0, v_0, w_0 at $z = 0$ can then be computed using Eq. (26). Next, assume a value for $\Delta\omega$, and repeatedly compute u_0, v_0, w_0 for $\omega = n\Delta\omega(n = 0, 1, 2, \ldots)$. According to PEM, the ground response PSDs would be

$$S_{uu} = u_0^* \cdot u_0, \quad S_{vv} = v_0^* \cdot v_0, \quad S_{ww} = w_0^* \cdot w_0 \tag{30}$$

7. Numerical examples

A solid consists of two layers, whose parameters are taken from the Gutenberg model (Aki and Richards 1980), as shown in Table 1. The viscoelastic parameters are $p_0 = 1.0$, $p_1 = 0.05$, $q_0 = 1$, $q_1 = 0.1$ and the wave-numbers are $\kappa_x = 0.0002(\mathrm{m}^{-1})$ and $\kappa_y = 0.0003(\mathrm{m}^{-1})$. The spectral densities of the stationary components of the evolutionary bedrock excitations (white noises) are $S_{xg} = 1.0\ \mathrm{m}^2/\mathrm{s}^3$, $S_{yg} = 1.0\ \mathrm{m}^2/\mathrm{s}^3$ and $S_{zg} = 1.0\ \mathrm{m}^2/\mathrm{s}^3$. Assume $\omega \in [0, 6]\mathrm{s}^{-1}$, the frequency step is $\Delta\omega = 0.02\mathrm{s}^{-1}$ and the time step is $\eta = 0.5\mathrm{s}$. Take the non-uniformly modulated function as the following form

$$A(\omega, t) = \exp\left(-\eta_0 \frac{\omega t}{\omega_a t_a}\right) g_1(t), \quad \omega_a t_a = 100.0$$

Table 1 Parameters of the soil

Layer	λ (10^{10} N/m^2)	G (10^{10} N/m^2)	ρ (10^3 Kg/m^3)	Thickness (10^4 m)
1	3.3	3.5	2.74	1.9
2	4.4	4.3	3.00	1.9

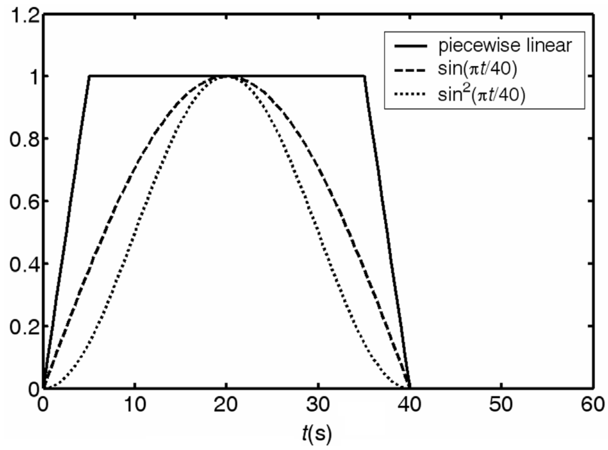

Fig. 2 Modulation functions

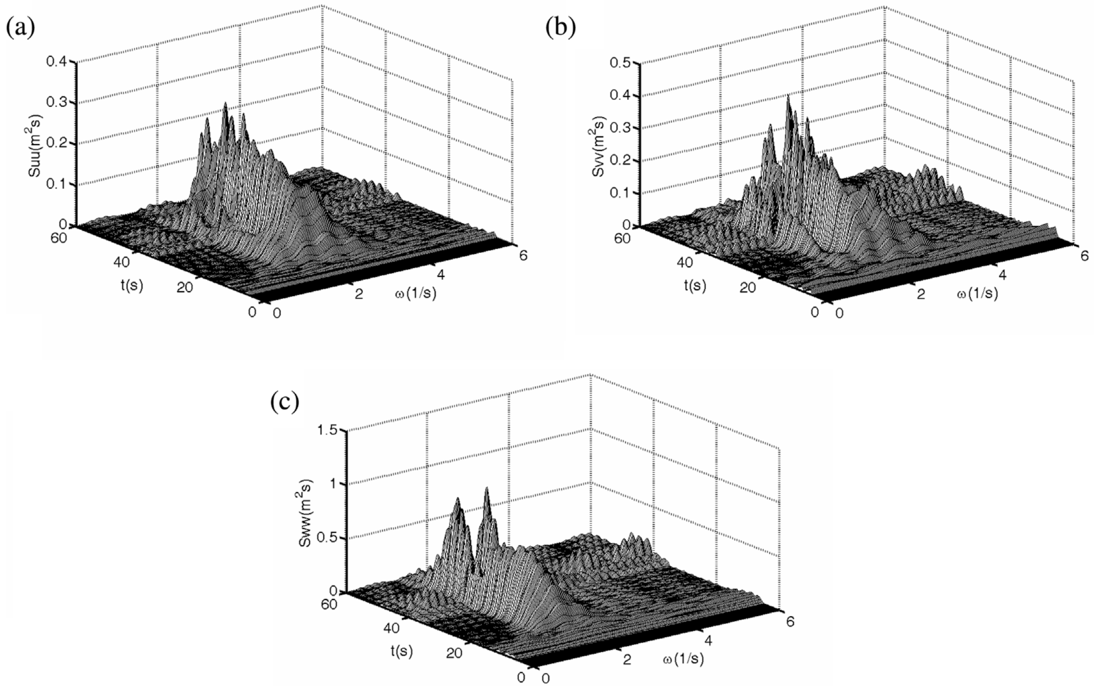

Fig. 3 Response PSDs for piecewise linear modulation function: (a) PSD of u, (b) PSD of v and, (c) PSD of w

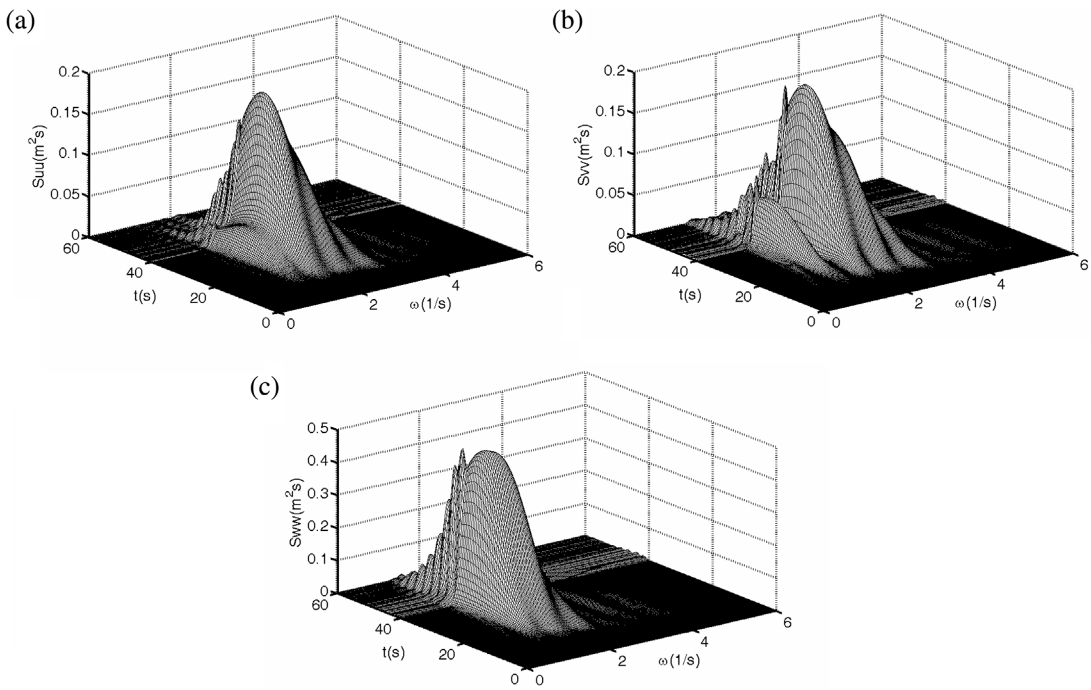

Fig. 4 Response PSDs for sine modulation function: (a) PSD of u, (b) PSD of v and, (c) PSD of w

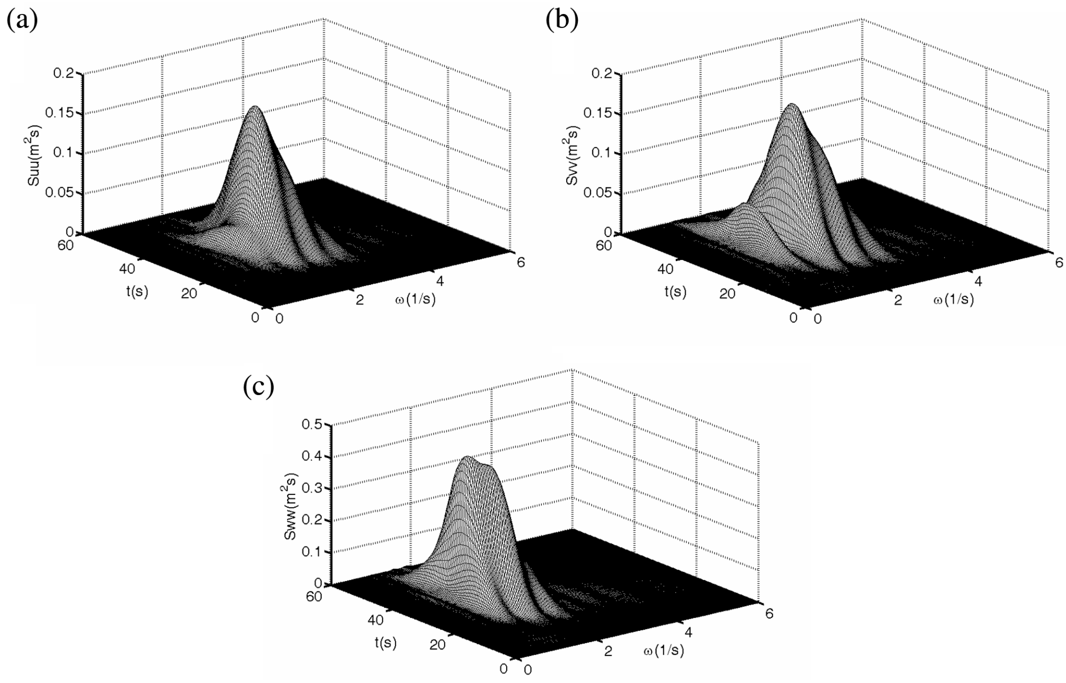

Fig. 5 Response PSDs for the square of sine modulation function: (a) PSD of u, (b) PSD of v and, (c) PSD of w

Case 1. Consider $\eta_0 = 0.0$, that is the case for uniform modulation. Let $g_1(t)$ take three different functions, as shown in Fig. 2, which are, in the time interval $t \in [0, 40]$ s, (1) the piecewise linear modulation functions composed by solid lines; (2) the sinusoidal function $\sin(\pi t/40)$ plotted by a broken curve and; (3) the squared sinusoidal function $\sin^2(\pi t/40)$ plotted by a dotted curve. When $t > 40$s, all $g_1(t)$ equals zero. The PSDs of ground displacement responses are shown in Figs. 3, 4 and 5.

In all these modulation functions, the numbers of participating modes required range from 100 to 250. Therefore the efficiency of the algorithm is of great importance. The proposed PEM based algorithm has proved to be quite effective in this sort of application elsewhere (Lin *et al.* 1994, 1997, 2005).

There exists a stationary duration lasting for 30s in the piecewise linear modulation function, whereas the displacement response PSDs don't reach the stationary state within this duration. This shows that the non-stationary effect must be considered in the analysis of the propagation of random waves.

It can be seen from Fig. 3 that the response PSD surfaces oscillate dramatically near the time when the first derivative of the modulation function becomes severely discontinuous. (See also Lin *et al.* 1995c). In Fig. 4 this phenomenon still exists, but the oscillations are not so strong. In Fig. 5 such oscillations almost disappear, since the discontinuity of the derivative of the modulation function at $t = 40$s is not so severe as for the previous ones.

Case 2 If $\eta_0 = 0.0$ is replaced by $\eta_0 = 2.0$ or $\eta_0 = 4.0$, the ground displacement response PSD surfaces for the piecewise linear modulation function in Case 1 will be replaced by the surfaces

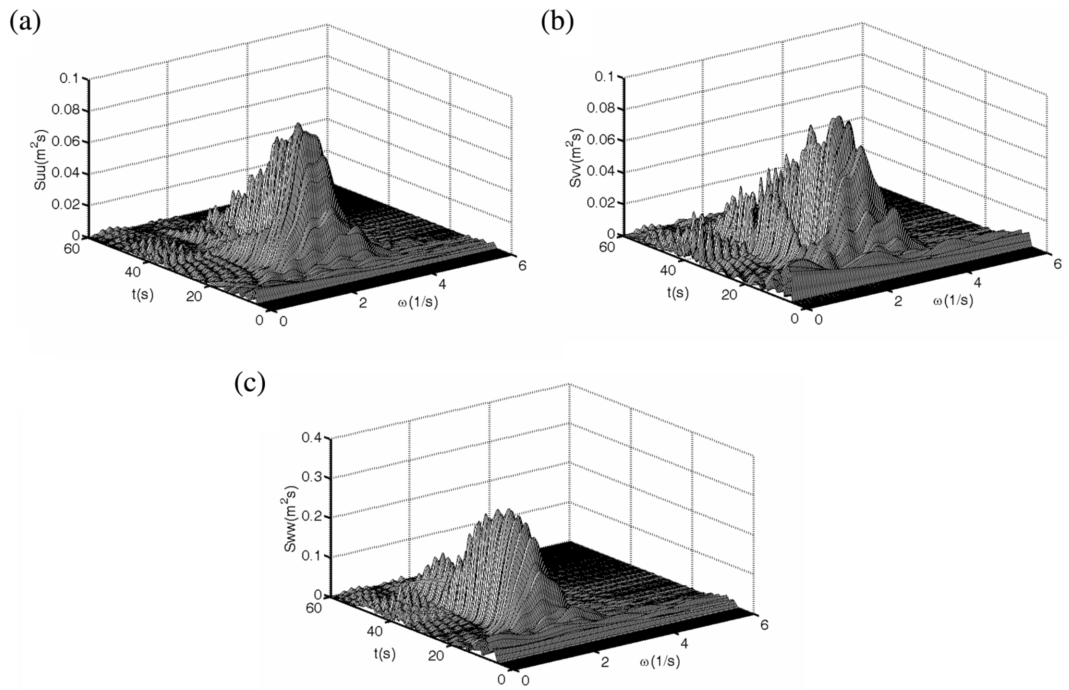

Fig. 6 Response PSDs with $\eta_0 = 2.0$ for piecewise linear modulation function: (a) PSD of *u*, (b) PSD of *v* and, (c) PSD of *w*

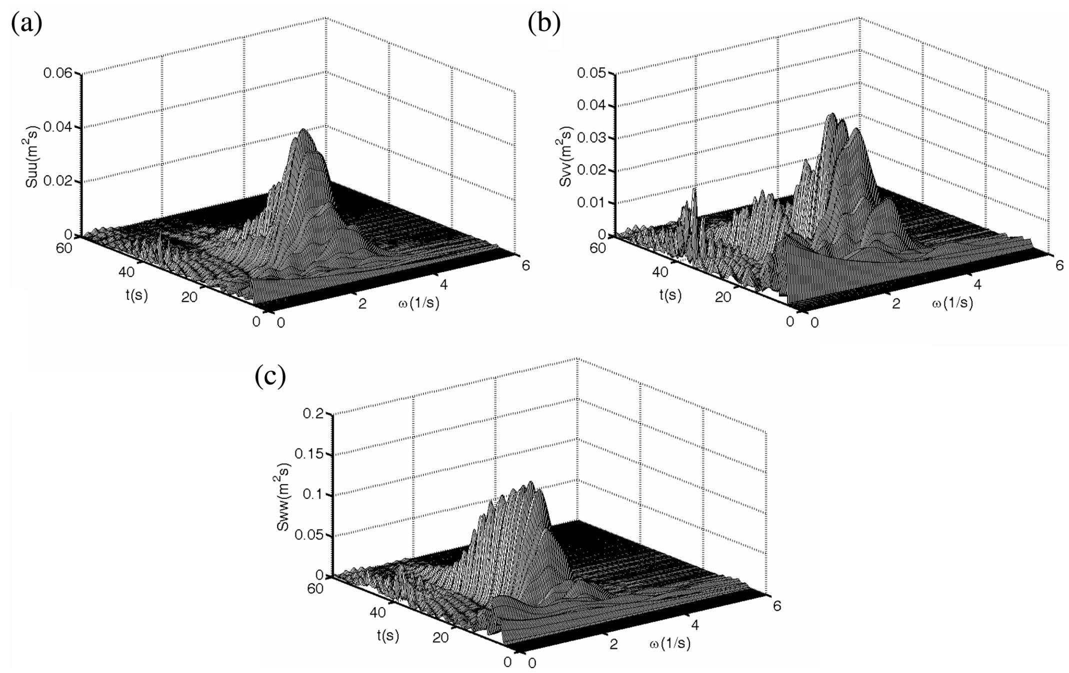

Fig. 7 Response PSDs with $\eta_0 = 4.0$ for piecewise linear modulation function: (a) PSD of *u*, (b) PSD of *v* and, (c) PSD of *w*

shown in Fig. 6 or Fig. 7. When $\eta_0 \neq 0$, the modulation function $A(\omega, t)$ decays exponentially and it will therefore approach zero more quickly for a bigger η_0. As a result, the response PSDs will also decay more quickly as η_0 increases. Comparing Figs. 2, 6 and 7, it can be observed that the effect of a bigger η_0 is to cause the higher frequency response components to decay more quickly in the time domain. This agreement also justifies the method proposed in this paper.

The participating eigenvalues of the system distribute quite uniformly, therefore the complete quadratic combination (CQC) (Lin 1992, 2005) method must be used in the analysis. The proposed PEM - based algorithm provides a strict and efficient CQC method.

8. Conclusions

A new approach to deal with the propagation of non-uniformly modulated evolutionary random waves in transversely isotropic materials is proposed. The time dependent governing differential equations are derived in terms of the pseudo-excitation method in the frequency and wave-number domain and are solved using the eigen-solution expansion method. The precise integration algorithm in combination with the extended W-W algorithm is applied to the extraction of the eigen-solutions of the ordinary differential equation. The displacement responses of the ground subjected to non-uniformly modulated evolutionary random excitations of the bedrock are investigated. Numerical analyses show that the extended W-W algorithm combined with the precise integration method and the pseudo excitation method is an effective way to deal with such problems.

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