Structural Engineering and Mechanics, Vol. 22, No. 6 (2006) 741-759
DOI: http://dx.doi.org/10.12989/sem.2006.22.6.741

On the theory of curved anisotropic plate

Yih-Cherng Chiang†

Department of Mechanical Engineering, Chinese Culture University, No. 55, Hua-Kang Rd., Taipei, Taiwan

(Received September 22, 2005, Accepted January 9, 2006)

Abstract. A general theory which describes the elastic response of a curved anisotropic plate subjected to stretching and bending will be developed by considering the nonlinear effect that reflecting the non-flat geometry of the structure. By applying a newly derived 6×6 matrix constitutive relation between force resultants, moment resultants, mid-plane strains and deformed curvatures, the governing differential equations for a curved anisotropic plate is developed in the usual manner, namely, by consideration of the constitutive relation and equilibrium equations. Solutions are obtained for simply-supported boundary conditions and compared to corresponding solutions that neglecting the nonlinear effect in the analysis. The comparisons indicate that the nonlinear terms in the equations that caused by the curvature of the structure is crucial for the curved plate analysis. Under certain curved plate geometries the unreasonable results will be induced by neglecting the nonlinear effect in the analysis.

Keywords: curved anisotropic plate; nonlinear; constitutive relation.

1. Introduction

There are a number of theories for curved plates (known as shells). Novozhilov (1959), Vlasov (1964), Kraus (1972) and Ambartsumyan (1964) have developed the equilibrium equations by Sokolnikoff's (1964) strain-displacement tensor relations and Hamilton's principle for thin flat and curved plates made of isotropic and homogeneous materials. Furthermore, Flügge (1967) used force balance method to derive the equilibrium equations of curved plate and obtained the solution for some special case about elliptical shell. On the subject of anisotropic plates, Lekhnitskii (1986) published many remarkable and valuable works on anisotropic plate subjected to bending and concentrated loading. Ambartsumyan (1964) also solved the problems of orthotropic thin flat and curved plates by various loading. Dong *et al.* (1962) formulated a theory for thin flat and curved plates consisting of laminated anisotropic materials. By applying Flügge's theory for isotropic material, Cheng and Ho (1963) presented a curved plate analysis for laminated anisotropic material. Whitney (1987) and Reddy (2004) have used double Fourier series to solve some problems of thin laminated anisotropic plates and shells. The effect of transverse shear deformation on a curved plate analysis was considered by Gulati and Essenberg (1967), Zukas and Vinson (1971). By further considering transverse normal strain and expansional strains except to transverse shear deformation, Whitney and Sun (1974) developed a shear deformation theory for laminated shells. Reddy (1984), Reddy and Wang (2000) presented a generalization of the first-order shear deformation theory of

† Associate Professor, E-mail: ycchiang@staff.pccu.edu.tw

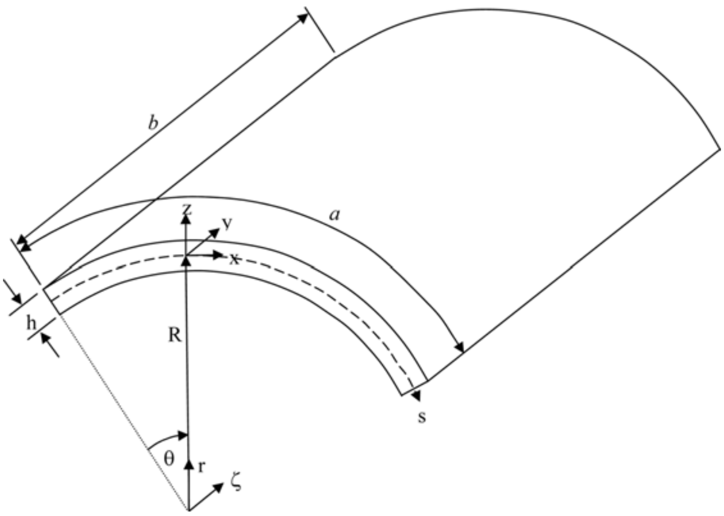

Fig. 1 Geometry of a curved plate

shells, also known as the Sanders shell theory (Sanders 1959), for laminated, doubly-curved anisotropic shells. Due to the difficulty to obtain an analytical solution for the problems of laminated anisotropic plates or shells, a numerical method of Chebyshev polynomials associated with boundary conditions was proposed to obtain more general form of the laminated anisotropic plate and shell problems (Kjellmert 1997).

In general, the analyses mentioned above are limited to the curved plate with shallow curvature such that the classic constitutive relation of a flat anisotropic or laminated plate can still be applied in the analysis. For example, in the work of Whitney (1987), the constitutive relation of a curved laminate with shallow curvature was developed by simplifying the nonlinear terms induced by the geometrical curvature of the structure into the linear terms. As a result, the derived constitutive relation was of the same form as the 6×6 ABD matrix constitutive relation of the classic lamination theory for a flat laminate. The difference of the curved laminate analysis by Whitney from the flat laminate analysis is the retaining the displacement terms of w/R, where w is the displacement in the z direction and R is the radius of the curved plate as seen in Fig. 1, in the analysis. However, if the curved plate is so shallow (i.e., $h/R \rightarrow 0$; h: plate thickness) that the terms z/R ($-h/2 \leqq z \leqq h/2$) can be neglected to simplify the nonlinear terms of $z/(1 + z/R)$ into the linear terms of z, the w/R terms should be ignored according to the analysis assumption that the plate deformation shall be small than the plate thickness h. As a result, the Whitney analysis that neglecting the terms of z/R but retaining the terms of w/R will predict unreasonable results under some curved plate geometries which will be illustrated and discussed in the paper. Reddy (2004) has advanced the Whitney analysis by considering the effect of transverse shear stresses in the curved laminate analysis with moderate plate thickness. However, the nonlinear effect that reflecting the non-flat geometry of the structure has also not been taken into account in the analysis. Bickford (1998) has considered the nonlinear effect in the curved beam analysis and indicated that the nonlinear effect becomes notable as the beam thickness is no longer negligible. In the present analysis, a new curved anisotropic plate theory will be developed by retaining the nonlinear terms and, therefore, the effect of the non-flat geometry can be properly taken into account in the analysis.

Actually, the difficulty and challenge on the analyses of curved structures are ascribed to the nonlinear nature of the structure.

The present paper will devote to derive the equations that describe the elastic response of a curved anisotropic plate subjected to stretching and bending by taking into account the nonlinear distributions of the stresses and strains caused by the non-flat geometry of the structure. Assumed that the curved anisotropic plate deforms according to the Kirchhoff-Love hypothesis, the strains can be expressed in terms of the mid-plane strains and the deformed curvatures in the nonlinear forms. Subsequently, the stresses can be evaluated from the plane stress constitutive relations. By integrating the stresses through the plate thickness, a new 6×6 matrix constitutive relation between stress resultants, moment resultants, mid-plane strains and deformed curvatures has been formulated by Chiang (2005) for a curved plate. As similar to the classic 6×6 ABD matrix constitutive relation of a flat laminate (Vinson and Sierakowski 1987, Herakovich 1998), this new constitutive relation for curved plates will provide the fundamental basis to the analyses of curved structures (e.g., curved beams and plates etc.) composing of either isotropic or anisotropic materials. The anisotropic effect is exhibited on the existence of the stretching-shearing and bending-shearing couplings by comparison with the isotropic plate.

Similar to the biharmonic equation $D\nabla^2(\nabla^2 w) = p$ for a flat isotropic plate, the governing differential equations for a curved anisotropic plate will be developed in the usual manner, namely, by consideration of the constitutive relation and equilibrium equations. By applying the newly derived 6×6 matrix constitutive relation and the strain-displacement relations, the force and the moment resultants can be expresses in terms of the displacements. By substituting the above resultant-displacement relations into the equilibrium equations, the governing differential equations can be derived in terms of displacements. Solutions are obtained for curved anisotropic plates with simply-supported boundary conditions. The nonlinear effect is demonstrated by comparing the present solutions to corresponding solutions that neglected the nonlinear effect in the analysis. The comparisons indicate that some unreasonable results will be incurred by the governing differential equations derived by simplification of the nonlinear terms as the linear terms.

2. Curved plate theory

Consider a curved plate of thickness h as depicted in Fig. 1. Here, the x-axis is passing everywhere through the centroid of the section and tangent to a circular arc of radius R, that is, $ds = Rd\theta$, where θ is the angular variable associated with a change in location along the curved section. The z-axis lies along the local direction of the radius R with the y-axis such that a right-handed coordinate system is formed.

2.1 Strain-displacement relations

Under small deformation, a curved plate is deformed by the following assumptions (Reddy 2004):

(1) The transverse normal is inextensible (i.e., $\varepsilon_z = 0$).

(2) Normals to the mid-plane (i.e., xy plane) before deformation remain straight after deformation (Kirchhoff-Love hypothesis).

(3) The curved plate deformations are small and strains are infinitesimal.

(4) The plane stress assumption can be invoked due to the negligence of transverse normal stress.

Based upon the foregoing assumptions, the strain-displacement relations of a curved plate have been derived by Chiang (2005) by considering the nonlinear terms caused by geometrical curvature of the structure:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} 0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{Bmatrix} + \frac{z}{1 + \kappa z} \begin{Bmatrix} \kappa_x^1 \\ 0 \\ \kappa_{xy}^1 \end{Bmatrix} \tag{1}$$

where the mid-plane strains $\{\varepsilon^0\}$ and the deformed curvatures $\{\kappa^0\}$ and $\{\kappa^1\}$ are given by

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial s} + \kappa w \tag{2}$$

$$\kappa_x^1 = -\left(\frac{\partial^2 w}{\partial s^2} + \kappa^2 w\right) \tag{3}$$

$$\varepsilon_y^0 = \frac{\partial v_0}{\partial y} \tag{4}$$

$$\kappa_y^0 = -\frac{\partial^2 w}{\partial y^2} \tag{5}$$

$$\gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial s} \tag{6}$$

$$\kappa_{xy}^0 = -\left(\frac{\partial^2 w}{\partial s \partial y} - \kappa \frac{\partial u_0}{\partial y}\right) \tag{7}$$

$$\kappa_{xy}^1 = -\left(\frac{\partial^2 w}{\partial s \partial y} + \kappa \frac{\partial v_0}{\partial s}\right) \tag{8}$$

where κ (i.e., $\kappa = 1/R$) is the geometric curvature of the curved plate; and u_0 and v_0 are, respectively, the mid-plane displacement in the x and y direction; and w is the displacement in the z direction. Eq. (1) indicates that the planar strains $\{\varepsilon\}$ at any z-location in the curved plate are in terms of the mid-plane strains $\{\varepsilon^0\}$ and the deformed curvatures $\{\kappa^0\}$ and $\{\kappa^1\}$; it is a fundamental equation of curved plate theory.

2.2 Stress-strain relations

The stresses at any z-location can, then, be determined by substituting the strain-displacement relations of Eq. (1) into the plane stress-strain relations, and lead to

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\bar{Q}] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{9}$$

where $[\bar{Q}]$ is the stiffness matrix of the material which can be either isotropic or anisotropic. Combining Eqs. (1) and (9), a general expression for stresses at z-location in the plate is given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\bar{Q}]\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + [\bar{Q}]z\begin{Bmatrix} 0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{Bmatrix} + [\bar{Q}]\frac{z}{1+\kappa z}\begin{Bmatrix} \kappa_x^1 \\ 0 \\ \kappa_{xy}^1 \end{Bmatrix} \tag{10}$$

The first term in Eq. (10) corresponds to the stresses associated with the mid-plane strains, and the second and third terms correspond to the stresses associated with deformed curvatures. It is noted that $\{\varepsilon^0\}$, $\{\kappa^0\}$ and $\{\kappa^1\}$, which are only associated with the geometric curvature κ and the displacements of u_0, v_0 and w, are independent of z location.

2.3 Curved plate constitutive relation

The force resultants $\{N\}$ refer to the stresses integrated over the thickness of the plate. A similar interpretation can be given to the moment resultants $\{M\}$. By carrying out the integrations, the fundamental equation of the curved plate theory can be written in the following matrix form

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} h\bar{Q}_{11} & h\bar{Q}_{12} & h\bar{Q}_{16} & 0 & 0 & 0 & I_1\bar{Q}_{11} & I_1\bar{Q}_{12} & I_1\bar{Q}_{16} \\ h\bar{Q}_{12} & h\bar{Q}_{22} & h\bar{Q}_{26} & 0 & 0 & 0 & I_1\bar{Q}_{12} & I_1\bar{Q}_{22} & I_1\bar{Q}_{26} \\ h\bar{Q}_{16} & h\bar{Q}_{26} & h\bar{Q}_{66} & 0 & 0 & 0 & I_1\bar{Q}_{16} & I_1\bar{Q}_{26} & I_1\bar{Q}_{66} \\ 0 & 0 & 0 & I\bar{Q}_{11} & I\bar{Q}_{12} & I\bar{Q}_{16} & I_2\bar{Q}_{11} & I_2\bar{Q}_{12} & I_2\bar{Q}_{16} \\ 0 & 0 & 0 & I\bar{Q}_{12} & I\bar{Q}_{22} & I\bar{Q}_{26} & I_2\bar{Q}_{12} & I_2\bar{Q}_{22} & I_2\bar{Q}_{26} \\ 0 & 0 & 0 & I\bar{Q}_{16} & I\bar{Q}_{26} & I\bar{Q}_{66} & I_2\bar{Q}_{16} & I_2\bar{Q}_{26} & I_2\bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ 0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \\ \kappa_x^1 \\ 0 \\ \kappa_{xy}^1 \end{Bmatrix} \tag{11}$$

where $I = h^3/12$; and I_1 and I_2 are defined by

$$I_1 = \int_{-h/2}^{h/2} \frac{z}{1+\kappa z} dz = \frac{1}{\kappa}\left(h - \frac{1}{\kappa}\ln\frac{1+\kappa h/2}{1-\kappa h/2}\right) \tag{12}$$

$$I_2 = \int_{-h/2}^{h/2} \frac{z^2}{1+\kappa z} dz = -\frac{1}{\kappa^2}\left(h - \frac{1}{\kappa}\ln\frac{1+\kappa h/2}{1-\kappa h/2}\right) = -\frac{I_1}{\kappa} \tag{13}$$

The 6×9 matrix form of Eq. (11) is not suitable for mathematical operations. Recalling Eqs. (6-8), the mid-plane strain of γ_{xy}^0 can relate to the deformed curvatures of κ_{xy}^0 and κ_{xy}^1 by

$$\kappa_{xy}^0 = \kappa\gamma_{xy}^0 + \kappa_{xy}^1 \tag{14}$$

Substituting Eq. (14) into Eq. (11) and rearranging the matrix, the constitutive relation of Eq. (11) becomes

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} h\overline{Q}_{11} & h\overline{Q}_{12} & h\overline{Q}_{16} & I_1\overline{Q}_{11} & 0 & I_1\overline{Q}_{16} \\ h\overline{Q}_{12} & h\overline{Q}_{22} & h\overline{Q}_{26} & I_1\overline{Q}_{12} & 0 & I_1\overline{Q}_{26} \\ h\overline{Q}_{16} & h\overline{Q}_{26} & h\overline{Q}_{66} & I_1\overline{Q}_{16} & 0 & I_1\overline{Q}_{66} \\ 0 & 0 & \kappa I\overline{Q}_{16} & I_2\overline{Q}_{11} & I\overline{Q}_{12} & (I_2+I)\overline{Q}_{16} \\ 0 & 0 & \kappa I\overline{Q}_{26} & I_2\overline{Q}_{12} & I\overline{Q}_{22} & (I_2+I)\overline{Q}_{26} \\ 0 & 0 & \kappa I\overline{Q}_{66} & I_2\overline{Q}_{16} & I\overline{Q}_{26} & (I_2+I)\overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x^1 \\ \kappa_y^0 \\ \kappa_{xy}^1 \end{Bmatrix} \tag{15}$$

This new curved plate constitutive relation will provide the fundamental basis to the analyses of curved structures (e.g., curved beams and plates etc.) of isotropic or anisotropic materials. For the curved anisotropic plates, the normal responses are coupling to the shear responses by the existence of $\overline{Q}_{16}$ and $\overline{Q}_{26}$ terms.

3. Governing equations

The equilibrium equations of elasticity without body forces for the curved plate under r-θ-ζ cylindrical coordinate system, as shown in Fig. 1, can be written as

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta\varsigma}}{\partial \varsigma} + \frac{\partial \tau_{\theta r}}{\partial r} + \frac{2\tau_{\theta r}}{r} = 0 \tag{16}$$

$$\frac{1}{r}\frac{\partial \tau_{\theta\varsigma}}{\partial \theta} + \frac{\partial \sigma_{\varsigma}}{\partial \varsigma} + \frac{\partial \tau_{\varsigma r}}{\partial r} + \frac{\tau_{\varsigma r}}{r} = 0 \tag{17}$$

$$\frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{\varsigma r}}{\partial \varsigma} + \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{18}$$

By specializing the equilibrium equations in the cylindrical coordinate system into the x-y-z coordinate system (Fig. 1), the equilibrium equations become

$$\frac{1}{1+\kappa z}\frac{\partial \sigma_x}{\partial s} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \frac{2\kappa}{1+\kappa z}\tau_{xz} = 0 \tag{19}$$

$$\frac{1}{1+\kappa z}\frac{\partial \tau_{xy}}{\partial s} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \frac{\kappa}{1+\kappa z}\tau_{yz} = 0 \tag{20}$$

$$\frac{1}{1+\kappa z}\frac{\partial \tau_{xz}}{\partial s} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \frac{\kappa}{1+\kappa z}(\sigma_z - \sigma_x) = 0 \tag{21}$$

The integrated equations of equilibrium can be obtained by carrying out the integrations of the equations through the thickness. Multiplying Eqs. (19-21) by $(1 + \kappa z)$ and integrating the equations results in

$$\frac{\partial}{\partial s}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x dz + \frac{\partial}{\partial y}\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} dz + \frac{\partial}{\partial z}\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} dz + \kappa\frac{\partial}{\partial y}\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} z dz + \kappa\frac{\partial}{\partial z}\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} z dz + 2\kappa\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} dz = 0 \quad (22)$$

$$\frac{\partial}{\partial s}\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} dz + \frac{\partial}{\partial y}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y dz + \frac{\partial}{\partial z}\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz} dz + \kappa\frac{\partial}{\partial y}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y z dz + \kappa\frac{\partial}{\partial z}\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz} z dz + \kappa\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz} dz = 0 \quad (23)$$

$$\frac{\partial}{\partial s}\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} dz + \frac{\partial}{\partial y}\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz} dz + \frac{\partial}{\partial z}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_z dz + \kappa\frac{\partial}{\partial y}\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz} z dz + \kappa\frac{\partial}{\partial z}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_z z dz + \kappa\int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_z - \sigma_x) dz = 0 \quad (24)$$

Here, we define the shear force resultants $\{Q\}$ referring to the shear stresses of τ_{xz} and τ_{yz} integrated over the thickness of the plate as

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} dz \quad (25)$$

Plus the definitions of the force resultants and the moment resultants, Eqs. (22-24) can be written as:

$$\frac{\partial N_x}{\partial s} + \frac{\partial N_{xy}}{\partial y} + (\tau_{1x} - \tau_{2x}) + \kappa\frac{\partial M_{xy}}{\partial y} + \frac{\kappa h}{2}(\tau_{1x} + \tau_{2x}) + \kappa Q_x = 0 \quad (26)$$

$$\frac{\partial N_{xy}}{\partial s} + \frac{\partial N_y}{\partial y} + (\tau_{1y} - \tau_{2y}) + \kappa\frac{\partial M_y}{\partial y} + \frac{\kappa h}{2}(\tau_{1y} + \tau_{2y}) = 0 \quad (27)$$

$$\frac{\partial Q_x}{\partial s} + \frac{\partial Q_y}{\partial y} + (p_1 - p_2) + \frac{\kappa h}{2}(p_1 + p_2) - \kappa N_x = 0 \quad (28)$$

where

$$\tau_{xz}\left(\frac{h}{2}\right) = \tau_{1x} \quad \text{and} \quad \tau_{xz}\left(-\frac{h}{2}\right) = \tau_{2x} \quad (29a)$$

$$\tau_{yz}\left(\frac{h}{2}\right) = \tau_{1y} \quad \text{and} \quad \tau_{yz}\left(-\frac{h}{2}\right) = \tau_{2y} \quad (29b)$$

$$\sigma_z\left(\frac{h}{2}\right) = p_1 \quad \text{and} \quad \sigma_z\left(-\frac{h}{2}\right) = p_2 \quad (29c)$$

In addition to the integrated force equilibrium equations, two integrated moment equilibrium equations are also needed. Multiplying Eqs. (19-20) by $(1 + \kappa z)$, multiplying by z and integrating through the thickness results in the following:

$$\frac{\partial}{\partial s}\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_x z dz+\frac{\partial}{\partial y}\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{xy} z dz+\frac{\partial}{\partial z}\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{xz} z dz+\kappa\frac{\partial}{\partial y}\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{xy} z^2 dz+\kappa\frac{\partial}{\partial z}\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{xz} z^2 dz+2\kappa\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{xz} z dz = 0 \quad (30)$$

$$\frac{\partial}{\partial s}\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{xy} z dz+\frac{\partial}{\partial y}\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_y z dz+\frac{\partial}{\partial z}\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{yz} z dz+\kappa\frac{\partial}{\partial y}\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_y z^2 dz+\kappa\frac{\partial}{\partial z}\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{yz} z^2 dz+\kappa\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{yz} z dz = 0 \quad (31)$$

With the definitions of the force, moment and shear force resultants, Eqs. (30-31) can be expressed as

$$\frac{\partial M_x}{\partial s}+\frac{\partial M_{xy}}{\partial y}+\frac{h}{2}(\tau_{1x}+\tau_{2x})-Q_x+\kappa\frac{\partial}{\partial y}\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{xy} z^2 dz = 0 \quad (32)$$

$$\frac{\partial M_{xy}}{\partial s}+\frac{\partial M_y}{\partial y}+\frac{h}{2}(\tau_{1y}+\tau_{2y})-Q_y+\kappa\frac{\partial}{\partial y}\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_y z^2 dz = 0 \quad (33)$$

Thus, Eqs. (26-28) and (32-33) are five equilibrium equations for the stretching and bending of a curved plate.

The equilibrium equations of Eqs. (26-28) and (32-33) can be further combined to eliminate the terms of surface shear stresses and shear force resultants. Substituting Eq. (32) into Eq. (26), taking derivatives with respect to s results in

$$\frac{\partial^2 N_x}{\partial s^2}+\frac{\partial^2 N_{xy}}{\partial s\partial y}+\kappa\frac{\partial^2 M_x}{\partial s^2}+2\kappa\frac{\partial^2 M_{xy}}{\partial s\partial y}+\kappa^2\frac{\partial^2}{\partial s\partial y}\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{xy} z^2 dz = 0 \quad (34)$$

Taking derivative of Eq. (27) with respect to y yields

$$\frac{\partial^2 N_{xy}}{\partial s\partial y}+\frac{\partial^2 N_y}{\partial y^2}+\kappa\frac{\partial^2 M_y}{\partial y^2} = 0 \quad (35)$$

Eqs. (32) and (33) can be substituted into Eq. (28) with the result of

$$\frac{\partial^2 M_x}{\partial s^2}+2\frac{\partial^2 M_{xy}}{\partial s\partial y}+\frac{\partial^2 M_y}{\partial y^2}+\kappa\frac{\partial^2}{\partial s\partial y}\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{xy} z^2 dz+\kappa\frac{\partial^2}{\partial y^2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_y z^2 dz-\kappa N_x =$$
$$-(p_1-p_2)-\frac{\kappa h}{2}(p_1+p_2) \quad (36)$$

Comparing Eqs. (34-36) to the equilibrium equations for a flat plate, it can be found that the difference is the existence of the terms comprising the curvature κ in Eqs. (34-36). Thus, as the curvature $\kappa\to 0$ (i.e., $R\to\infty$), the terms comprising the curvature κ approach to zero and Eqs. (34-36) reduce to the equilibrium equations of a flat plate.

Eqs. (34-36) are derived from equilibrium consideration alone. The next step is to express the equilibrium equations in terms of displacements. By substituting the strain-displacement relations given by Eqs. (2-8) into the constitutive equations, the force resultants and moment resultants can be expressed in terms of displacements as

$$N_x = h\bar{Q}_{11}\left(\frac{\partial u_0}{\partial s} + \kappa w\right) + h\bar{Q}_{12}\left(\frac{\partial v_0}{\partial y}\right) + h\bar{Q}_{16}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial s}\right)$$
$$- I_1\bar{Q}_{11}\left(\frac{\partial^2 w}{\partial s^2} + \kappa^2 w\right) - I_1\bar{Q}_{16}\left(\frac{\partial^2 w}{\partial s\partial y} + \kappa\frac{\partial v_0}{\partial s}\right) \tag{37a}$$

$$N_y = h\bar{Q}_{12}\left(\frac{\partial u_0}{\partial s} + \kappa w\right) + h\bar{Q}_{22}\left(\frac{\partial v_0}{\partial y}\right) + h\bar{Q}_{26}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial s}\right)$$
$$- I_1\bar{Q}_{12}\left(\frac{\partial^2 w}{\partial s^2} + \kappa^2 w\right) - I_1\bar{Q}_{26}\left(\frac{\partial^2 w}{\partial s\partial y} + \kappa\frac{\partial v_0}{\partial s}\right) \tag{37b}$$

$$N_{xy} = h\bar{Q}_{16}\left(\frac{\partial u_0}{\partial s} + \kappa w\right) + h\bar{Q}_{26}\left(\frac{\partial v_0}{\partial y}\right) + h\bar{Q}_{66}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial s}\right)$$
$$- I_1\bar{Q}_{16}\left(\frac{\partial^2 w}{\partial s^2} + \kappa^2 w\right) - I_1\bar{Q}_{66}\left(\frac{\partial^2 w}{\partial s\partial y} + \kappa\frac{\partial v_0}{\partial s}\right) \tag{37c}$$

$$M_x = \kappa I\bar{Q}_{16}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial s}\right) - I_2\bar{Q}_{11}\left(\frac{\partial^2 w}{\partial s^2} + \kappa^2 w\right) - I\bar{Q}_{12}\left(\frac{\partial^2 w}{\partial y^2}\right)$$
$$- (I + I_2)\bar{Q}_{16}\left(\frac{\partial^2 w}{\partial s\partial y} + \kappa\frac{\partial v_0}{\partial s}\right) \tag{37d}$$

$$M_y = \kappa I\bar{Q}_{26}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial s}\right) - I_2\bar{Q}_{12}\left(\frac{\partial^2 w}{\partial s^2} + \kappa^2 w\right) - I\bar{Q}_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)$$
$$- (I + I_2)\bar{Q}_{26}\left(\frac{\partial^2 w}{\partial s\partial y} + \kappa\frac{\partial v_0}{\partial s}\right) \tag{37e}$$

$$M_{xy} = \kappa I\bar{Q}_{66}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial s}\right) - I_2\bar{Q}_{16}\left(\frac{\partial^2 w}{\partial s^2} + \kappa^2 w\right) - I\bar{Q}_{26}\left(\frac{\partial^2 w}{\partial y^2}\right)$$
$$- (I + I_2)\bar{Q}_{66}\left(\frac{\partial^2 w}{\partial s\partial y} + \kappa\frac{\partial v_0}{\partial s}\right) \tag{37f}$$

The stress σ_y in Eq. (36) and the shear stress τ_{xy} in Eqs. (34) and (36) can be expressed in terms of displacements by the stress-strain relations and the strain-displacement relations given by Eqs. (1-9). Furthermore, the integral involving rational algebraic function in Eqs. (34-36) is given in the following:

$$\int_{-h/2}^{h/2} \frac{z^3}{1 + \kappa z}dz = \frac{I}{\kappa} + \frac{1}{\kappa^3}\left(h - \frac{1}{\kappa}\ln\frac{1 + \kappa h/2}{1 - \kappa h/2}\right) = I_3 = \frac{(I - I_2)}{\kappa} \tag{38}$$

Substituting Eqs. (37a-37f) into Eqs. (34-36) and carrying out the integrations, the equations of equilibrium can be expressed in terms of displacements as

$$h\bar{Q}_{11}\frac{\partial^3 u_0}{\partial s^3} + (2\kappa^2 I + 2h)\bar{Q}_{16}\frac{\partial^3 u_0}{\partial s^2 \partial y} + (3\kappa^2 I + h)\bar{Q}_{66}\frac{\partial^3 u_0}{\partial s \partial y^2}$$

$$+ h\bar{Q}_{16}\frac{\partial^3 v_0}{\partial s^3} + h(\bar{Q}_{12} + \bar{Q}_{66})\frac{\partial^3 v_0}{\partial s^2 \partial y} + (\kappa^2 I + h)\bar{Q}_{26}\frac{\partial^3 v_0}{\partial s \partial y^2} - 2\kappa I\bar{Q}_{16}\frac{\partial^4 w}{\partial s^3 \partial y}$$

$$- \kappa I(\bar{Q}_{12} + 3\bar{Q}_{66})\frac{\partial^4 w}{\partial s^2 \partial y^2} - 2\kappa I\bar{Q}_{26}\frac{\partial^4 w}{\partial s \partial y^3} + \kappa h\bar{Q}_{11}\frac{\partial^2 w}{\partial s^2} + \kappa h\bar{Q}_{16}\frac{\partial^2 w}{\partial s \partial y} = 0 \tag{39}$$

$$h\bar{Q}_{16}\frac{\partial^3 u_0}{\partial s^2 \partial y} + h(\bar{Q}_{66} + \bar{Q}_{12})\frac{\partial^3 u_0}{\partial s \partial y^2} + (\kappa^2 I + h)\bar{Q}_{26}\frac{\partial^3 u_0}{\partial y^3}$$

$$+ (\kappa^2 I_2 + h)\bar{Q}_{66}\frac{\partial^3 v_0}{\partial s^2 \partial y} + 2h\bar{Q}_{26}\frac{\partial^3 v_0}{\partial s \partial y^2} + h\bar{Q}_{22}\left(\frac{\partial^3 v_0}{\partial y^3}\right)$$

$$+ \kappa I_2\bar{Q}_{16}\frac{\partial^4 w}{\partial s^3 \partial y} + (\kappa I_2\bar{Q}_{66} - \kappa I\bar{Q}_{26})\frac{\partial^4 w}{\partial s^2 \partial y^2} - \kappa I\bar{Q}_{22}\frac{\partial^4 w}{\partial y^4}$$

$$+ \kappa(\kappa^2 I_2 + h)\bar{Q}_{16}\frac{\partial^2 w}{\partial s \partial y} + \kappa h\bar{Q}_{12}\frac{\partial^2 w}{\partial y^2} = 0 \tag{40}$$

$$2\kappa I\bar{Q}_{16}\frac{\partial^3 u_0}{\partial s^2 \partial y} + \kappa I(3\bar{Q}_{66} + \bar{Q}_{12})\frac{\partial^3 u_0}{\partial s \partial y^2} + 2\kappa I\bar{Q}_{26}\frac{\partial^3 u_0}{\partial y^3} - \kappa h\bar{Q}_{11}\frac{\partial u_0}{\partial s} - \kappa h\bar{Q}_{16}\frac{\partial u_0}{\partial y}$$

$$- \kappa I_2\bar{Q}_{16}\frac{\partial^3 v_0}{\partial s^3} - \kappa I_2\bar{Q}_{66}\frac{\partial^3 v_0}{\partial s^2 \partial y} + \kappa I\bar{Q}_{26}\frac{\partial^3 v_0}{\partial s \partial y^2} + \kappa I\bar{Q}_{22}\frac{\partial^3 v_0}{\partial y^3} - \kappa(h + \kappa^2 I_2)\bar{Q}_{16}\frac{\partial v_0}{\partial s} - \kappa h\bar{Q}_{12}\frac{\partial v_0}{\partial y}$$

$$- I_2\bar{Q}_{11}\frac{\partial^4 w}{\partial s^4} - 2(I + I_2)\bar{Q}_{16}\frac{\partial^4 w}{\partial s^3 \partial y} - (2I\bar{Q}_{12} + (3I + I_2)\bar{Q}_{66})\frac{\partial^4 w}{\partial s^2 \partial y^2} - 4I\bar{Q}_{26}\frac{\partial^4 w}{\partial s \partial y^3} - I\bar{Q}_{22}\frac{\partial^4 w}{\partial y^4}$$

$$-2\kappa^2 I_2\bar{Q}_{11}\frac{\partial^2 w}{\partial s^2} - 2\kappa^2 I_2\bar{Q}_{16}\frac{\partial^2 w}{\partial s \partial y} - \kappa^2(\kappa^2 I_2 + h)\bar{Q}_{11} w$$

$$= -(p_1 - p_2) - \frac{\kappa h}{2}(p_1 + p_2) \tag{41}$$

These are the governing differential equations for the stretching and bending of a curved plate composed of an anisotropic material. As the radius R becomes infinite (i.e., $k \to 0$) for an isotropic material, $I_2 \to I$ and Eq. (41) reduces to the biharmonic equation $D\nabla^2(\nabla^2 w) = p$ for a flat isotropic plate. These governing differential equations can be solved analytically for certain boundary conditions.

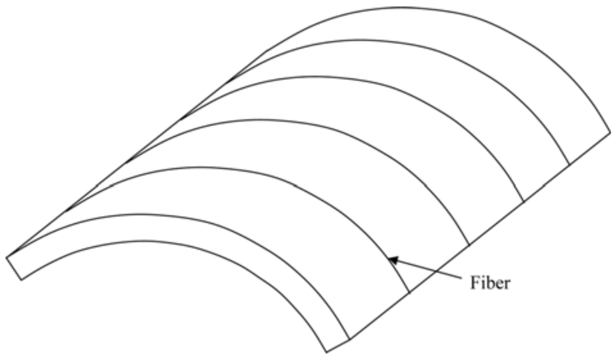

Fig. 2 Fiber placement of the curved orthotropic plate

4. Simply-supported curved orthotropic plates

Here, we consider the curved anisotropic plate of a special orthotropic class of which fiber placement related to coordinate system is shown in Fig. 2. Then, the stiffness matrix $[\bar{Q}]$ is given by

$$[\bar{Q}] = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 \\ 0 & 0 & \bar{Q}_{66} \end{bmatrix} = \begin{bmatrix} \dfrac{E_{11}}{1-\nu_{12}\nu_{21}} & \dfrac{\nu_{21}E_{11}}{1-\nu_{12}\nu_{21}} & 0 \\ \dfrac{\nu_{12}E_{22}}{1-\nu_{12}\nu_{21}} & \dfrac{E_{22}}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \tag{42}$$

where E, ν and G are Young's modulus, Poisson ratio and shear modulus, respectively, and subscripts 1 and 2 indicate the fiber and transverse directions.

The simply-supported boundary conditions for a curved plate, as shown in Fig. 1, can be stated as

at $s = 0$ and a

$$v_0 = N_y = 0 \tag{43a}$$

$$w = M_y = 0 \tag{43b}$$

at $y = 0$ and b

$$u_0 = N_x = 0 \tag{44a}$$

$$w = M_x = 0 \tag{44b}$$

The governing differential equations for a curved orthotropic plate obtained by substituting $\bar{Q}_{16} = 0$ and $\bar{Q}_{26} = 0$ into Eqs. (39-41) with the simply-supported boundary conditions of Eqs. (43-44) can be solved by using a Fourier-series expansion. For external pressure loading $p = \sigma_z(h/2) = p_1$, p can be expanded in a Fourier series as

$$p(s,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P_{mn}\sin\frac{m\pi s}{a}\sin\frac{n\pi y}{b} \tag{45}$$

where a and b are the dimensions of the curved plate in the s and y directions, as shown in Fig. 1. The coefficient P_{mn} can be determined in the usual way from Eq. (45) as

$$P_{mn} = \frac{4}{ab}\int_0^a\int_0^b p(s,y)\sin\frac{m\pi s}{a}\sin\frac{n\pi y}{b}ds dy \tag{46}$$

The displacements u_0, v_0 and w can be expanded in a Fourier series as

$$u_0(s,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn}\cos\frac{m\pi s}{a}\sin\frac{n\pi y}{b} \tag{47}$$

$$v_0(s,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn}\sin\frac{m\pi s}{a}\cos\frac{n\pi y}{b} \tag{48}$$

$$w(s,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\sin\frac{m\pi s}{a}\sin\frac{n\pi y}{b} \tag{49}$$

Substituting Eqs. (47-49) and (45) into the governing differential equations for a curved orthotropic plate results in the following equation in matrix form:

$$\begin{bmatrix} \Gamma_{11mn} & \Gamma_{12mn} & \Gamma_{13mn} \\ \Gamma_{12mn} & \Gamma_{22mn} & \Gamma_{23mn} \\ \Gamma_{13mn} & \Gamma_{23mn} & \Gamma_{33mn} \end{bmatrix} \begin{bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \left(1+\frac{\kappa h}{2}\right)P_{mn} \end{bmatrix} \tag{50}$$

where if $\lambda_m = m\pi/a$ and $\lambda_n = n\pi/b$

$$\Gamma_{11mn} = h\overline{Q}_{11}\lambda_m^2 + (3\kappa^2 I + h)\overline{Q}_{66}\lambda_n^2 \tag{51a}$$

$$\Gamma_{12mn} = h(\overline{Q}_{12} + \overline{Q}_{66})\lambda_m\lambda_n \tag{51b}$$

$$\Gamma_{13mn} = -\kappa(h\overline{Q}_{11}\lambda_m + I(3\overline{Q}_{66} + \overline{Q}_{12})\lambda_m\lambda_n^2) \tag{51c}$$

$$\Gamma_{22mn} = (\kappa^2 I_2 + h)\overline{Q}_{66}\lambda_m^2 + h\overline{Q}_{22}\lambda_n^2 \tag{51d}$$

$$\Gamma_{23mn} = \kappa(I_2\overline{Q}_{66}\lambda_m^2\lambda_n - I\overline{Q}_{22}\lambda_n^3 - h\overline{Q}_{12}\lambda_n) \tag{51e}$$

$$\Gamma_{33mn} = I_2\overline{Q}_{11}\lambda_m^4 + (2I\overline{Q}_{12} + (3I + I_2)\overline{Q}_{66})\lambda_m^2\lambda_n^2 + I\overline{Q}_{22}\lambda_n^4 - 2\kappa^2 I_2\overline{Q}_{11}\lambda_m^2 + \kappa^2(\kappa^2 I_2 + h)\overline{Q}_{11} \tag{51f}$$

In the analysis of Whitney (1987) in which the nonlinear terms of $z/(1 + z/R)$ were simplified into linear terms of z under the assumption of z/R<<1 but retaining the terms of w/R, the matrix form for a curved orthotropic plate was given by

$$\begin{bmatrix} \Gamma_{11mn} & \Gamma_{12mn} & \Gamma_{13mn} \\ \Gamma_{12mn} & \Gamma_{22mn} & \Gamma_{23mn} \\ \Gamma_{13mn} & \Gamma_{23mn} & \Gamma_{33mn} \end{bmatrix} \begin{bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ P_{mn} \end{bmatrix} \tag{52}$$

where

$$\Gamma_{11mn} = h\bar{Q}_{11}\lambda_m^2 + 2\bar{Q}_{66}\lambda_n^2 \tag{53a}$$

$$\Gamma_{12mn} = h(\bar{Q}_{12} + \bar{Q}_{66})\lambda_m\lambda_n \tag{53b}$$

$$\Gamma_{13mn} = -\kappa h\bar{Q}_{11}\lambda_m \tag{53c}$$

$$\Gamma_{22mn} = h\bar{Q}_{66}\lambda_m^2 + h\bar{Q}_{22}\lambda_n^2 \tag{53d}$$

$$\Gamma_{23mn} = -\kappa h\bar{Q}_{12}\lambda_n \tag{53e}$$

$$\Gamma_{33mn} = I\bar{Q}_{11}\lambda_m^4 + 2(I\bar{Q}_{12} + 2I\bar{Q}_{66})\lambda_m^2\lambda_n^2 + I\bar{Q}_{22}\lambda_n^4 \tag{53f}$$

Consider the case of $p(s, y) = p_0$, that is, the case of a uniformly load on the outer surface. Evaluating the integrals of Eq. (46) gives

$$P_{mn} = \frac{16p_0}{mn\pi^2} \quad \text{if } m \text{ odd and } n \text{ odd} \tag{54a}$$

or

$$P_{mn} = 0 \quad \text{otherwise} \tag{54b}$$

then,

$$U_{mn} = \frac{\Gamma_{12mn}\Gamma_{23mn} - \Gamma_{22mn}\Gamma_{13mn}}{\Lambda_{mn}}\left(1 + \frac{\kappa h}{2}\right)\frac{16p_0}{mn\pi^2} \quad \text{if } m \text{ odd and } n \text{ odd} \tag{55}$$

$$V_{mn} = \frac{\Gamma_{12mn}\Gamma_{13mn} - \Gamma_{11mn}\Gamma_{23mn}}{\Lambda_{mn}}\left(1 + \frac{\kappa h}{2}\right)\frac{16p_0}{mn\pi^2} \quad \text{if } m \text{ odd and } n \text{ odd} \tag{56}$$

$$W_{mn} = \frac{\Gamma_{11mn}\Gamma_{22mn} - \Gamma_{12mn}^2}{\Lambda_{mn}}\left(1 + \frac{\kappa h}{2}\right)\frac{16p_0}{mn\pi^2} \quad \text{if } m \text{ odd and } n \text{ odd} \tag{57}$$

where Λ_{mn} is the determinant of the matrix in Eq. (50) and given by

$$\Lambda_{mn} = \Gamma_{11}\Gamma_{22}\Gamma_{33} + 2\Gamma_{12}\Gamma_{13}\Gamma_{23} - (\Gamma_{11}\Gamma_{23}^2 + \Gamma_{22}\Gamma_{13}^2 + \Gamma_{33}\Gamma_{12}^2) \tag{58}$$

Then, the displacements u_0, v_0 and w are, respectively, given by

$$u_0(s, y) = \left(1 + \frac{\kappa h}{2}\right)\frac{16p_0}{\pi^2}\sum_{m=1,3,5,\ldots}\sum_{n=1,3,5,\ldots}\frac{\Gamma_{12mn}\Gamma_{23mn} - \Gamma_{22mn}\Gamma_{13mn}}{mn\Lambda_{mn}}\cos\lambda_m s\sin\lambda_n y \tag{59}$$

$$v_0(s, y) = \left(1 + \frac{\kappa h}{2}\right)\frac{16p_0}{\pi^2}\sum_{m=1,3,5,\ldots}\sum_{n=1,3,5,\ldots}\frac{\Gamma_{12mn}\Gamma_{13mn} - \Gamma_{11mn}\Gamma_{23mn}}{mn\Lambda_{mn}}\sin\lambda_m s\cos\lambda_n y \tag{60}$$

$$w(s, y) = \left(1 + \frac{\kappa h}{2}\right)\frac{16p_0}{\pi^2}\sum_{m=1,3,5,\ldots}\sum_{n=1,3,5,\ldots}\frac{\Gamma_{11mn}\Gamma_{22mn} - \Gamma_{12mn}^2}{mn\Lambda_{mn}}\sin\lambda_m s\sin\lambda_n y \tag{61}$$

The well-known solution of transverse displacement $w(x, y)$ for the bending of a flat isotropic plate (Bickford 1998) can be recovered by Eq. (61) through allowing $\kappa\to0$ and applying the stiffness matrix of isotropic material:

$$w(x, y) = \frac{16p_0}{D\pi^2}\sum_{m=1,3,5,\ldots}\sum_{n=1,3,5,\ldots}\frac{1}{mn(\lambda_m^2 + \lambda_n^2)^2}\sin\lambda_m x\sin\lambda_n y \tag{62}$$

where $D = Eh^3/12(1 - \nu^2)$ is referred to as the flexural rigidity.

5. Results and discussions

The AS4/3501-6 carbon/epoxy composite, of which material properties are E_{11} = 128 GPa, E_{22} = 11.1 GPa, G_{12} = 6.55 GPa, ν_{12} = 0.28 and $\nu_{21} = (E_{22}/E_{11})\nu_{12}$ (Swanson 1997), is used for case study. The transverse displacements w for the simply-supported curved orthotropic plate given by Eq. (61) are compared to the corresponding solutions of Eq. (52) by the Whitney analysis. By taking only the first term of the series and assuming $a = b$, the maximum transverse displacements w occurring at the center $s = a/2$ and $y = b/2$ are predicted as a function of the h/R ratio for the present and

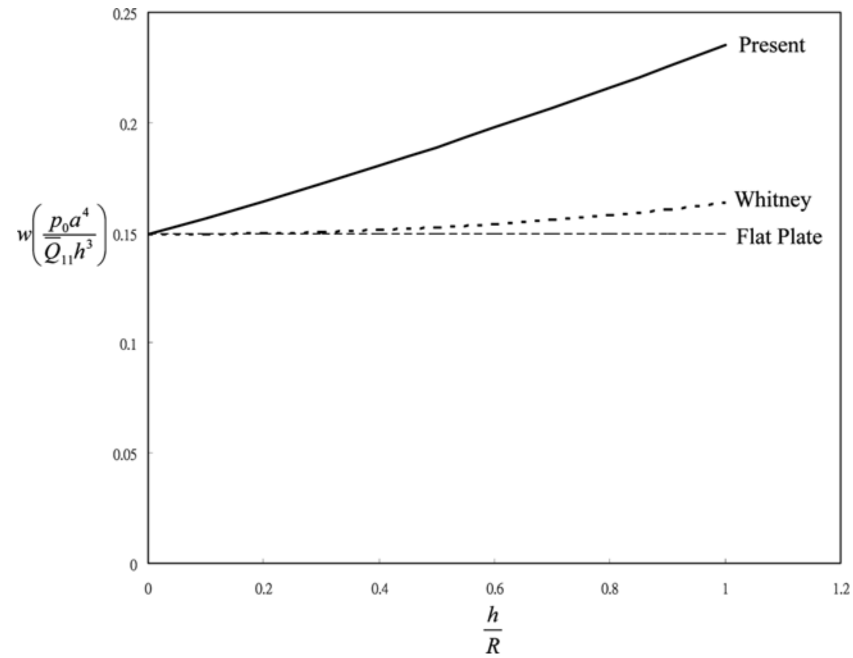

Fig. 3 The maximum transverse displacement w as a function of h/R for the case of $a = h$

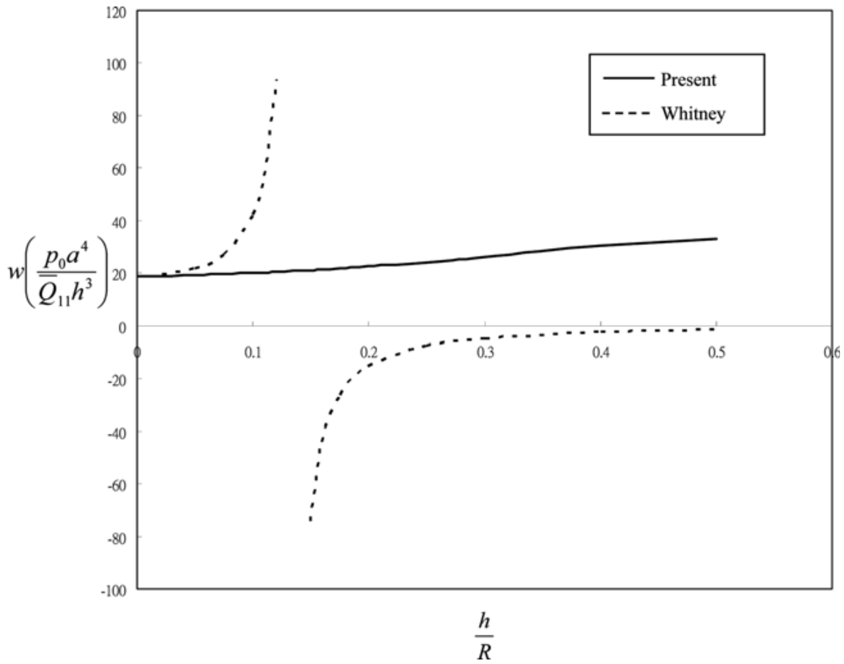

Fig. 4 The maximum transverse displacement w as a function of h/R for the case of a = 5h

Whitney analyses. For the geometry of a = h, Fig. 3 shows that the present and Whitney analyses approach to the identical result of a flat orthotropic plate as $h/R \to 0$ while the discrepancy between two analyses becomes larger as the h/R ratio increases. In the case of a = 5h, Fig. 4 indicates that the Whitney results will diverge and discontinue at the particulate h/R ratio and the results will turn into different sign as the h/R ratio is larger than the discontinued site. The unreasonable results are induced from the Whitney analysis that simplified the nonlinear terms of $z/(1 + z/R)$, where $-h/2 \leq z \leq h/2$, into the linear items of z under the assumption of $h/R \ll 1$ for the shallow curved plate. However, in order to differ from the flat plate analysis the displacement terms of w/R were retained in the Whitney analysis. This is illogical because the displacement w should be smaller than the thickness of the plate h under Kirchhoff-Love hypothesis for curved plate deformation. Therefore, if the curved plate is so shallow that z/R can be ignored, the terms of w/R should also be neglected and the analysis reduces to the flat plate analysis. This is the reason why some unreasonable results will be induced by the Whitney analysis under some curved plate geometries. It implies that the nonlinear effect that reflecting the geometric curvature of the structure cannot be neglected for the curved plate analysis.

The nonlinear effect can also be illustrated by the comparisons of the predictions of the displacements u_0 and v_0 between the present and Whitney analyses. As shown in Figs. 5 and 6, where the maximum mid-plane displacements u_0 and v_0 are, respectively, illustrated as a function of the h/R ratio for the case of a = h, the present and Whitney analyses approach to the same result as $h/R \to 0$ while the discrepancy between two analyses becomes larger as the h/R ratio increases. In the situation of a = 5h, Figs. 7-8 show that the maximum mid-plane displacements u_0 and v_0 by the Whitney analysis will diverge and discontinue at the h/R ratio at which the maximum displacements w diverges. As similar to the maximum displacements w as shown in Fig. 4, the results will turn

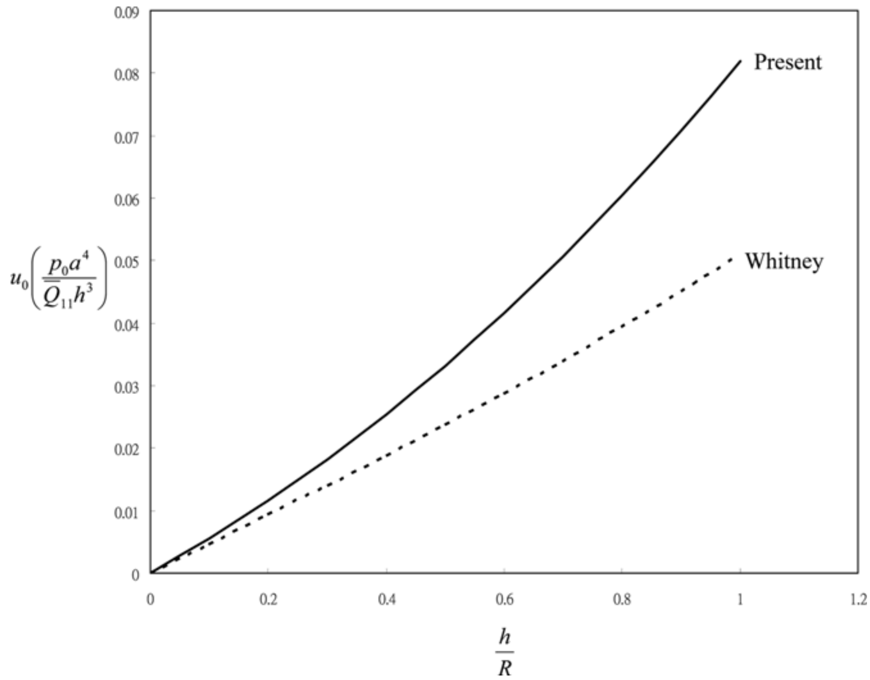

Fig. 5 The maximum mid-plane displacement in the x direction u_0 as a function of h/R for the case of $a = h$

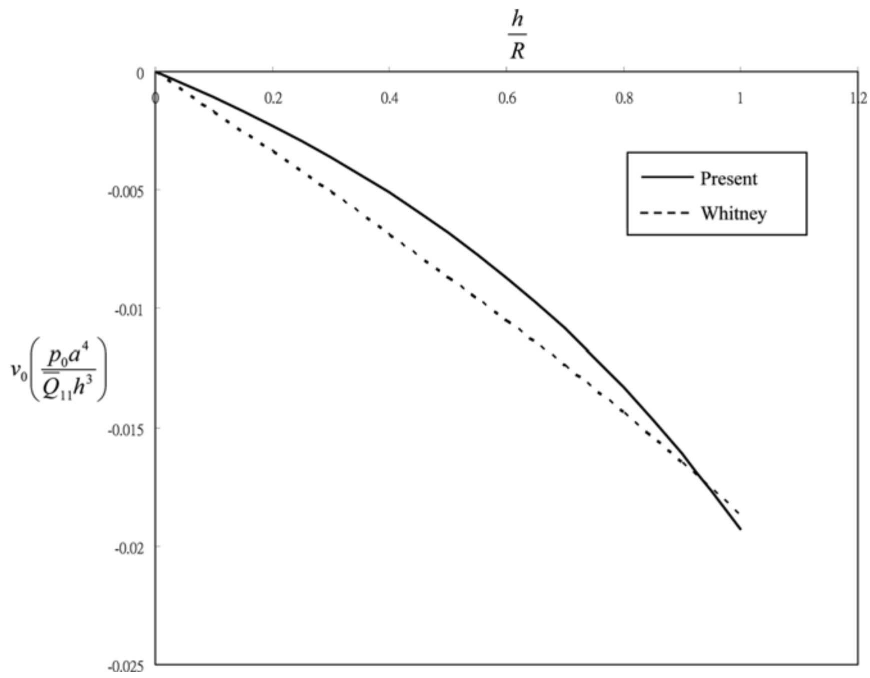

Fig. 6 The maximum mid-plane displacement in the y direction v_0 as a function of h/R for the case of $a = h$

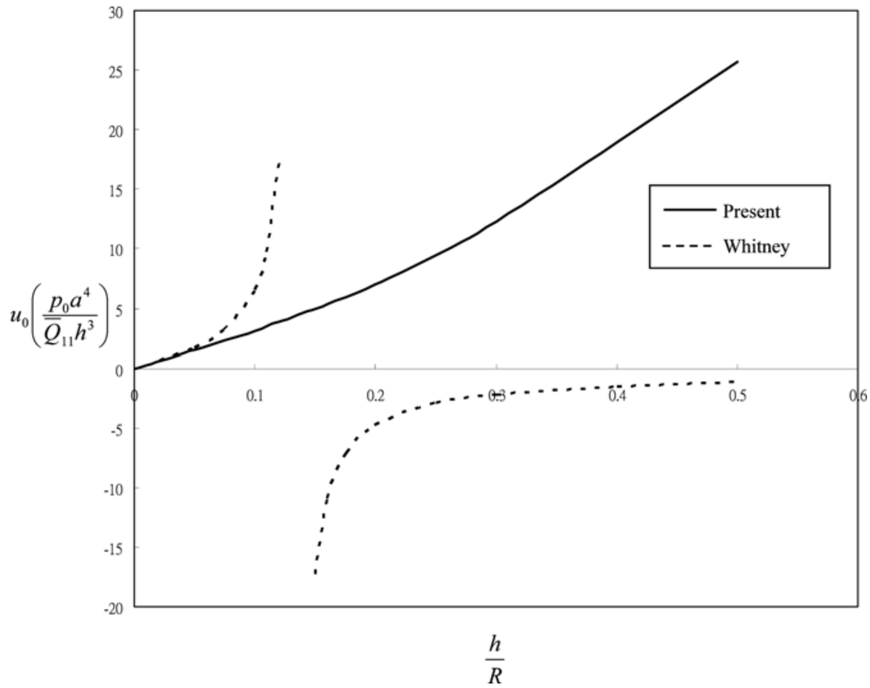

Fig. 7 The maximum mid-plane displacement in the x direction u_0 as a function of h/R for the case of $a = 5h$

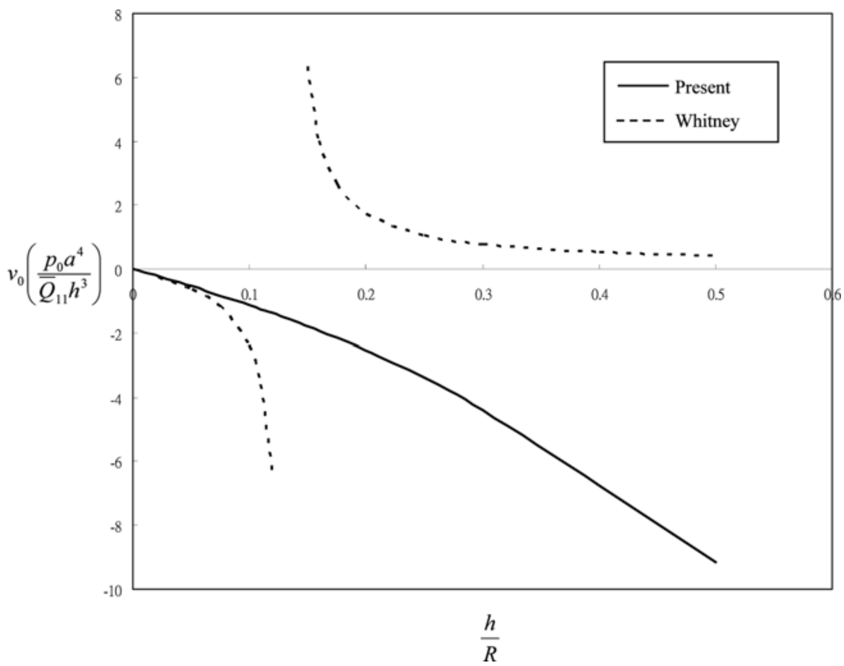

Fig. 8 The maximum mid-plane displacement in the y direction v_0 as a function of h/R for the case of $a = 5h$

into different sign as the h/R ratio is larger than the discontinued point.

After obtaining the displacements, force and moment resultants can be given by substituting Eqs. (59-61) into the constitutive relation of Eq. (15). In addition, the stresses can be obtained by substituting Eqs. (59-61) into the strain-displacement relations and the stress-strain relations. The solutions of displacements will converge more quickly than those of the stresses.

6. Conclusions

1. A general theory for a curved anisotropic plate has been developed by considering the nonlinear effect that reflecting the non-flat geometry of the structure. By applying the newly derived 6×6 matrix constitutive relation by which the force and the moment resultants can be expresses in terms of the displacements, the governing differential equations for a curved anisotropic plate have been developed in the usual manner, namely, by consideration of the constitutive relation and equilibrium equations. As the radius R becomes infinite (i.e., $\kappa \rightarrow 0$) for an isotropic material, the governing differential equations can be reduced to the well-known biharmonic equation $D\nabla^2(\nabla^2 w) = p$ for the bending of a flat isotropic plate.
2. Solutions are obtained for the anisotropic curved plate of a special orthotropic class with simply-supported boundary conditions and compared to corresponding solutions that neglecting the nonlinear effect in the analysis. In situation where the thickness-radius (h/R) ratio approaches zero, the present and Whitney analyses approach to the identical result of a flat plate. However, when the h/R ratio is getting larger, the discrepancy between two analyses broadens. Under some curved plate geometries, unreasonable results will be induced by the Whitney analysis, as indicated in Figs. 4 and 7-8. It implies that the nonlinear effect reflecting the non-flat geometry of the structure cannot be neglected on the curved plate analysis.
3. The effect of the transverse shear stress may need to be considered in addition to the effect of the nonlinear variation of stress-strain through the thickness as the h/R ratio is getting larger. However, the nonlinear effect on the curved plate analysis can be indicated by the comparison between the analyses that both didn't account for the effect of the transverse shear stress, as shown in Figs. 3-8.

References

Ambartsumyan, S.A. (1964), *Theory of Anisotropic Shells*, Moscow, 1961, English Translation, NASA-TT-F118.
Ambartsumyan, S.A. (1964), *Theory of Anisotropic Shells*, NASA Report TT F-118.
Bickford, W.B. (1998), *Advance Mechanics of Materials*, 1st ed. Addison Wesley Longman, Inc., Menlo Park.
Chaudhuri, R.A. (1986), "Arbitrarily laminated, anisotropic cylindrical shell under internal pressure", *AIAA J.*, **24**(11), 1851-1858.
Cheng, S. and Ho, B.P.C. (1963), "Stability of heterogeneous aeolotropic cylindrical shells under combined loading", *AIAA J.*, **1**(4), 892-898.
Chiang, Y.C. (2005), "The poisson effect on the curved beam analysis", *Struct. Eng. Mech.*, **19**(6), 707-720.
Dong, S.B., Pister, K.S. and Taylor, R.L. (1962), "On the theory of laminated anisotropic shells and plates," *J. of the Aerospace Sci.*, **29**, 969-975.
Flügge, W. (1967), *Stress in Shells,* Springer, Berlin.
Gulati, S.T. and Essenberg, F. (1967), "Effects of anisotropic in axisymmetric cylindrical shells", *J. Appl. Mech.*, **34**, 659-666.

Herakovich, C.T. (1998), *Mechanics of Fibrous Composites*, New York: John Wiley & Sons, Inc..
Kjellmert, B. (1997), "A Chebyshev collocation multidomain method to solve the Reissner-Mindlin equations for the transient response of an anisotropic plate subjected to impact", *Int. J. Numer. Meth. Eng.*, **40**, 3689-3702.
Koiter, W.T. (1959), "A consistent first approximation in the general theory of thin elastic shells," in *Proc. of the IUTAM Symposium on The Theory of Thin Elastic Shells*, Delft, 24-28 August, 1959, Edited by W.T. Koiter, North-Holland Publishing Co., Amsterdam, 12-33.
Kraus, H. (1972), *Thin Elastic Shells: An Introduction to the Theoretical Foundations and the Analysis of Their Static and Dynamic Behavior,* John Wiley and Sons, Inc..
Lekhnitskii, S.G. (1986), *Anisotropic Plates*, Gordon and Breach Science Publishers, New York.
Novozhilov, V.V. (1959), *The Theory of Thin Shells*, Translated by Lowe, P.G., Edited by Radok, J.R.M. Groningen: P. Noordhoff.
Reddy, J.N. (1984), "Exact solutions of moderately thick laminated shells", *J. Eng. Mech.*, **110**(5), 794-809.
Reddy, J.N. (2004), *Mechanics of Laminated Composite Plates and Shells Theory and Analysis*, 2nd Ed., CRC Press.
Reddy, J.N. and Wang, C.M. (2000), "An overview of the relationships between solutions of the classical and shear deformation plate theories", *Comp. Sci. Tech.*, **60**, 2327-2335.
Sanders Jr., J.L. (1959), "An improved first approximation theory for thin shells", NASA TR-R24.
Sokolnikoff, I.S. (1964), *Tensor Analysis: Theory and Applications to Geometry and Mechanics of Continua*, John Wiley and Sons, Inc..
Swanson, S.R. (1997), *Introduction to Design and Analysis with Advanced Composite Materials*, Prentice-Hall, Inc..
Timoshenko, S. and Woinowsky-Krieger, S. (1959), *Theory of Plates and Shells*, Second Edition, McGraw-Hill Book Company, Inc..
Vinson, J.R. and Sierakowski, R.L. (1987), *The Behavior of Structures Composed of Composite Materials*, Kluwer Adademic Publishers.
Vlasov, V.Z. (1964), *General Theory of Shells and Its Application to Engineering*, Moscow-Leningrad, 1949, NASA Technical Translation NASA-TT-F-99.
Whitney, J.M. (1987), *Structure Analysis of Laminated Anisotropic Plates*, Technomic Publishing Co. Inc..
Whitney, J.M. and Sun, C.-T. (1974), "A refined theory for laminated anisotropic, cylindrical shells", *J. Appl. Mech.*, **41**(2), 471-476.
Zukas, J.A. and Vinson, J.R. (1971), "Laminated transversely isotropic cylindrical shells", *J. Appl. Mech.*, **38**, 400-407.