Structural Engineering and Mechanics, Vol. 20, No. 4 (2005) 477-493
DOI: http://dx.doi.org/10.12989/sem.2005.20.4.477

Added effect of uncertain geometrical parameter on the response variability of Mindlin plate

Hyuk Chun Noh†

Department of Civil Engineering and Engineering Mechanics, Columbia University, New York, NY 10027, USA

Chang Koon Choi‡

Department of Civil and Environmental Engineering, Korea Advanced Institute of Science and Technology, Daejeon 305-701, Korea

(Received February 16, 2005, Accepted May 18, 2005)

Abstract. In case of Mindlin plate, not only the bending deformation but also the shear behavior is allowed. While the bending and shear stiffness are given in the same order in terms of elastic modulus, they are in different order in case of plate thickness. Accordingly, bending and shear contributions have to be dealt with independently if the stochastic finite element analysis is performed on the Mindlin plate taking into account of the uncertain plate thickness. In this study, a formulation is suggested to give the response variability of Mindlin plate taking into account of the uncertainties in elastic modulus as well as in the thickness of plate, a geometrical parameter, and their correlation. The cubic function of thickness and the correlation between elastic modulus and thickness are incorporated into the formulation by means of the modified auto- and cross-correlation functions, which are constructed based on the general formula for *n*-th joint moment of random variables. To demonstrate the adequacy of the proposed formulation, a plate with various boundary conditions is taken as an example and the results are compared with those obtained by means of classical Monte Carlo simulation.

Key words: stochastic FEM; weighted integral method; plates; modified cross- and auto-correlation function; response variability; Monte Carlo simulation.

1. Introduction

The ultimate concern of stochastic mechanics might be divided into two main issues. The one is to incorporate as many uncertain system parameters as possible in the analysis, and the other is the evaluation of possible maximum (so-called, the bounds) of response variability. As far as the aim of stochastic mechanics is concerned, these two concerns are complementary to each other to give, say, final results on the response variability of stochastic systems.

From the viewpoint of system parameters, the other name of structural system is the stochastic system as virtually all the system parameters contains, it its nature, the uncertainty. Accordingly, it

† Ph.D., Research Associate, Corresponding author, E-mail: hcn2101@columbia.edu
‡ Institute Chair Professor, E-mail: cck@kaist.ac.kr

is highly demanded to include as many uncertain system parameters as possible in the analysis based on rigorous mathematical and statistical terms so as the stochastic analysis to give practical outcomes for actual application and to reveal the nature and degree of influences of these parameters on the response variability. Many researches have been performed to accomplish this in various ways with various objectives: response variability due to uncertain elastic modulus (Choi 1993, 1996a, Deodatis 1991, Falsone 2002, Liu 1986, Shinozuka 1988) and uncertain geometrical parameters (Choi 1996a, 2000, Graham 2001); response statistics in time domain due to uncertain random excitation with deterministic (Choi 1996b, To 1986) or stochastic material parameters (Manjuprasad 2003); random eigenvalue analysis (Graham 2001, Zhu 1992) and soil-structure interaction (Sarkani 1999). The application to homogenization of composite material is also performed (Kaminski 2001). In addition, various mathematical formulations for model problems are also pursued (Babuska 1972, Philipp 2005). In case of the bounds of response variability, some research works have been proposed recently (Deodatis 2003, Papadopoulos 2005) where an effort to establish spectral- and probability-distribution-free upper bounds are achieved for stochastic beams.

In addition to the non-statistical or analytic methods, the statistical methodology, where the Monte Carlo simulation (MCS) approach stands at the center, is also of concerns (Choi 2000, Shinozuka 1972, Stefanou 2004). In this area, the random sample generation techniques for Gaussian and non-Gaussian, homogeneous (stationary) and inhomogeneous (non-stationary) stochastic fields (Deodatis 1996, Popescu 1998, Yamazaki 1990) of nVmD (n-variate m-dimensional) are of great importance. As far as the MCS is concerned, the accurate and efficient random field generation technique is an utmost crucial factor since the analysis results and the analysis efforts as well are heavily depend on the random field employed.

In this paper, as an approach related to the first item of ultimate concern of stochastic mechanics, mentioned above, a formulation to evaluate the response statistics due to multiple uncertain parameters is presented for the Mindlin plate in the context of weighted integral method. The kinematics of plate can be divided into two main ones: Poisson-Kirchhoff and Reissner-Mindlin. In the Poisson-Kirchhoff assumption, the shear deformation is not included and requires C^1-continuity. Without consideration on shear contribution, the stochastic formulation can be achieved with relative ease. However, in case of Reissner-Mindlin type of plate element, where only C^0-continuity is required, special concern has to be taken to take into account of the shear effect for the evaluation of the response variability. This is caused by the different order of contributions of thickness, a geometrical parameter, in bending and shear stiffness. Accordingly, the correlation between elastic modulus and thickness of plate affects bending and shear stiffness in different ways. In addition, there also appears the correlation effect between bending and shear behaviors. The assessment of the effect of uncertain plate thickness in plate bending problem has been investigated in (Choi 1996a). In that study, however, only the sole effect of this random parameter and only in bending stiffness were investigated without any consideration of the correlation with other random parameter such as elastic modulus and of correlation between bending and shear stiffness.

The correlations between uncertain parameters and the cubic functions of thickness are incorporated into the formulation by means of the general formula for n-th joint moment of Gaussian random variables (Lin 1967). Accordingly, it has to be mentioned that the uncertain material parameters are assumed implicitly to follow the Gaussian distribution. The degree of correlation between two uncertain parameters is defined by introducing the cross-correlation factor (CCF) $\gamma_{Et} \in [-1, 1]$, where $\gamma_{Et} = -1.0,\ 0.0$ and 1.0 means perfect negative, zero and perfect positive correlation, respectively.

2. Uncertain parameters in plate bending

If we put the attention only on the structure excluding the factors in environment such as boundary conditions and applied loads, the parameters involving uncertainty are material and geometrical ones, i.e., Elastic modulus E, Poisson's ratio ν and thickness of plate t. The general expression for these uncertain parameters can be written as

$$S(\mathbf{x}) = \bar{S}[1 + f(\mathbf{x})] \tag{1}$$

where $\bar{S}$ denotes the ensemble mean of position dependent stochastic parameter $S(\mathbf{x})$. The stochastic field function $f(\mathbf{x})$ is assumed to be homogeneous with zero mean, as implied in Eq. (1). The position vector $\mathbf{x}$, which determines the position dependent value of stochastic parameters in the domain, belongs to the domain of structure, $\mathbf{x} \in \Omega_{str}$.

The element stiffness in plate bending with Mindlin assumption has these three uncertain parameters as ingredients. In case of plate thickness, however, it is noticed that this parameter has different contributions in bending and shear, respectively. Furthermore, the Poisson's ratio is seen to be difficult to include in the formulation, though the sole effect of this parameter on the structural response variability in plate bending problem is given recently (Noh 2004). The difficulties arise from the higher order terms involved in the polynomial expansion on the Poisson's ratio which makes it almost not plausible to realize the so-called modified auto-correlation functions when it is correlated with higher order terms in stochastic field of plate thickness t, which contributes the element stiffness as a cubic function, another higher order term. Accordingly, in this study, only the uncertainties in elastic modulus and in plate thickness are taken into account.

The constitutive matrices for bending and shear in Mindlin plate are given as follows:

$$\mathbf{D}_b(E, t) = \frac{Et^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & \\ \nu & 1 & \\ & & \dfrac{1-\nu}{2} \end{bmatrix}, \quad \mathbf{D}_s(E, t) = \kappa t\begin{bmatrix} G & \\ & G \end{bmatrix} \tag{2}$$

where, κ is a shear correction factor, 5/6, and G is a shear elastic modulus which is given as $E/2(1+\nu)$. If use is made of the general equation for stochastic parameter in Eq. (1) to elastic modulus and thickness, the constitutive matrices can be rearranged as

$$\begin{gathered} \mathbf{D}_b(E, t) = \frac{\bar{E}(1+f_E)\bar{t}^3(1+f_t)^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & \\ \nu & 1 & \\ & & \dfrac{1-\nu}{2} \end{bmatrix} \\ \mathbf{D}_s(E, t) = \kappa\bar{E}(1+f_E)\bar{t}(1+f_t)\begin{bmatrix} \dfrac{1}{2(1+\nu)} & \\ & \dfrac{1}{2(1+\nu)} \end{bmatrix} \end{gathered} \tag{3}$$

With Eq. (3), we can divide each constitutive matrix into the deterministic and deviatoric ones as

$$\begin{aligned}\mathbf{D}_b(E, t) &= \mathbf{D}_b^{\det} + \Delta\mathbf{D}_b = \mathbf{D}_b(\bar{E}, \bar{t}) + f_b(\mathbf{x})\mathbf{D}_b(\bar{E}, \bar{t}) \\ \mathbf{D}_s(E, t) &= \mathbf{D}_s^{\det} + \Delta\mathbf{D}_s = \mathbf{D}_s(\bar{E}, \bar{t}) + f_s(\mathbf{x})\mathbf{D}_s(\bar{E}, \bar{t})\end{aligned} \tag{4}$$

where, $f_b(\mathbf{x}) = f_E + F_t + F_{Et}$, $f_s(\mathbf{x}) = f_E + f_t + f_E f_t$ and $F_t = 3f_t + 3f_t^2 + f_t^3$, $F_{Et} = f_E F_t$. The two functions, $f_E(\mathbf{x}), f_t(\mathbf{x})$, are the original stochastic field functions which represent the uncertain field of elastic modulus and plate thickness.

3. Stochastic element stiffness

From the minimization of elastic strain energy functional, the element stiffness based on the displacement formulation can be derived as follows:

$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{D}(s) \mathbf{B} dV \tag{5}$$

where, 's' is used to designate that the constitutive matrix is a function of stochastic parameters. In case of Mindlin plate, the element stiffness is divided into two parts: bending (b) and shear (s). Accordingly, the genuine form of element stiffness is given as

$$\mathbf{k} = \int_V \mathbf{B}_b^T \mathbf{D}_b(E, t) \mathbf{B}_b dV + \int_V \mathbf{B}_s^T \mathbf{D}_s(E, t) \mathbf{B}_s dV \tag{6}$$

Substituting Eq. (4) into Eq. (6), the element stiffness can be divided into the deterministic and deviatoric ones as follows:

$$\begin{aligned}\mathbf{k} &= \int_V \mathbf{B}_b^T \{\mathbf{D}_b(\bar{E}, \bar{t}) + f_b(\mathbf{x})\mathbf{D}_b(\bar{E}, \bar{t})\} \mathbf{B}_b dV + \int_V \mathbf{B}_s^T \{\mathbf{D}_s(\bar{E}, \bar{t}) + f_s(\mathbf{x})\mathbf{D}_s(\bar{E}, \bar{t})\} \mathbf{B}_s dV \\ &= \mathbf{k}_b^{\det} + \Delta\mathbf{k}_b + \mathbf{k}_s^{\det} + \Delta\mathbf{k}_s \\ &= \mathbf{k}_{\det} + \Delta\mathbf{k}\end{aligned} \tag{7}$$

In the weighted integral stochastic methods, the random variable X is defined as an integration of unknown stochastic function $f(\mathbf{x})$ multiplied (i.e., weighted) by a known deterministic function $g(\mathbf{x})$.

$$X = \int_V f(\mathbf{x}) g(\mathbf{x}) dV \tag{8}$$

In Eq. (7), the random variables can be seen explicitly if the strain displacement matrices $\mathbf{B}_b$ and $\mathbf{B}_s$ are expressed as the sum of constant matrix $\mathbf{B}_{ci}$ multiplied by an independent polynomial p_{ci}, as ($c = b$ or s)

$$\mathbf{B}_b = \sum_{i=1}^{b_n} \mathbf{B}_{bi} p_{bi}, \quad \mathbf{B}_s = \sum_{i=1}^{s_n} \mathbf{B}_{si} p_{si} \tag{9}$$

where, b_n, s_n denote the number of independent polynomials in strain-displacement matrices $\mathbf{B}_b$, $\mathbf{B}_s$ respectively. Substituting the decomposed strain-displacement matrices into Eq. (7), the deviatoric stiffness is given as a function of random variables as follows:

$$\begin{aligned}\Delta\mathbf{k} &= \Delta\mathbf{k}_b + \Delta\mathbf{k}_s \\ &= \int_V f_b(\mathbf{x})\mathbf{B}_b^T\mathbf{D}_b(\bar{E}, \bar{t})\mathbf{B}_b dV + \int_V f_s(\mathbf{x})\mathbf{B}_s^T\mathbf{D}_s(\bar{E}, \bar{t})\mathbf{B}_s dV \\ &= \mathbf{B}_{bi}^T\mathbf{D}_b(\bar{E}, \bar{t})\mathbf{B}_{bj}X_{bij}^{(e)} + \mathbf{B}_{sk}^T\mathbf{D}_s(\bar{E}, \bar{t})\mathbf{B}_{sl}X_{skl}^{(e)}\end{aligned} \tag{10}$$

where,

$$X_{cij}^{(e)} = \int_V f_c(\mathbf{x})p_{ci}p_{cj}dV; \quad c = b \text{ or } s; \quad i,j = 1, 2, \ldots, b_n \text{ or } s_n \tag{11}$$

The superscript 'e' is appended to designate that the random variables are related with the specific element 'e' under consideration. From Eq. (10), it is noticed that the number of random variables, N_{RV}, is evaluated to be

$$\begin{aligned}N_{RV} &= \frac{1}{2}b_n(b_n + 1) + \frac{1}{2}s_n(s_n + 1) \\ &= N_{RV^{(b)}} + N_{RV^{(s)}}\end{aligned} \tag{12}$$

where, $N_{RV^{(b)}}$ is the number of random variables related to the bending stiffness and $N_{RV^{(s)}}$ to the shear stiffness. All the random variables in the domain of analysis, with N_e number of finite elements, can be written in a vector form as follows:

$$\begin{aligned}\mathbf{X} &= \langle X_{RV_b}^e, X_{RV_s}^e:\ RV_b = 1, \ldots, N_{RV^{(b)}};\ RV_s = 1, \ldots, N_{RV^{(s)}};\ e = 1, \ldots, N_e\rangle^T \\ &\overset{\text{or}}{=} \langle X_1^1, X_2^1, \ldots, X_{N_{RV}}^1, X_1^2, X_2^2, \ldots, X_{N_{RV}}^2, \ldots, X_1^{N_e}, X_2^{N_e}, \ldots, X_{N_{RV}}^{N_e}\rangle^T\end{aligned} \tag{13}$$

3.1 Mean stiffness

As it is seen in Eq. (4), the stochastic field functions $f_b(\mathbf{x}), f_s(\mathbf{x})$ are given as a joint function of f_E and f_t. As a result, the mean of element stiffness in Eq. (7) is not given as the deterministic stiffness but has some contribution from the deviatoric stiffness. That is, the mean stiffness is $\bar{\mathbf{k}} = \mathbf{k}_{\det} + E[\Delta\mathbf{k}]$. The additional term $E[\Delta\mathbf{k}]$ can be evaluated as follows:

$$E[\Delta\mathbf{k}] = \int_V E[f_b(\mathbf{x})]\mathbf{B}_b^T\mathbf{D}_b(\bar{E}, \bar{t})\mathbf{B}_b dV + \int_V E[f_s(\mathbf{x})]\mathbf{B}_s^T\mathbf{D}_s(\bar{E}, \bar{t})\mathbf{B}_s dV \tag{14}$$

From the definition of $f_b(\mathbf{x}), f_s(\mathbf{x})$, the expectations in Eq. (14) can be written as

$$\begin{aligned}E[f_b(\mathbf{x})] &= E[3f_t^2 + 3f_E f_t + f_E f_t^3] \\ E[f_s(\mathbf{x})] &= E[f_E f_t]\end{aligned} \tag{15}$$

As can be noted from the general formula for n-th joint moment (Lin 1967), the expectation on the joint stochastic function in odd number vanishes, and therefore the Eq. (15) is resulted in. Furthermore, the expectation in Eq. (15) can be replaced by specific auto- and cross-correlation functions as follows:

$$E[f_b(\mathbf{x})] = R_b(\xi) = 3R_{tt}(\xi) + 3R_{Et}(\xi) + 3R_{tt}(\xi)R_{Et}(\xi)$$
$$E[f_s(\mathbf{x})] = R_s(\xi) = R_{Et}(\xi) \tag{16}$$

Consequently, the element stiffness can be written as the sum of mean stiffness $\bar{\mathbf{k}}$ and deviatoric stiffness $\delta\mathbf{k}$ as follows:

$$\mathbf{k} = \bar{\mathbf{k}} + \delta\mathbf{k} = (\mathbf{k}_{\mathrm{det}} + E[\Delta\mathbf{k}]) + (\Delta\mathbf{k} - E[\Delta\mathbf{k}]) \tag{17}$$

It has to be noted here that the mean stiffness $\bar{\mathbf{k}}$ is used for the evaluation of the mean displacement $\bar{\mathbf{U}}$. This means that the mean displacement is dependent on the correlation distance of stochastic fields. In deriving Eq. (16) from Eq. (15), the general formula for the n-th joint moment, as given in Eq. (18), is employed.

$$E[X_1 X_2 \ldots X_n] = \sum_{k,\,k \neq j} E[X_{r_1} X_{r_2} \ldots X_{r_{n-2}}] E[X_k X_j] \tag{18}$$

In general, if $n = 2m$, Eq. (18) results in N terms of m pairs of $E[X_i X_j]$, where N is evaluated as $N = n!/m!2^m$.

4. Evaluation of the first and second statistical moments

Since the deviatoric stiffness $\Delta\mathbf{k}$, or $\delta\mathbf{k}$ in a more stringent sense, is given as a function of random variable $X^e_{RV}(RV = 1, \ldots, N_{RV};\ e = 1, \ldots, N_e)$, the displacement vector $\mathbf{U}$ can be expanded in the first order Taylor's series centered by the mean random variable $\bar{X}^e_{RV}$ as follows:

$$\mathbf{U} \approx \bar{\mathbf{U}}(\bar{X}^e_{RV_b}, \bar{X}^e_{RV_s}) + \sum_{e=1}^{N_e} \sum_{RV_b=1}^{N_{RV(b)}} (X^e_{RV_b} - \bar{X}^e_{RV_b}) \left[\frac{\partial\mathbf{U}}{\partial X^e_{RV_b}}\right]_E + \sum_{e=1}^{N_e} \sum_{RV_s=1}^{N_{RV(s)}} (X^e_{RV_s} - \bar{X}^e_{RV_s}) \left[\frac{\partial\mathbf{U}}{\partial X^e_{RV_s}}\right]_E \tag{19}$$

Employing the partial differentiation on the equilibrium equation, $\mathbf{K}(s)\mathbf{U}(s) = \mathbf{F}$, the derivative of displacement vector with respect to the random variable can be replaced by that of stiffness. Accordingly, Eq. (19) can be modified as follows:

$$\mathbf{U} \approx \bar{\mathbf{U}}(\bar{X}^e_{RV_b}, \bar{X}^e_{RV_s})$$
$$- \sum_{e=1}^{N_e} \sum_{RV_b=1}^{N_{RV(b)}} (X^e_{RV_b} - \bar{X}^e_{RV_b}) \bar{\mathbf{K}}^{-1} \left[\frac{\partial\mathbf{K}}{\partial X^e_{RV_b}}\right]_E \bar{\mathbf{U}} - \sum_{e=1}^{N_e} \sum_{RV_s=1}^{N_{RV(s)}} (X^e_{RV_s} - \bar{X}^e_{RV_s}) \bar{\mathbf{K}}^{-1} \left[\frac{\partial\mathbf{K}}{\partial X^e_{RV_s}}\right]_E \bar{\mathbf{U}} \tag{20}$$

4.1 Mean of displacement

Since the mean operation on the displacement vector in Eq. (20) gives $E[X^e_{RV_b}] = \bar{X}^e_{RV_b}$ and $E[X^e_{RV_s}] = \bar{X}^e_{RV_s}$, the mean displacement is resulted in as

$$E[\mathbf{U}] = \bar{\mathbf{U}} \tag{21}$$

As noted in the preceding section, $\bar{\mathbf{U}}$ is obtained using the mean stiffness $\bar{\mathbf{k}}$ in Eq. (17).

4.2 Covariance of displacement

The covariance is defined as an expectation on the square of mean centered deviation of variable under consideration. Therefore, with $\Delta\mathbf{U} = \mathbf{U} - \bar{\mathbf{U}}$ and symbolic replacements of

$$\begin{aligned}
\Sigma &= \sum_{e=1}^{N_e}\sum_{RV_b=1}^{N_{RV(b)}} X_{RV_b}^{e}\bar{\mathbf{K}}^{-1}\left[\frac{\partial\mathbf{K}}{\partial X_{RV_b}^{e}}\right]_E\bar{\mathbf{U}} + \sum_{e=1}^{N_e}\sum_{RV_s=1}^{N_{RV(s)}} X_{RV_s}^{e}\bar{\mathbf{K}}^{-1}\left[\frac{\partial\mathbf{K}}{\partial X_{RV_s}^{e}}\right]_E\bar{\mathbf{U}} \\
&= \sum_{e=1}^{N_e}\sum_{RV=1}^{N_{RV}} X_{RV}^{e}\bar{\mathbf{K}}^{-1}\left[\frac{\partial\mathbf{K}}{\partial X_{RV}^{e}}\right]_E\bar{\mathbf{U}} \\
\bar{\Sigma} &= \sum_{e=1}^{N_e}\sum_{RV_b=1}^{N_{RV(b)}} \bar{X}_{RV_b}^{e}\bar{\mathbf{K}}^{-1}\left[\frac{\partial\mathbf{K}}{\partial X_{RV_b}^{e}}\right]_E\bar{\mathbf{U}} + \sum_{e=1}^{N_e}\sum_{RV_s=1}^{N_{RV(s)}} \bar{X}_{RV_s}^{e}\bar{\mathbf{K}}^{-1}\left[\frac{\partial\mathbf{K}}{\partial X_{RV_s}^{e}}\right]_E\bar{\mathbf{U}} \\
&= \sum_{e=1}^{N_e}\sum_{RV=1}^{N_{RV}} \bar{X}_{RV}^{e}\bar{\mathbf{K}}^{-1}\left[\frac{\partial\mathbf{K}}{\partial X_{RV}^{e}}\right]_E\bar{\mathbf{U}}
\end{aligned} \tag{22}$$

the covariance of displacement can be given as follows:

$$\begin{aligned}
Cov[\mathbf{U}] &= E[\Delta\mathbf{U}\Delta\mathbf{U}^T] \\
&= E[(-\Sigma+\bar{\Sigma})(-\Sigma+\bar{\Sigma})^T] \\
&= E[\Sigma\Sigma^T] - E[\bar{\Sigma}\bar{\Sigma}^T]
\end{aligned} \tag{23}$$

where, it is noticed that $E[\Sigma\bar{\Sigma}^T] = E[\bar{\Sigma}\Sigma^T] = E[\bar{\Sigma}\bar{\Sigma}^T]$.

The first term of covariance in Eq. (23) can be rearranged, with aid of Eq. (22), as follows:

$$\begin{aligned}
E[\Sigma\Sigma^T] &= E\left[\left(\sum_{ei=1}^{N_e}\sum_{RVi=1}^{N_{RV}} X_{RVi}^{ei}\bar{\mathbf{K}}^{-1}\left[\frac{\partial\mathbf{K}}{\partial X_{RVi}^{ei}}\right]_E\bar{\mathbf{U}}\right)\left(\sum_{ej=1}^{N_e}\sum_{RVj=1}^{N_{RV}} X_{RVj}^{ej}\bar{\mathbf{K}}^{-1}\left[\frac{\partial\mathbf{K}}{\partial X_{RVj}^{ej}}\right]_E\bar{\mathbf{U}}\right)^T\right] \\
&= \bar{\mathbf{K}}^{-1}E\left[\sum_{ei}\Delta\mathbf{k}^{ei}\bar{\mathbf{U}}\bar{\mathbf{U}}^T\sum_{ej}\Delta\mathbf{k}^{ej}\right]\bar{\mathbf{K}}^{-T} \\
&= \bar{\mathbf{K}}^{-1}\sum_{ei}\sum_{ej}\bar{\bar{\mathbf{F}}}_{eiej,E}\bar{\mathbf{K}}^{-T}
\end{aligned} \tag{24}$$

After some manipulations, the term $\bar{\bar{\mathbf{F}}}_{eiej,E}$, a load-equivalent covariance matrix, in Eq. (24) can be written as follows:

$$\bar{\bar{\mathbf{F}}}_{eiej,E} = E[\Delta\mathbf{k}^{ei}\bar{\mathbf{U}}\bar{\mathbf{U}}^T\Delta\mathbf{k}^{ej}] \tag{25}$$

The load-equivalent covariance matrix in Eq. (25), between two distinct elements *ei* and *ej*, can be written in integral form as follows:

$$
\begin{aligned}
\bar{\bar{\mathbf{F}}}_{eiej,E} &= E[\int_{V^{ei}}(f_b(\mathbf{x})\tilde{\mathbf{k}}_b^{ei}+f_s(\mathbf{x})\tilde{\mathbf{k}}_s^{ei})dV^{ei}\bar{\mathbf{U}}\bar{\mathbf{U}}^T\int_{V^{ej}}(f_b(\mathbf{x})\tilde{\mathbf{k}}_b^{ej}+f_s(\mathbf{x})\tilde{\mathbf{k}}_s^{ej})dV^{ej}] \\
&= \int_{V^{ei}}\int_{V^{ej}}\{E[f_b(\mathbf{x}_{ei})f_b(\mathbf{x}_{ej})]\tilde{\mathbf{k}}_b^{ei}\bar{\mathbf{U}}\bar{\mathbf{U}}^T\tilde{\mathbf{k}}_b^{ej}\}dV^{ej}dV^{ei} \\
&+ \int_{V^{ei}}\int_{V^{ej}}\{E[f_b(\mathbf{x}_{ei})f_s(\mathbf{x}_{ej})]\tilde{\mathbf{k}}_b^{ei}\bar{\mathbf{U}}\bar{\mathbf{U}}^T\tilde{\mathbf{k}}_s^{ej}\}dV^{ej}dV^{ei} \\
&+ \int_{V^{ei}}\int_{V^{ej}}\{E[f_s(\mathbf{x}_{ei})f_b(\mathbf{x}_{ej})]\tilde{\mathbf{k}}_s^{ei}\bar{\mathbf{U}}\bar{\mathbf{U}}^T\tilde{\mathbf{k}}_b^{ej}\}dV^{ej}dV^{ei} \\
&+ \int_{V^{ei}}\int_{V^{ej}}\{E[f_s(\mathbf{x}_{ei})f_s(\mathbf{x}_{ej})]\tilde{\mathbf{k}}_s^{ei}\bar{\mathbf{U}}\bar{\mathbf{U}}^T\tilde{\mathbf{k}}_s^{ej}\}dV^{ej}dV^{ei}
\end{aligned} \tag{26}
$$

The expectations on the stochastic field functions in Eq. (26) can be transformed into modified auto- and cross-correlation functions as given in Appendix. The $\tilde{\mathbf{k}}_b$ and $\tilde{\mathbf{k}}_s$ in Eq. (26) denote $\tilde{\mathbf{k}}_b = \mathbf{B}_b^T\mathbf{D}_b(\bar{E},\bar{t})\mathbf{B}_b$ and $\tilde{\mathbf{k}}_s = \mathbf{B}_s^T\mathbf{D}_s(\bar{E},\bar{t})\mathbf{B}_s$ as given previously in Eq. (7).

The second term, which is related to the terms having mean of random variable, is

$$
\begin{aligned}
&E[\bar{\Sigma}\bar{\Sigma}^T] \\
&= E\left[\left(\sum_{ei=1}^{N_e}\sum_{RVi=1}^{N_{RV}}\bar{X}_{RVi}^{ei}\bar{\mathbf{K}}^{-1}\left[\frac{\partial\mathbf{K}}{\partial X_{RVi}^{ei}}\right]_E\bar{\mathbf{U}}\right)\left(\sum_{ej=1}^{N_e}\sum_{RVj=1}^{N_{RV}}\bar{X}_{RVj}^{ej}\bar{\mathbf{K}}^{-1}\left[\frac{\partial\mathbf{K}}{\partial X_{RVj}^{ej}}\right]_E\bar{\mathbf{U}}\right)^T\right] \\
&= \bar{\mathbf{K}}^{-1}\left(\sum_{e=1}^{N_e}\int_V(R_b(\xi)\Delta\tilde{\mathbf{k}}_b^e+R_s(\xi)\Delta\tilde{\mathbf{k}}_s^e)dV\bar{\mathbf{U}}\right)\left(\sum_{e=1}^{N_e}\int_V(R_b(\xi)\Delta\tilde{\mathbf{k}}_b^e+R_s(\xi)\Delta\tilde{\mathbf{k}}_s^e)dV\bar{\mathbf{U}}\right)^T\bar{\mathbf{K}}^{-T}
\end{aligned} \tag{27}
$$

It has to be noted that the auto-correlation functions for bending and shear in Eq. (27), $R_b(\xi)$, $R_s(\xi)$, are defined in Eq. (16).

5. Numerical applications

5.1 Monte Carlo Simulation and auto-correlation function

The 2V2D stochastic fields for uncertain elastic modulus and plate thickness are generated using the statistical preconditioning scheme and are used in Monte Carlo simulation (MCS). Noting that $f_E(\mathbf{x}) = f_t(\mathbf{x})$ when $\gamma_{Et} = 1.0$ and $f_E(\mathbf{x}) = -f_t(\mathbf{x})$ when $\gamma_{Et} = -1.0$, special concern is taken to save the computation time for numerical realization of 2V2D random fields. The auto-correlation functions used in random field generation are as follows:

$$
R_{EE}(\xi) = \sigma_{EE}^2\exp\left\{-\frac{|\xi_x|}{d_x}-\frac{|\xi_y|}{d_y}\right\},\quad R_{tt}(\xi) = \sigma_{tt}^2\exp\left\{-\frac{|\xi_x|}{d_x}-\frac{|\xi_y|}{d_y}\right\} \tag{28}
$$

where, σ_{EE}, σ_{tt} denote the coefficient of variation of random field of elastic modulus and plate thickness, respectively. The correlation distances in x and y directions are denoted as d_i, and ξ_i stands for components of separation vector ξ. The function for cross-correlation between two uncertain parameters is assumed to be the same as the auto-correlation function of each random parameter as follows:

$$R_{Et}(\xi) = \gamma_{Et}\kappa_{Et}\exp\left\{-\frac{|\xi_x|}{d_x}-\frac{|\xi_y|}{d_y}\right\} \tag{29}$$

where, κ_{Et} denotes the covariance between uncertain elastic modulus and plate thickness and is defined as $\kappa_{Et} = E[(X_E - \bar{X}_E)(X_t - \bar{X}_t)]$. The degree of correlation between two random parameters is controlled by the cross-correlation factor (CCF) γ_{Et}. The values of $\gamma_{Et} = -1.0$, 1.0 and 0.0 represent negative perfect, positive perfect and zero correlations, respectively. Obviously, the auto- and cross-correlation functions given in Eqs. (28) and (29) are also employed in the weighted integral scheme suggested in this study. In performing MCS, a local average scheme is adopted. The local average means that the representative random number for each finite element is evaluated as an average of all random numbers over the domain of finite element under consideration.

5.2 Description of example structure

A square plate in dimension of 20×20 carrying distributed load $q = 1.0$ or concentrated load $p = 1.0$ at the center of plate (point A in Fig. 1) is chosen as an example. As boundary conditions, simple and clamp support are assumed. The mean material and geometrical parameters are: elastic modulus $\bar{E} = 10920.0$, Poisson's ratio $\bar{\nu} = 0.25$ and thickness $\bar{t} = 1.0$ unless mentioned otherwise. Here, all the parameters are given without units so that any units can be specified as long as they are used consistently. Considering the symmetries in geometry and applied load, a quarter model with 6×6 elements is employed as shown in Fig. 1. The coefficient of variation (COV) of response is extracted from point A in Fig. 1 for all the boundary and loading conditions. The COV, an indicator for the variability of structural response, is given as a ratio of square root of variance to the mean of response R, as given in Eq. (30).

$$COV = \frac{\sqrt{Var[R]}}{|\bar{R}|} \tag{30}$$

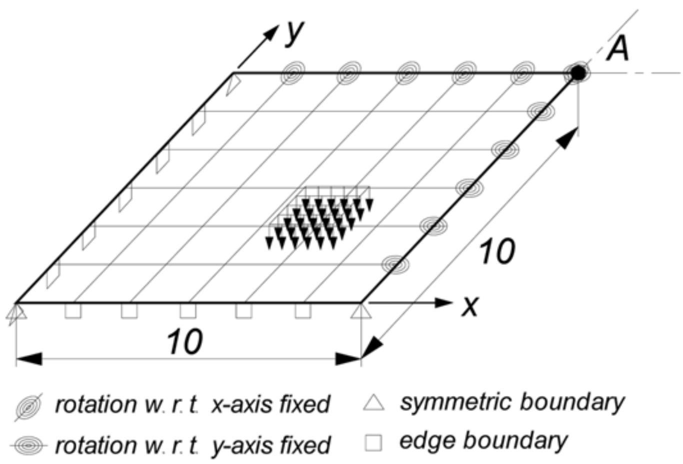

Fig. 1 Example plate structure

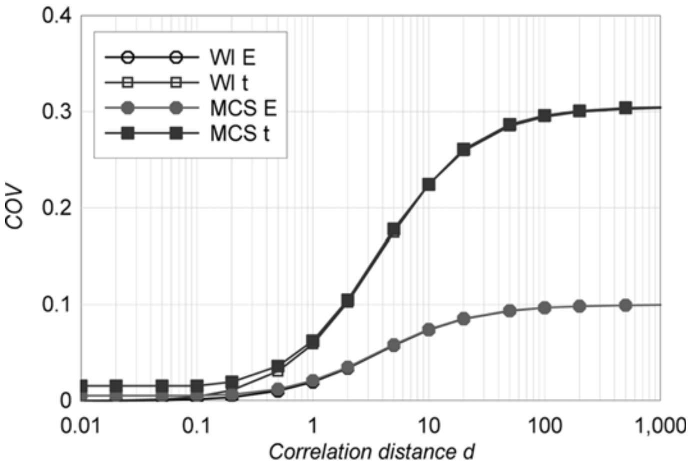

Fig. 2 Response variability due to single uncertain parameter: d vs. COV

5.3 Response variability due to individual uncertainties

Though the formulation suggested in this study deals with the multiple uncertain parameters and their correlation, it should also have the capability of reproducing the response variability due to single uncertain parameter, i.e., elastic modulus or plate thickness. The analysis results for single uncertain parameter are shown in Fig. 2 for simply supported plate. The intensity of uncertainty is taken as $\sigma_{EE} = \sigma_{tt} = 0.1$ for each analysis. The MCS results are also given for comparison. It is examined that the response variability is in good agreement between the two analyses.

5.4 Influence of cross-correlation and boundary condition

For the comparison of influence of boundary condition, plates with simple and clamped boundary conditions are analyzed. As far as the correlation is concerned, it is not clear how the two uncertain parameters, elastic modulus and plate thickness, are correlated each other. And there is not any experimental data for this and so are the mathematical or theoretical evidences. At most, we can assume a weak negative correlation between elastic modulus and plate thickness if we refer to the so-called *link theory*, which states that the longer the link, the weaker the link becomes. Accordingly, the three cases, $\gamma_{Et} = -1.0$, 0.0 and +1.0, are investigated.

In Fig. 3, the COVs of response for plates with simple (SB) and clamped (CB) boundary are given as functions of correlation distance d. The results of MCS are given only for plate with simple boundary to avoid complexity of figure. The values of cross-correlation factor (CCF) are given in parenthesis. This figure shows clearly not only the correlation effect between two uncertain parameters but also the good agreement between the two analyses schemes: proposed weighted integral scheme and simulation.

As seen in Fig. 3 and Table 1, the increase in COV for positive correlation is smaller than the decrease for negative correlation when it is seen from COV with zero correlation. Even though there appears slightly curved relation for negative correlation, as shown in Fig. 4, it does not mean, however, that the effect of negative correlation is greater than that of positive correlation, but just indicates that even in zero correlation the response variability is affected to a great degree. This fact

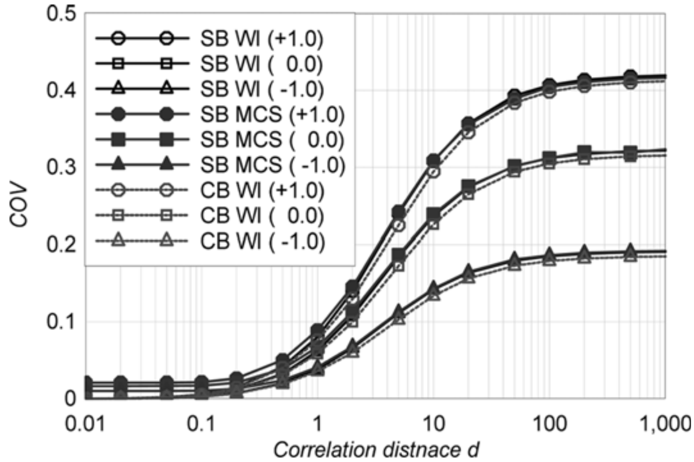

Fig. 3 Effect of correlation between uncertain parameters for plates with simple and clamp boundary

Table 1 Comparison of COVs at $d = 1000.0$

Boundary condition	γ_{Et} (CCF)				Differences				Ratio
	(a) −1.0	(b)* ×	(c) 0.0	(d) 1.0	(b)-(a)	(d)-(b)	(c)-(a)	(d)-(c)	(c)/(b)
Simple	0.1911	0.3039	0.3218	0.4195	0.1128	0.1156	0.1307	0.0977	1.0589
Clamp	0.1850	0.2968	0.3154	0.4120	0.1118	0.1152	0.1304	0.0966	1.0627

Note: *Column (b) contains COV for the case when only the plate thickness has uncertainty, i.e., the elastic modulus is assumed as deterministic

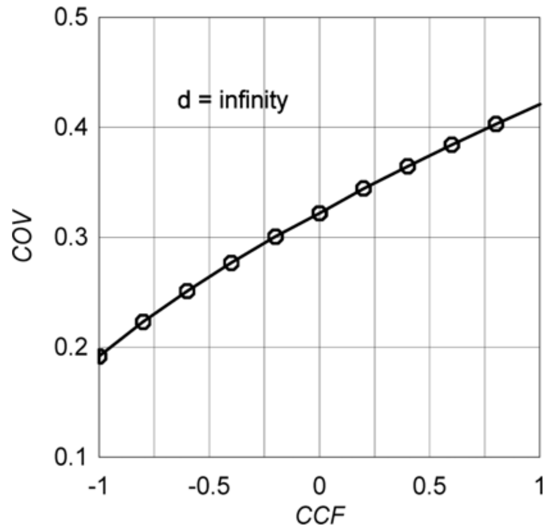

Fig. 4 COV as a function of cross correlation factor CCF

Table 2 Comparison of COV for uncertain plate thickness (d = infinity)

Boundary condition	Coefficient of variation (COV) of center displacement					
	Present		Choi (1996)		Lawrence (Choi 1996)	
	WI	MCS*	WI	MCS**	BRV	MCS#
Simple	0.3068	0.3050	0.3072	0.3145	0.2905	0.3062
Clamped	0.3062	0.3045	0.3074	0.3122	0.2902	0.3032

Notes: *Local averaging scheme with 11,520 samples; **720 samples; # 10,000 samples

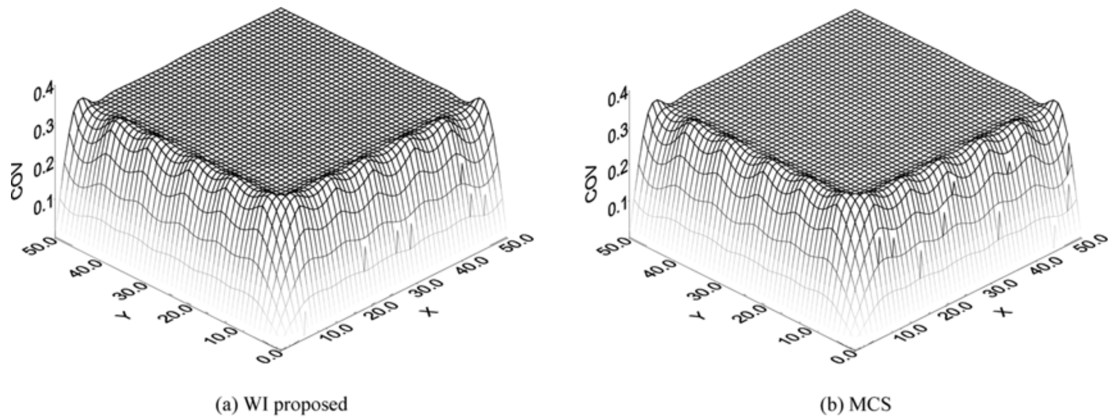

Fig. 5 Comparison of COV distribution over the structural domain (d = infinity)

can be noticed in the columns for 'Differences' and 'Ratio' in Table 1. As seen in columns (b) and (c) in Table 1 or from the comparison of Figs. 2 and 3, the response variability is increased about 6% even when no correlation is assumed between two uncertain parameters.

5.5 Plate with concentrated load

In this section, a square plate with dimension of 100×100 is analyzed. The material and geometrical constants are as follows: $\bar{E} = 1.0 \times 10^7$, $\bar{t} = 1.0$, $\nu = 0.3$. For the purpose of comparison with the precedent research works (Choi 1996), only the plate thickness is assumed as uncertain with coefficient of variation of 0.1. A concentrated load $p = 100.0$ is applied at the center of plate. As shown in Table 2, the results compare well to those of the precedent research works.

Fig. 5 compares the COV distribution over the structural domain obtained by means of suggested weighted integral method with that of MCS. As seen in the figure, they are not only in good agreement with each other but also indicate that the COV is constant over the domain of structure. The COV distribution in constant state is also investigated in the other boundary and loading conditions.

5.6 Effect of varying thickness

Fig. 6 shows the performance of plate finite element in the deterministic analysis as a function of aspect ratio $Log(L/t)$ when Poisson's ratio $\nu = 0.3$ and $L = 20.0$. Employing the thin plate theory

(Timoshenko 1959), the center deflections of simply supported square plate with side length L are given as follows:

$$w_q = \alpha\frac{qL^4}{D}, \quad \alpha = 0.00406$$
$$w_p = \beta\frac{pL^4}{D}, \quad \beta = 0.01106 \tag{31}$$

where, D denotes the flexural rigidity given as $Et^3/12(1-\nu^2)$. As seen in Fig. 6, the behavior is desirable even when the aspect ratio is relatively large.

To demonstrate whether the performance of the suggested stochastic analysis scheme depends on the aspect ratio of plate, the COV is evaluated for various values of aspect ratio and is shown in Fig. 7. As seen in the figure, irregular variation in COV appears when the aspect ratio is greater

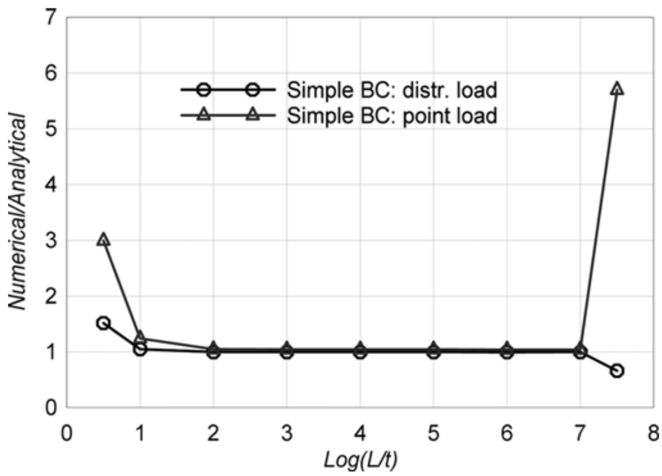

Fig. 6 Deterministic performance of element in terms of aspect ratio

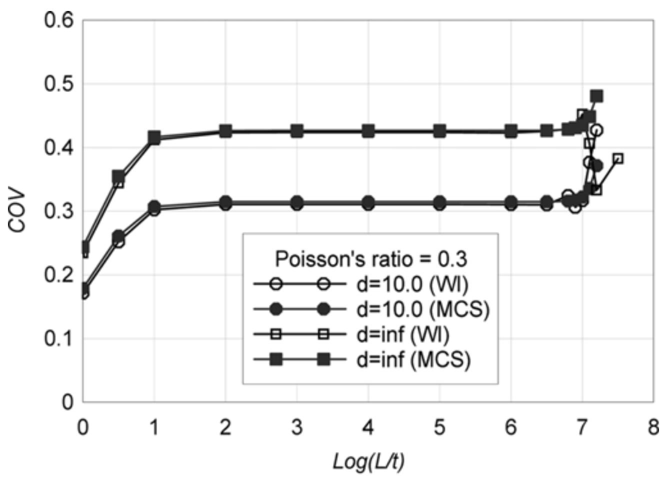

Fig. 7 Influence of aspect ratio on the response variability: simple support

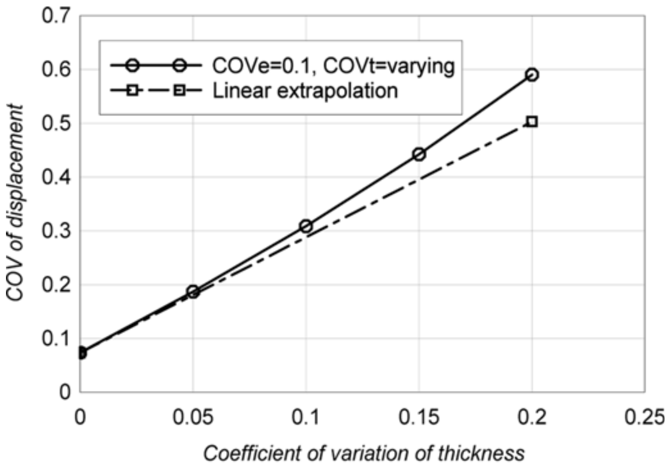

Fig. 8 Effect of coefficient of variation of plate thickness on the response variability

than 6.5. This is common in proposed weighted integral method and MCS, though the trend of variation is different to each other. It has to be noticed here that the steadiness of COV shown in Fig. 7 is destroyed if the deterministic stiffness is employed instead of the mean stiffness of Eq. (17) in assessing the mean and covariance of response. The deterioration commences at the aspect ratio of 2.5 and the COV reaches to over 5,000 when the aspect ratio is 5. This investigation shows the importance of the formulation proposed.

5.7 Sensitivity to the coefficient of variation in plate thickness

Fig. 8 shows the variation of COV in response when the coefficient of variation of plate thickness varies from 0 to 0.2. The coefficient of variation of elastic modulus remains constant as 0.1 and correlation distance d is 10.0, CCF is 1.0. As seen in the figure, the COV of response increases with positive rate of change as the coefficient of variation in thickness is increased. This is caused by the fact that the thickness is involved in the plate stiffness as a cubic function in bending part. This phenomenon is also observed in the case when the Poisson ratio, which has also a nonlinear form in its mathematical expression, is taken as an uncertain parameter (Noh 2004). As expected, the COV of response is a linear function of coefficient of variation of elastic modulus.

6. Conclusions

In this paper, the response variability due not only to the uncertain elastic modulus but also to the uncertain plate thickness is investigated in the context of weighted integral stochastic finite element method taking into account of the correlation between these two uncertain parameters. The correlation between two uncertain parameters is incorporated adopting the cross-correlation function and cross-correlation factor (CCF, γ_{Et}). The incorporation of correlation between two random parameters and the higher order joint moments is made possible by means of the general formula for n-th joint moment, which is valid for Gaussian random variables. Accordingly, two uncertain parameters, elastic modulus and plate thickness, are implicitly assumed to follow the Gaussian probabilistic distribution. Through the numerical example analysis on the plate with simple and clamped boundaries, it is demonstrated that the response variability obtained by means of the proposed scheme shows good agreement with that of MCS and of precedent research works. To provide some insight as to the statistical behavior of plate structure with uncertainties, the influences of aspect ratio and varying coefficient of variation in plate thickness on the response variability are also examined, which are also in good agreement with MCS.

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Appendix

The modified auto- and cross-correlation functions in Eq. (26).

$$
\begin{aligned}
E[f_b^i f_b^j] &= E[(3f_t^i + 3f_t^{ii} + f_t^{iii} + f_E^i + 3f_E^i f_t^i + 3f_E^i f_t^{ii} + f_E^i f_t^{iii}) \\
&\quad \cdot (3f_t^j + 3f_t^{jj} + f_t^{jjj} + f_E^j + 3f_E^j f_t^j + 3f_E^j f_t^{jj} + f_E^j f_t^{jjj})] \\
&= R_{EE}^{ij} + 6R_{Et}^{ij} + 9R_{tt}^{ij} \\
&\quad + 3(R_{EE}^{ij}R_{tt}^{ii} + 3R_{EE}^{ij}R_{tt}^{ij} + R_{EE}^{ij}R_{tt}^{jj}) \\
&\quad + 3(2R_{Et}^{ij}R_{Et}^{ii} + 3R_{Et}^{ij}R_{Et}^{ij} + 2R_{Et}^{ij}R_{Et}^{jj} + 3R_{Et}^{ii}R_{Et}^{jj}) \\
&\quad + 6(2R_{Et}^{ij}R_{tt}^{ii} + 6R_{Et}^{ij}R_{tt}^{ij} + 2R_{Et}^{ij}R_{tt}^{jj} + 3R_{Et}^{ii}R_{tt}^{ij} + 3R_{Et}^{ii}R_{tt}^{jj} + 3R_{Et}^{jj}R_{tt}^{ij} + 3R_{Et}^{jj}R_{tt}^{ii}) \\
&\quad + 9(R_{tt}^{ij}R_{tt}^{ii} + 2R_{tt}^{ij}R_{tt}^{ij} + R_{tt}^{ij}R_{tt}^{jj} + R_{tt}^{ii}R_{tt}^{jj}) \\
&\quad + 9(R_{Et}^{ij}R_{Et}^{ij}R_{tt}^{jj} + 4R_{Et}^{ij}R_{Et}^{ij}R_{tt}^{ij} + R_{Et}^{ij}R_{Et}^{ij}R_{tt}^{ii}) \\
&\quad + 9(2R_{Et}^{ij}R_{Et}^{ii}R_{tt}^{jj} + 2R_{Et}^{ij}R_{Et}^{ii}R_{tt}^{ij} + 2R_{Et}^{jj}R_{Et}^{ij}R_{tt}^{ii} + 2R_{Et}^{jj}R_{Et}^{ij}R_{tt}^{ij}) \\
&\quad + 9(R_{Et}^{jj}R_{Et}^{ii}R_{tt}^{ii} + 4R_{Et}^{jj}R_{Et}^{ii}R_{tt}^{ij} + R_{Et}^{jj}R_{Et}^{ii}R_{tt}^{jj}) \\
&\quad + 18(R_{tt}^{ij}R_{tt}^{ij}R_{Et}^{ii} + 2R_{tt}^{ij}R_{tt}^{ij}R_{Et}^{ij} + R_{tt}^{ij}R_{tt}^{ij}R_{Et}^{jj}) \\
&\quad + 18(R_{tt}^{ij}R_{tt}^{ii}R_{Et}^{ij} + R_{tt}^{ij}R_{tt}^{ii}R_{Et}^{jj}) \\
&\quad + 18(R_{tt}^{ij}R_{tt}^{jj}R_{Et}^{ij} + R_{tt}^{ij}R_{tt}^{jj}R_{Et}^{ii}) \\
&\quad + 9(R_{tt}^{ij}R_{tt}^{ij}R_{EE}^{ij} + R_{tt}^{ij}R_{tt}^{ii}R_{EE}^{ij} + R_{tt}^{jj}R_{tt}^{ij}R_{EE}^{ij} + R_{tt}^{jj}R_{tt}^{ii}R_{EE}^{ij}) \\
&\quad + 3(3R_{tt}^{jj}R_{tt}^{ij}R_{tt}^{ii} + 2R_{tt}^{ij}R_{tt}^{ij}R_{tt}^{ij})
\end{aligned}
$$

$$
\begin{aligned}
E[f_b^i f_s^j] &= E[(3f_t^i + 3f_t^{ii} + f_t^{iii} + f_E^i + 3f_E^i f_t^i + 3f_E^i f_{ti}^{ii} + f_E^i f_t^{iii})(f_E^j + f_t^j + f_E^j f_t^j)] \\
&= R_{EE}^{ij} + 4R_{Et}^{ij} + 3R_{tt}^{ij} + 3R_{Et}^{ii}R_{Et}^{jj} + 3R_{EE}^{ij}R_{tt}^{ij} + 3R_{Et}^{ij}R_{Et}^{ij} + 3R_{Et}^{jj}R_{tt}^{ii} + 6R_{Et}^{ij}R_{tt}^{ij} \\
&\quad + 6R_{Et}^{ij}R_{Et}^{ii} + 3R_{EE}^{ij}R_{tt}^{ii} + 6R_{Et}^{ii}R_{tt}^{ij} + 6R_{Et}^{ij}R_{tt}^{ii} + 3R_{tt}^{ii}R_{tt}^{ij} \\
&\quad + 3R_{Et}^{jj}R_{Et}^{ii}R_{tt}^{ii} + 6R_{Et}^{ii}R_{Et}^{ij}R_{tt}^{ij} + 3R_{Et}^{ij}R_{Et}^{ij}R_{tt}^{ii} + 3R_{EE}^{ij}R_{tt}^{ii}R_{tt}^{jj}
\end{aligned}
$$

$$
\begin{aligned}
E[f_s^i f_b^j] &= E[(f_E^i + f_t^i + f_E^i f_t^i)(3f_t^j + 3f_t^{jj} + f_t^{jjj} + f_E^j + 3f_E^j f_t^j + 3f_E^j f_{ti}^{jj} + f_E^j f_t^{jjj})] \\
&= R_{EE}^{ij} + 4R_{Et}^{ij} + 3R_{tt}^{ij} + 3R_{Et}^{jj}R_{Et}^{ii} + 3R_{EE}^{ij}R_{tt}^{ij} + 3R_{Et}^{ij}R_{Et}^{ij} + 3R_{Et}^{ii}R_{tt}^{jj} + 6R_{Et}^{ij}R_{tt}^{ij} \\
&\quad + 6R_{Et}^{ij}R_{Et}^{jj} + 3R_{EE}^{ij}R_{tt}^{ii} + 6R_{Et}^{ii}R_{tt}^{ij} + 6R_{Et}^{ij}R_{tt}^{jj} + 3R_{tt}^{jj}R_{tt}^{ij} \\
&\quad + 3R_{Et}^{ii}R_{Et}^{jj}R_{tt}^{jj} + 6R_{Et}^{jj}R_{Et}^{ij}R_{tt}^{ij} + 3R_{Et}^{ij}R_{Et}^{ij}R_{tt}^{jj} + 3R_{EE}^{ij}R_{tt}^{jj}R_{tt}^{ij}
\end{aligned}
$$

$$
\begin{aligned}
E[f_s^i f_s^j] &= E[(f_E^i + f_t^i + f_E^i f_t^i)(f_E^j + f_t^j + f_E^j f_t^j)] \\
&= R_{EE}^{ij} + 2R_{Et}^{ij} + R_{tt}^{ij} + R_{Et}^{jj}R_{Et}^{ii} + R_{tt}^{ij}R_{EE}^{ij} + R_{Et}^{ij}R_{Et}^{ij}
\end{aligned}
$$

where, $f_\alpha^{n-fold\ \mathrm{of}\ k} = \overbrace{f_\alpha(\mathbf{x}_k)\ldots f_\alpha(\mathbf{x}_k)}^{n-fold}$; $k = i$ on j, and $\alpha = E$ or t

$R_{\alpha\beta}^{kl} = R_{\alpha\beta}(\xi_{kk}, \xi_{kl}, \xi_{ll}) = E[f_\alpha(\mathbf{x}_k) f_\beta(\mathbf{x}_l)]$

The separation vector ξ_{ij} is defined between two elements i and j under consideration, and ξ_{ii}, ξ_{jj} are defined in the element i and j respectively.

Special case I ($\sigma_{EE} \neq 0$, $\sigma_{tt} = 0$): When only the uncertain elastic modulus is taken into account, auto-correlation function for random thickness and cross-correlation function vanish, $R_{tt} = R_{Et} = 0$, therefore,

$$E[f_b^i f_b^j] = E[f_E^i f_E^j] = R_{EE}^{ij}$$

$$E[f_b^i f_s^j] = E[f_E^i f_E^j] = R_{EE}^{ij}$$

$$E[f_s^i f_b^j] = E[f_E^i f_E^j] = R_{EE}^{ij}$$

$$E[f_s^i f_s^j] = E[f_E^i f_E^j] = R_{EE}^{ij}$$

Special case II ($\sigma_{EE} = 0$, $\sigma_{tt} \neq 0$): When only the uncertain thickness is taken into account, auto-correlation function for random elastic modulus and cross-correlation function vanish, $R_{EE} = R_{Et} = 0$, therefore,

$$
\begin{aligned}
E[f_b^i f_b^j] &= E[(3f_t^i + 3f_t^{ii} + f_t^{iii})(3f_t^j + 3f_t^{jj} + f_t^{jjj})] \\
&= 9R_{tt}^{ij} + 9(R_{tt}^{ij}R_{tt}^{ii} + 2R_{tt}^{ij}R_{tt}^{ij} + R_{tt}^{ij}R_{tt}^{ij} + R_{tt}^{ii}R_{tt}^{jj}) \\
&\quad + 3(3R_{tt}^{jj}R_{tt}^{ij}R_{tt}^{ii} + 2R_{tt}^{ij}R_{tt}^{ij}R_{tt}^{ij})
\end{aligned}
$$

$$E[f_b^i f_s^j] = E[(3f_t^i + 3f_t^{ii} + f_t^{iii})(f_t^j)] = 3R_{tt}^{ij} + 3R_{tt}^{ii}R_{tt}^{ij}$$

$$E[f_s^i f_b^j] = E[(f_t^i)(3f_t^j + 3f_t^{jj} + f_t^{jjj}) = 3R_{tt}^{ij} + 3R_{tt}^{jj}R_{tt}^{ij}$$

$$E[f_s^i f_s^j] = E[f_t^i f_t^j] = R_{tt}^{ij}$$