Structural Engineering and Mechanics, Vol. 20, No. 2 (2005) 191-207
DOI: http://dx.doi.org/10.12989/sem.2005.20.2.191

Active control of a flexible structure with time delay

Guo-Ping Cai†

Department of Engineering Mechanics, Shanghai Jiaotong University, Shanghai 200240, China

Simon X. Yang‡

Advanced Robotics and Intelligent Systems (ARIS) Lab, School of Engineering, University of Guelph, Guelph, Ontario N1G 2W1, Canada

(Received January 27, 2004, Accepted March 14, 2005)

Abstract. Time delay exists inevitably in active control, which may not only degrade the system performance but also render instability to the dynamic system. In this paper, a novel active controller is developed to solve the time delay problem in flexible structures. By using the independent modal space control method, the differential equation of the controlled mode with time delay is obtained from the time-delay system dynamics. Then it is discretized and changed into a first-order difference equation without any explicit time delay by augmenting the state variables. The modal controller is derived based on the augmented system using the discrete variable structure control method. The switching surface is determined by minimizing a discrete quadratic performance index. The modal coordinate is extracted from sensor measurements and the actuator control force is converted from the modal one. Since the time delay is explicitly included throughout the entire controller design without any approximation, the system performance and stability are guaranteed. Numerical simulations show that the proposed controller is feasible and effective in active vibration control of dynamic systems with time delay. If the time delay is not explicitly included in the controller design, instability may occur.

Key words: flexible structure; variable structure control; time delay.

1. Introduction

Flexible structures have been widely used in many engineering applications, such as aerospace, aviation and robotics. In general, flexible structures have a small material-damping ratio. When the structures are excited by external excitations, the structural vibration will last for a very long time if no controllers are used. It will not only affect the normal performance but also result in the premature fatigue breaking of the structures, and consequently shorten the service life of the structures. Passive control is the earliest control strategy used for structural vibration suppression. This strategy has the characteristics of easy implementation, low cost and simple structure. However it is short of maneuverability in control. In addition, its control effectiveness highly depends on the characteristics of the external excitations. In modern control systems, more and more attentions are

† Associate Professor, Corresponding author, E-mail: caigp@sjtu.edu.cn
‡ Associate Professor, E-mail: syang@uoguelph.ca

being paid to the vibration suppression using active control strategies. Active controllers have the advantages that the control performance is hardly affected by the characteristics of external excitations, and evidently superior to that of passive controllers. Therefore, vibration suppression of flexible structures using active controllers has been a very important research topic during the past several decades (Baz and Poh 1999, Darby and Pellegrino 1999, Schafer and Holzach 1985, Wang and Huang 2003, Meirovitch and Baruh 1982, Zee and Hughes 2000, Gennaro 1998, Bailey and Hubbard 1985, Zhang *et al.* 1996). In the active controller, the independent modal space control (IMSC) method may implement separate control of each controlled mode, so it has become a main analytical method in modal control strategies. There are many studies using the IMSC method for vibration control of flexible structures (Baz and Poh 1999, Schafer and Holzach 1985, Wang and Huang 2003, Meirovitch and Baruh 1982, Bailey and Hubbard 1985, Zhang *et al.* 1996) and modal controller is usually designed using linear optimal control method.

On the other hand, time delay exists inevitably in active control systems. Many factors, such as measurement of system variables, calculation of controller and processes for actuators to build up required control force, may result in non-synchronization of control force. Although time delay in most cases is small, it still makes actuator apply energy to the control system even when no energy is needed. This may cause the degradation of control efficiency and even render the system unstable (Hu 1997, Cai and Huang 2002a). So far time delay problems are mainly investigated in mathematics and control systems and most studies are focused on stability and maximum time delay for stability of time-delay system (Qin 1987, Chen 1995). For active control of structures, time delay has been usually neglected to avoid incomprehensible complexity of control design. But due to the inevitable existence of time delay, big errors often occur between theoretical control results and experimental ones if time delay is not considered in control design. So many researchers have made studies on time delay problem in structural control and some treating methods were developed (Chung *et al.* 1988, Yang *et al.* 1990, Abdel-Mooty and Roorda 1991, Agrawal *et al.* 1993, Chung *et al.* 1995, Wong 2005). In these methods, the expansion of Taylor series and the technique of phase shift are the two that are widely employed to deal with time delay (Abdel-Mooty and Roorda 1991, Chung *et al.* 1995, Cai 2002). However, these two methods are only available for very small time delay (Cai and Huang 2002a, Chung *et al.* 1995, Cai 2002, Cai and Huang 2002b). For example, for the technique of phase shift, active controller is firstly designed without considering time delay, the control gain is then revised referring to natural frequency of system with and without considering time delay in order to obtain the controller with time delay. When biggish time delay exists, this technique fails to compensate time delay and renders the control system unstable (Cai and Huang 2002a, Chung *et al.* 1995, Cai 2002, Cai and Huang 2002b). Recently, Cai and Huang (Cai and Huang 2002a, Cai 2002, Cai and Huang 2002b) developed a treating method for time delay problem, which is applied for vibration control of seismic-excited building structures. In their study, active controller is designed directly from time-delay differential equation and no approximation and estimation are made in the process of control design, so system performance and stability are guaranteed. Thus this controller is suitable for an arbitrary time delay. However, for vibration control of flexible structures, there has been no study on time delay problem up to now. Therefore, it is essential to study time delay problem for flexible structures.

In this paper, the IMSC strategy is applied to active vibration control of a flexible structure with time delay, in which time delay is compensated using the method adopted in (Cai and Huang 2002a, Cai 2002, Cai and Huang 2002b). Modal controller is designed using a discrete variable structure control (DVSC) method. A discrete switching surface is designed for the proposed controller. Numerical simulation studies are carried out to demonstrate the effectiveness and efficiency of the proposed controller in the end of this paper.

This paper is organized as follows. Section 2 briefly presents the system motion dynamics with explicit time delay in the differential equation, and the differential equation of the controlled mode. The proposed controller design is given in Section 3, including the discretization and standardization of the modal equation, the determination of discrete switching surface, the design of DVSC modal controller, the estimation of modal coordinates from sensor measurements and the conversion of actuator control forces from the modal control forces. Section 4 provides numerical simulation studies of a flexible cantilever beam using the proposed control algorithm. Finally, a concluding remark is given in Section 5.

2. Motion equation

The flexible structure considered in this paper is a flexible cantilever beam. The transverse vibration control problem of the Bernoulli-Euler beam is studied. The beam has a constant cross-section area with every center inertia axis being in the same plane *xoy* (see in Fig. 1) and an external load acting in this plane. The motion equation of the beam can be expressed as

$$EJ\frac{\partial^4 y(x,t)}{\partial x^4} + \rho A\frac{\partial^2 y(x,t)}{\partial t^2} = P(x,t) \tag{1}$$

where $y(x, t)$ represents the transverse displacement of the point that is x from the origin o at the moment t. Parameter E is the Young's modulus of the beam material, J is the area moment of inertia of the beam cross-section, ρ is the mass per unit volume, A is the cross-section area, and $P(x, t)$ is the external distributed force.

Defining the following dimensionless parameters

$$y^* = \frac{y(x,t)}{L},\ x^* = \frac{x}{L},\ t^* = \frac{\sqrt{EJ/\rho A}}{L^2}t,\ P^* = \frac{L^3}{EJ}P(x,t) \tag{2}$$

where L is the length of the beam, and P^* is the dimensionless distributed force, then (1) can be transformed into the following dimensionless form

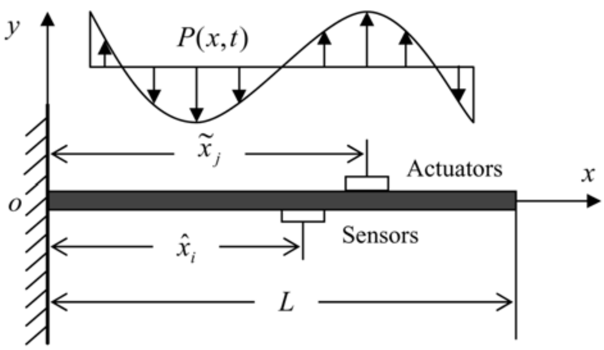

Fig. 1 Cantilever beam model

$$\frac{\partial^4 y^*}{\partial x^{*4}} + \frac{\partial^2 y^*}{\partial t^{*2}} = P^* \tag{3}$$

For convenience in theoretical analysis and numerical simulation, the dimensionless motion equation in (3) of the flexible beam is used as the analytical object in this study.

When using the IMSC method for the flexible beam, the distributed actuators are required to supply the modal control forces. However the distributed actuators are not always available in practice, and point actuators are usually used instead. Assume that the first r_1 modes of the beam are controlled. According to the IMSC method, r_1 actuators are required to be installed on the beam to control the r_1 modes separately. The control forces applied on the beam are assumed to have the same delayed time. Adding the time delay in the control forces in (3), we have

$$\frac{\partial^4 y^*}{\partial x^{*4}} + \frac{\partial^2 y^*}{\partial t^{*2}} = P^* + \sum_{j=1}^{r_1} \delta(x^* - \tilde{x}_j^*) F_j(t^* - \lambda) \tag{4}$$

where $\delta(\bullet)$ is a Dirac function defined as $\delta(x^* - \tilde{x}_j^*) = 1$ when $x^* = \tilde{x}_j^*$, and $\delta(x^* - \tilde{x}_j^*) = 0$ when $x^* \neq \tilde{x}_j^*$. Variable $\tilde{x}_j^*$ represents the location of the j-th actuator on the beam, $j = 1, 2, ..., r_1$; $F_j(t^* - \lambda)$ is the control force used to control the j-th mode of the beam; and λ is the time delay.

For (3), the normalized mode shape corresponding to the i-th mode can be written as

$$Y_i^*(x^*) = \cosh k_i x^* - \cos k_i x^* - \alpha_i(\sinh k_i x^* - \sin k_i x^*) \tag{5}$$

where $\alpha_i = (\cosh k_i + \cos k_i)/(\sinh k_i + \sin k_i)$. The eigenvalue k_i should satisfy the frequency equation $\cos k_i \cosh k_i = -1$. The natural frequency of the i-th mode of the flexible beam is $p_i = k_i^2$. The analytical solution of (3) can be written as

$$y^*(x^*, t^*) = \sum_{i=1}^{\infty} Y_i^*(x^*) \phi_i(t^*) \tag{6}$$

where $\phi_i(t^*)$ is the i-th modal coordinate. A set of uncoupled modal-controlled dynamic equation can be obtained as

$$\ddot{\phi}_i(t^*) + p_i^2 \phi_i(t^*) = f_i(t^*) + u_i(t^* - \lambda), \quad i = 1, 2, \ldots, r_1 \tag{7}$$

where

$$\begin{cases} f_i(t^*) = \int_0^1 Y_i^*(x^*) P^* dx^* \\ u_i(t^* - \lambda) = \sum_{j=1}^{r_1} Y_i(\tilde{x}_j^*) F_j(t^* - \lambda) \end{cases} \tag{8}$$

where $f_i(t^*)$ is the modal generalized force of the i-th mode and $u_i(t^* - \lambda)$ is the modal control force to be designed. An extra uncoupled modal damping ratio ζ_i is added to each modal equation to represent the beam damping factor. Thus, (7) can be written as

$$\ddot{\phi}_i(t^*) + 2\zeta_i p_i \dot{\phi}_i(t^*) + p_i^2 \phi_i(t^*) = f_i(t^*) + u_i(t^* - \lambda), \quad i = 1, 2, \ldots, r_1 \tag{9}$$

3. Independent modal space control

The IMSC method is used to suppress the vibration of the beam, in which the modal controller is designed using a DVSC method. Before the DVSC is presented, it is essential to introduce the characteristics of the VSC and the DVSC.

The goal of the VSC is to find a hypersurface in the system phase space. This hypersurface can divide the phase space into two parts: $s > 0$ and $s < 0$. Then the controller is designed based on a certain condition, which will force the state trajectories outside the hypersurface to reach the hypersurface in finite time, then slide on it and finally move to the system origin (Gao 1998). This hypersurface is called the *switching surface*. The motion of the state trajectories on the hypersurface is called *sliding mode*. The condition under which the state trajectories reach the switching surface is called the *reaching condition*. So the variable structure controller design includes two parts: the switching surface and the controller (Gao 1998). The remarkable feature of the VSC is its robustness and invariance to external disturbance and system parameter uncertainty. In addition, the VSC can be used for the controller design of nonlinear systems (Gao 1998, Gao and Hung 1993). The theory of the continuous VSC is well established. For a continuous VSC system, the state trajectories can reach the switching surface exactly.

The DVSC is similar to the continuous VSC. The discrete variable structure controller design also includes two parts: the discrete switching surface and discrete controller (Gao *et al.* 1995). But for a DVSC system, the state trajectories can seldom reach the switching surface exactly due to the discrete features of the DVSC (Gao *et al.* 1995). Possibly there does not exist the state $\tilde{z}(k)$ such that $s[\tilde{z}(k)] = 0$, where $\tilde{z}(k)$ is a system state vector. Thus for the DVSC, a *switching band* that contains the switching surface needs to be defined (Gao *et al.* 1995). The controller will force the state trajectories outside the switching band to approach the switching band and then move into the band. Once the state enters the band, it will stay in the band, and finally approach the system origin. The controller can cause zigzagging motion of the state trajectories about the switching surface instead of the motion on the surface. The motion of state trajectories in the switching band is called the *quasi-sliding mode*.

In the following section, the dynamic Eq. (9) with time delay is first discretized and changed into a standard discrete form that does not contain an explicit time delay by augmenting the state variables. Then discrete switching surface is designed and the DVSC modal controller is developed.

3.1 Discretization and standardization of the modal equation

The time delay can be written as

$$\lambda = l\bar{T} - \bar{m} \tag{10}$$

where $\bar{T}$ is data sampling period, $l > 0$ is a positive integral number, and $0 \le \bar{m} < \bar{T}$. The discretization method for the time-delay dynamic equation in (9) in the cases $\bar{m} = 0$ and $\bar{m} \ne 0$ is discussed in details in (Cai and Huang 2002a, Chung *et al.* 1995, Cai and Huang 2002b). For simplicity, only the case $\bar{m} = 0$ is discussed in this paper. When $\bar{m} = 0$, the time delay is integer times of the sampling period, and (9) can be discretized into the following form (Cai and Huang 2002a, Chung *et al.* 1995, Cai and Huang 2002b)

$$\boldsymbol{z}_i(k+1) = \boldsymbol{a}_i\boldsymbol{z}_i(k) + \boldsymbol{b}_i f_i(k) + \boldsymbol{d}_i u_i(k-l), \quad i = 1, 2, \ldots, r_1 \tag{11}$$

where $z_i(k) = [\phi_i(k), \dot{\phi}_i(k)]^T$ is a (2×1) state vector. The superscript T denotes the transpose of a matrix or vector. Parameter $\boldsymbol{a}_i$ is a (2×2) coefficient matrix, and $\boldsymbol{b}_i$ and $\boldsymbol{d}_i$ are (2×1) vectors.

Augmenting the following generalized state variables as

$$\begin{cases} z_{2+1}^{(i)}(k) = u_i(k-l) \\ z_{2+2}^{(i)}(k) = u_i(k-l+1) \\ \quad\vdots \\ z_{2+l}^{(i)}(k) = u_i(k-1) \end{cases} \tag{12}$$

and defining a new total generalized state vector as

$$\tilde{z}_i(k) = [z_i^T(k), z_{2+1}^{(i)}(k), \cdots, z_{2+l}^{(i)}(k)]^T \tag{13}$$

then (11) can be changed into the following standard discrete form without any explicit time delay

$$\tilde{z}_i(k+1) = \tilde{\boldsymbol{a}}_i\tilde{z}_i(k) + \tilde{\boldsymbol{b}}_i f_i(k) + \tilde{\boldsymbol{d}}_i u_i(k), \quad i = 1, 2, \ldots, r_1 \tag{14}$$

where $\tilde{z}_i(k)$ is a $(2 + l)$ vector. Parameters $\tilde{\boldsymbol{a}}_i$ is a $(2+l) \times (2+l)$ matrix, and $\tilde{\boldsymbol{b}}_i$ and $\tilde{\boldsymbol{d}}_i$ are $(2+l) \times 1$ vectors, which are given by

$$\tilde{\boldsymbol{a}}_i = \begin{bmatrix} \boldsymbol{a}_i & \boldsymbol{d}_i & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix}, \quad \tilde{\boldsymbol{b}}_i = \begin{bmatrix} \boldsymbol{b}_i \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \quad \tilde{\boldsymbol{d}}_i = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{15}$$

respectively. Therefore, with the transformation of (12) and (13), (11) with time delay has been re-written into the standard discrete form in (14) that contains no time delay. The discretization for case $\bar{m} \neq 0$ will result in the same form in (14) (for details, see (Cai and Huang 2002a, Chung *et al.* 1995, Cai and Huang 2002b)).

The sufficient condition for stability of system in (14) is that all the eigenvalues of $\tilde{\boldsymbol{a}}_i$ is within a unit circle. The system in (14) is controllable provided that the matrix $[\tilde{\boldsymbol{a}}_i, \tilde{\boldsymbol{d}}_i]$ is controllable.

3.2 Design of the discrete switching surface

As stated in Section 2, the first r_1 modes of the beam are controlled separately by r_1 actuators that require r_1 controllers. Thus r_1 discrete switching surfaces should be designed for the r_1 controllers. Each controller has its independent switching surface. The external excitation term can be neglected in the switching surfaces design (Yang *et al.* 1995), but will be considered later in the design of modal forces. Neglecting the external excitation term, (14) becomes

$$\tilde{z}_i(k+1) = \tilde{\boldsymbol{a}}_i\tilde{z}_i(k) + \tilde{\boldsymbol{d}}_i u_i(k), \quad i = 1, 2, \ldots, r_1 \tag{16}$$

The linear form of the discrete switching function can be considered as (Gao 1998, Gao *et al.* 1995)

$$s_i(k) = \boldsymbol{c}_i\tilde{\boldsymbol{z}}_i(k), \quad i = 1, 2, \ldots, r_1 \tag{17}$$

where $\boldsymbol{c}_i$ is an $1 \times (2 + l)$ coefficient vector of the switching function to be designed.

Let $\tilde{\boldsymbol{z}}_i(k)$ and $\tilde{\boldsymbol{a}}_i$ be partitioned as follows

$$\tilde{\boldsymbol{z}}_i(k) = \begin{bmatrix} \tilde{\boldsymbol{z}}_{i1}(k) \\ \tilde{z}_{i2}(k) \end{bmatrix}, \quad \tilde{\boldsymbol{a}}_i = \begin{bmatrix} \tilde{\boldsymbol{a}}_{i11} & \tilde{\boldsymbol{a}}_{i12} \\ \tilde{\boldsymbol{a}}_{i21} & \tilde{a}_{i22} \end{bmatrix} \tag{18}$$

where $\tilde{\boldsymbol{z}}_{i1}(k)$ is a $(l+1) \times 1$ vector and $\tilde{z}_{i2}(k)$ is a scalar. Parameter $\tilde{\boldsymbol{a}}_{i11}$ is a $(l+1)\times(l+1)$ matrix, and $\tilde{\boldsymbol{a}}_{i12}$ and $\tilde{\boldsymbol{a}}_{i21}$ are $(l+1) \times 1$ and $1 \times (l+1)$ vectors, respectively. Parameter $\tilde{a}_{i22}$ is a scalar. So (16) can be written as

$$\begin{cases} \tilde{\boldsymbol{z}}_{i1}(k+1) = \tilde{\boldsymbol{a}}_{i11}\tilde{\boldsymbol{z}}_{i1}(k) + \tilde{\boldsymbol{a}}_{i12}\tilde{z}_{i2}(k) \\ \tilde{z}_{i2}(k+1) = \tilde{\boldsymbol{a}}_{i21}\tilde{\boldsymbol{z}}_{i1}(k) + \tilde{a}_{i22}\tilde{z}_{i2}(k) + u_i(k) \end{cases} \tag{19}$$

The switching function in (17) can be written as

$$s_i(k) = \boldsymbol{c}_i\tilde{\boldsymbol{z}}_i(k) = \boldsymbol{c}_{i1}\tilde{\boldsymbol{z}}_{i1} + c_{i2}\tilde{z}_{i2} \tag{20}$$

where $\boldsymbol{c}_{i1}$ is an $1 \times (l+1)$ vector and c_{i2} is a scalar.

The system $[\tilde{\boldsymbol{a}}_{i11}, \tilde{\boldsymbol{a}}_{i12}]$ is controllable if $[\tilde{\boldsymbol{a}}_i, \tilde{\boldsymbol{d}}_i]$ is controllable (Gao 1998, Gao *et al.* 1995). Thus, for the subsystem given by the first equation in (19), there exists a feedback relationship

$$\tilde{z}_{i2}(k) = -\boldsymbol{v}_i\tilde{\boldsymbol{z}}_{i1}(k) \tag{21}$$

such that this subsystem is stable, where $\boldsymbol{v}_i$ is an $1 \times (l+1)$ vector. In addition, on the switching surface, we have $s_i(k) = 0$. From (20) and (21), we have $\boldsymbol{v}_i = c_{i2}^{-1}\boldsymbol{c}_{i1}$. So the coefficient vector of the switching function can be obtained as

$$\boldsymbol{c}_i = [\boldsymbol{c}_{i1}, c_{i2}] = c_{i2}[\boldsymbol{v}_i, 1], \quad i = 1, 2, \ldots, r_1 \tag{22}$$

For simplicity, $c_{i2} = 1$ can be chosen. The vector $\boldsymbol{v}_i$ may be designed using the discrete LQR method such that the following discrete performance index attaining minimum

$$J_i = \sum_{k=0}^{\infty} [\tilde{\boldsymbol{z}}_{i1}^T(k)\tilde{\boldsymbol{q}}_i\tilde{\boldsymbol{z}}_{i1}(k) + \tilde{r}_i\tilde{z}_{i1}^2(k)], \quad i = 1, 2, \ldots, r_1 \tag{23}$$

and is given by (Cai 2002, Kwakernaak and Sivan 1972)

$$\boldsymbol{v}_i = [\tilde{r}_i + \tilde{\boldsymbol{a}}_{i12}^T\tilde{\boldsymbol{s}}_i\tilde{\boldsymbol{a}}_{i12}]^{-1}\tilde{\boldsymbol{a}}_{i12}^T\tilde{\boldsymbol{s}}_i\tilde{\boldsymbol{a}}_{i11}, \quad i = 1, 2, \ldots, r_1 \tag{24}$$

The parameters $\tilde{\boldsymbol{q}}_i$ and $\tilde{r}_i$ in (23) are $(l+1) \times (l+1)$ positive-semidefinite symmetric coefficient matrix and positive scalar, respectively. The parameter $\tilde{\boldsymbol{s}}_i(k)$ in (24) is the solution of the following discrete Riccati algebraic equation which is a $(l+1) \times (l+1)$ matrix (Cai 2002, Kwakernaak and Sivan 1972)

$$\tilde{\boldsymbol{s}}_i = \tilde{\boldsymbol{a}}_{i11}^T\{\tilde{\boldsymbol{s}}_i - \tilde{\boldsymbol{s}}_i\tilde{\boldsymbol{a}}_{i12}[\tilde{r}_i + \tilde{\boldsymbol{a}}_{i12}^T\tilde{\boldsymbol{s}}_i\tilde{\boldsymbol{a}}_{i12}]^{-1}\tilde{\boldsymbol{a}}_{i12}^T\tilde{\boldsymbol{s}}_i\}\tilde{\boldsymbol{a}}_{i11} + \tilde{\boldsymbol{q}}_i \tag{25}$$

When $\boldsymbol{v}_i$ is obtained, from (22), the switching surface can be fully determined. Since the state variable $\tilde{z}_i(k)$ given in (13) contains the former l steps of controls, the obtained switching surface contains a linear combination of the former l steps of controls.

3.3 Design of the modal controller

The modal controller to be designed is a DVSC modal controller. This controller is designed to drive the state trajectory into the switching band. To achieve this goal, the discrete reaching condition in the form of approaching law is considered and written as (Gao *et al.* 1995)

$$s_i(k+1) - s_i(k) = -q_i\overline{T}s_i(k) - \varepsilon_i\overline{T}\operatorname{sgn}(s_i(k)), \quad i = 1, 2, \ldots, r_1 \tag{26}$$

where $q_i > 0$, $\varepsilon_i > 0$, $1 - q_i\overline{T} > 0$, and $\overline{T}$ is the data sampling period.

Substituting (14) into the left side of (26) and in consideration of (17), the DVSC modal controller can be obtained as

$$\begin{aligned} u_i(k) = &-(\boldsymbol{c}_i\tilde{\boldsymbol{d}}_i)^{-1}\{\boldsymbol{c}_i\tilde{\boldsymbol{a}}_i\tilde{\boldsymbol{z}}_i(k) + \boldsymbol{c}_i\tilde{\boldsymbol{b}}_i f_i(k) - \boldsymbol{c}_i\tilde{z}_i(k) \\ &+ q_i\overline{T}\boldsymbol{c}_i\tilde{\boldsymbol{z}}_i(k) + \varepsilon_i\overline{T}\operatorname{sgn}[\boldsymbol{c}_i\tilde{\boldsymbol{z}}_i(k)]\}, \quad i = 1, 2, \ldots, r_1 \end{aligned} \tag{27}$$

Same with the discrete switching surface, at every step of calculation, the controller contains a linear combination of the former l steps of control signals. For the i-th mode, the width of switching band is $\varepsilon_i\overline{T}$ (Gao *et al.* 1995).

3.4 Estimation of modal coordinates and conversion of actuator control forces

The DSVC modal controller given in (27) is a function of modal displacement and modal velocity. The modal displacement and velocity cannot be measured directly from sensors in practice. They should be estimated from physical sensor measurements, and then are used to calculate the modal control forces. Since the distributed type of sensors is not always available for the flexible beam in practice, discrete sensors are usually used instead. Then the required modal coordinates can be estimated from the sensor measurements. Here consider the case that the first r_2 modes of the beam are estimated, $r_2 > r_1$. Thus it requires sensors installed on the r_2 points of the beam to obtain the physical displacements and velocities of these points. The required modal coordinate of the i-th mode can be obtained using the following equation (Wang and Huang 2003).

$$\begin{cases} \phi_i(k) = \sum\limits_{j=1}^{r_2}(\boldsymbol{H}^{-1})_{ij}y^*(\hat{x}_j^*, k) \\ \dot{\phi}_i(k) = \sum\limits_{j=1}^{r_2}(\boldsymbol{H}^{-1})_{ij}\dot{y}^*(\hat{x}_j^*, k) \end{cases} \tag{28}$$

where $i = 1, 2, \ldots, r_1$, and $\hat{x}_j^*$ represents the location of the j-th sensor on the beam, $j = 1, 2, \ldots, r_2$. The transform matrix $\boldsymbol{H}$ is a $(r_2 \times r_2)$ matrix given by

$$\boldsymbol{H} = \begin{bmatrix} Y_1^*(\hat{x}_1^*) & Y_2^*(\hat{x}_1^*) & \dots & Y_{r_2}^*(\hat{x}_1^*) \\ Y_1^*(\hat{x}_2^*) & Y_2^*(\hat{x}_2^*) & \dots & Y_{r_2}^*(\hat{x}_2^*) \\ \vdots & \vdots & \dots & \vdots \\ Y_1^*(\hat{x}_{r_2}^*) & Y_2^*(\hat{x}_{r_2}^*) & \dots & Y_{r_2}^*(\hat{x}_{r_2}^*) \end{bmatrix} \tag{29}$$

After the modal coordinates are obtained, the modal control forces can be calculated.

When the modal filter given in (28) is used, observation spillover may occur, as in practice the number of modes is always greater than the number of sensors. Observation spillover occurs when the output of the limited number of sensors cannot synthesize the modal coordinates exactly, i.e., the unobserved mode responses are embedded into the observed mode responses and the sensors cannot give a clear picture of the structural vibration. The observation spillover may affect the precision of the estimated modal coordinates, or even cause instability of the control system. A useful and direct way to eliminate the observation spillover is to place the sensors on the nodes of the unobserved modes so as to reduce the affection of the unobserved modes. There are also some other techniques for eliminating the observation spillover, e.g., pre-filters (Balas 1980, 1982), augmented observers (Chait and Radeliffe 1989), and adding sufficiently large number of sensors on the beam (Meirovitch and Baruh 1985).

The control forces obtained using (27) are the modal control forces and not the physical control forces of actuators applied on the beam. After the modal control forces are obtained, they must be transformed into the actuator control forces. The actuator control force vector and the modal control force vector are denoted by $\boldsymbol{F}(k)$ and $\boldsymbol{U}(k)$, respectively. From (8) and (Wang and Huang 2003), the following relationship exists between $\boldsymbol{F}(k)$ and $\boldsymbol{U}(k)$

$$\boldsymbol{F}(k) = \boldsymbol{R}^{-1}\boldsymbol{U}(k) \tag{30}$$

where

$$\boldsymbol{F}(k) = \begin{bmatrix} F_1(k) \\ F_2(k) \\ \vdots \\ F_{r_1}(k) \end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} Y_1^*(\tilde{x}_1^*) & Y_1^*(\tilde{x}_2^*) & \dots & Y_1^*(\tilde{x}_{r_1}^*) \\ Y_2^*(\tilde{x}_1^*) & Y_2^*(\tilde{x}_2^*) & \dots & Y_2^*(\tilde{x}_{r_1}^*) \\ \vdots & \vdots & \dots & \vdots \\ Y_{r_1}^*(\tilde{x}_1^*) & Y_{r_1}^*(\tilde{x}_2^*) & \dots & Y_{r_1}^*(\tilde{x}_{r_1}^*) \end{bmatrix}, \quad \boldsymbol{U}(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_{r_1}(k) \end{bmatrix} \tag{31}$$

Variable $\tilde{x}_i^*$ represents the location of the i-th actuator on the beam, $i = 1, 2, ..., r_1$. The selection of $\tilde{x}_i^*$ should guarantee the existence of inverse matrix of the $(r_1 \times r_1)$ matrix $\boldsymbol{R}$, i.e., $\det(\boldsymbol{R}) \neq \boldsymbol{0}$.

When the flexible beam is controlled by the independent modal controller, while it suffers from the observation spillover, it may suffer from the control spillover as well. The control spillover occurs when the uncontrolled modes are excited by the control forces of the controlled modes. The control spillover may result in the deterioration of control performance. The control spillover may be reduced by placing the actuators on the locations that is far from the nodes of the controlled modes.

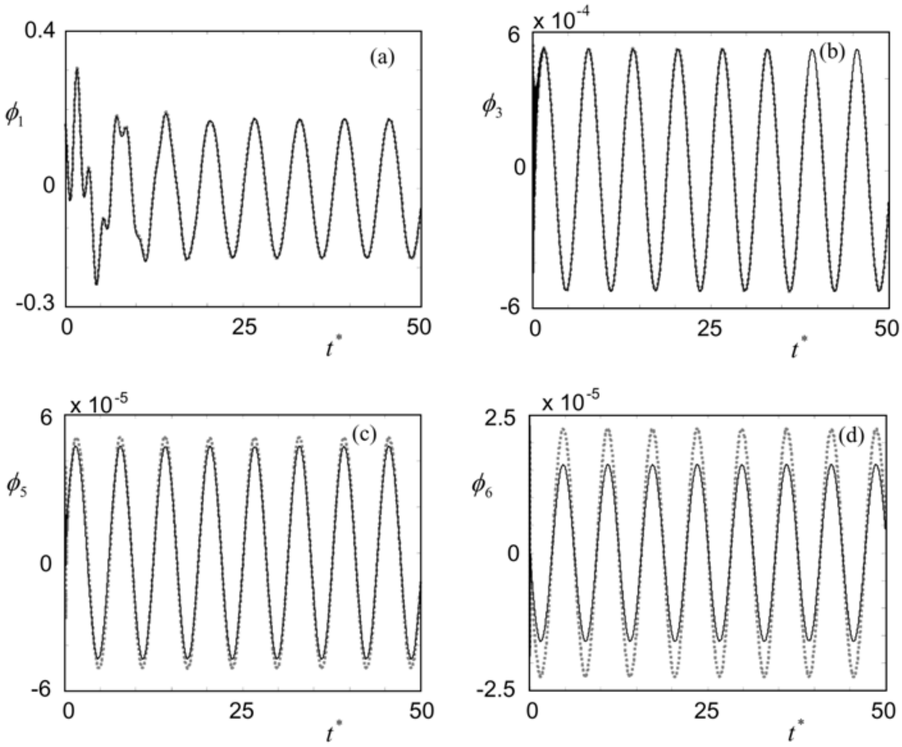

Fig. 2 Estimated responses of the first, third, fifth, and sixth modes using the modal filter without control for the beam (Actual result ············, Estimated result ————)

4. Numerical example

To demonstrate the effectiveness of the presented control method, numerical simulation studies are carried out. The flexible cantilever beam, given by (4), is used as the computational model. The initial condition, external excitation and modal parameters are chosen to be the same as those in (Wang and Huang 2003), given as follows. The initial condition of the flexible beam is given as $y^*(1,0) = 1/3$, $\dot{y}^*(1,0) = 0$. A sinusoidal excitation is applied at the tip of the beam, i.e., $P^* = \sin(t^*)$. The first ten modes of the beam are used to represent the actual vibration of the beam. The damping ratio of all the modes are chosen as $\zeta_i = 0.05, i = 1, 2, \ldots, 10$. The first four modes are used as the controlled modes, i.e., $r_1 = 4$. So four actuators are required to control these four controlled modes. The locations of the four actuators are chosen at the peak points of the first three modes, so we have (Wang and Huang 2003)

$$\tilde{x}_i^* = [0.291, 0.471, 0.694, 1.000]$$

The modal filter is used as the state estimator. The first six modes are estimated using the modal filter, i.e., $r_2 = 6$. So the sensors are installed on six points of the beam to measure the physical displacements and velocities at these six points. The locations of the sensors are chosen as the nodes of the seventh mode of the beam, which are given by (Wang and Huang 2003)

$$\hat{x}_j^* = [0.193, 0.346, 0.500, 0.654, 0.808, 0.949]$$

In the numerical computation, the sampling period is chosen as $\overline{T} = 0.01$.

The effect of the modal filter given in (28) is checked first. Consider the case without any controller for the beam. Figs. 2(a), (b), (c) and (d) show the time-history responses of the first, third, fifth and sixth modes, in which the solid line represents the results using the modal filter and the dashed line the theoretical results. From Fig. 2, it shows that the modal filter is effective in estimating the required modal coordinates. The estimation precise of the lower modes is better than that of the higher ones. The first and third estimated modes [shown in Figs. 2(a) and (b)] are almost the same as the theoretical results. The estimated second and fourth modes are almost the same as their theoretical results too (figures omitted herein). Small observation spillovers are observed in the fifth and sixth modes [shown in Figs. 2(c) and (d)].

Then consider the case with controller for the beam. First consider the case without the presence of any time delay in the control system, namely $\lambda = 0$. In this case, the state variables do not need to be augmented. The state vector $\tilde{z}_i$ in (14) is equal to z_i in (11), i.e., $\tilde{z}_i = z_i$. Then parameters $\tilde{\boldsymbol{a}}_{i11}$ and $\tilde{\boldsymbol{a}}_{i12}$ in (19), and $\boldsymbol{v}_i$ in (21) become scalars. Substituting (21) into (19) and setting $\tilde{a}_{i11} - \tilde{a}_{i12} v_i = 0.5$, $i = 1, 2, ..., 4$, the four coefficient vectors of the switching surfaces can be determined as

$$\begin{cases} \boldsymbol{c}_1 = [49.9949, 0.7499], & \boldsymbol{c}_2 = [49.7976, 0.7491] \\ \boldsymbol{c}_3 = [48.4037, 0.7474], & \boldsymbol{c}_4 = [43.7563, 0.7447] \end{cases}$$

In the design of DVSC modal controller using (27), set $q_i\overline{T} = 0.01$ and $\varepsilon_i\overline{T} = 0.0001$, where $i = 1, 2, ..., 4$. Thus the modal control forces can be obtained. Fig. 3 shows the time-history responses in displacement [Fig. 3 (a1)-(a4)] and velocity [Fig. 3 (b1)-(b4)] of the first four modes of the beam, where the solid and dashed lines represent the results with and without control, respectively. It is observed that the first four modes can be almost fully controlled by the DVSC method. It is also observed from Fig. 3 that, without control for the beam, magnitude of displacement and velocity of the first mode are much larger than those of the other three modes. Furthermore, the higher the mode order is, the smaller the magnitude of modal response is. In others words, response of the beam is dominated mainly by the lower-order modes of the beam, especially by the first-order mode. So control of the beam in the time domain may be changed to the control of several lower-order modes of the beam in the mode domain, especially the control of the first-order mode. The time-history responses in displacement and velocity of the tip position of the beam are illustrated in Figs. 4(a) and (b), respectively, where the solid line represents the result with control and the dashed line without control. It shows that the vibration response of the beam can be almost fully controlled. The advantage using the DVSC modal controller in the IMSC method for the flexible beam can be found evidently from Figs. 3 and 4.

Thirdly the case with the presence of time delay is considered. The delay is chosen to be $\lambda = 0.05$, thus $l = 5$ in (10). Our simulation results indicate that instability occurs when the active control system of the beam with $\lambda = 0.05$ is controlled by the controller designed in the case of no time delay. Therefore, the time delay problem needs a serious attention in the active control design. Otherwise the control system is susceptible to dynamic instability. Next consider the existence of $\lambda = 0.05$ in control design using the proposed control strategy. Here the matrix $\tilde{\boldsymbol{a}}_i$ in (14) is a (7×7) matrix, and all the eigenvalues lie inside the unit circle. The rank of $[\tilde{\boldsymbol{a}}_i, \tilde{\boldsymbol{d}}_i]$ is 7, i.e., $[\tilde{\boldsymbol{a}}_i, \tilde{\boldsymbol{d}}_i] = 7$. The matrix $\tilde{\boldsymbol{q}}_i$ in (23) is a (6×6) matrix, where $\tilde{\boldsymbol{q}}_i(1, 1) = 10^6$, with all the other

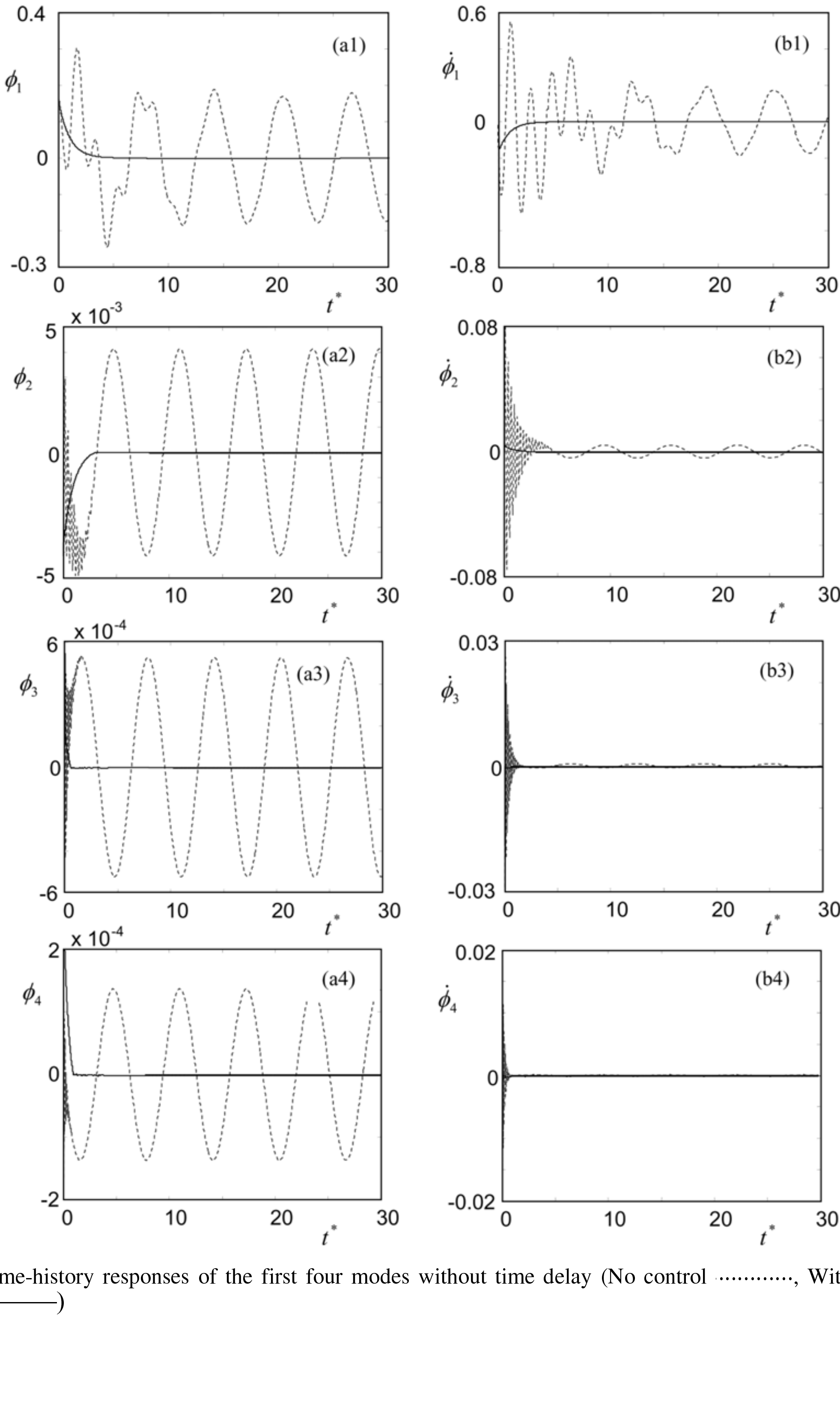

Fig. 3 Time-history responses of the first four modes without time delay (No control ·········, With control ———)

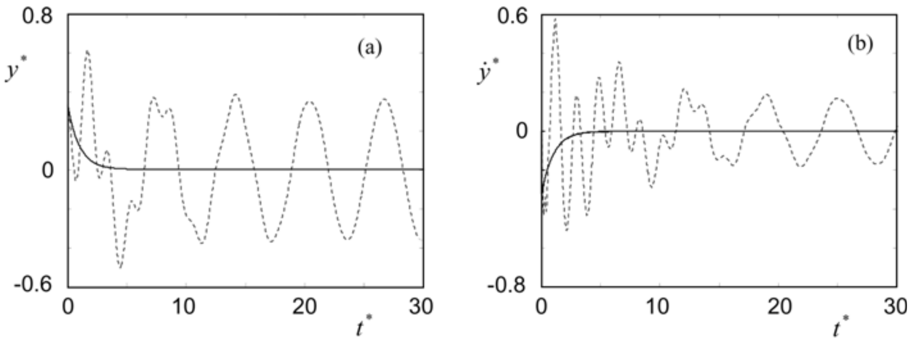

Fig. 4 Time-history response of the tip of the beam without time delay (No control ···········, With control ———)

elements being zero, i = 1, 2, ..., 4. The scalar $\tilde{r}_i$ in (23) is chosen to be $\tilde{r}_i$ = 0.01, i = 1, 2, ..., 4. Using the discrete LQR, the four coefficient vectors of the switching surfaces can be obtained as

$$\begin{cases} \boldsymbol{c}_1 = [4900.7, 295.3, 2.7, 2.2, 1.7, 1.2, 1.0] \\ \boldsymbol{c}_2 = [1441.9, 211.7, 2.1, 1.9, 1.6, 1.2, 1.0] \\ \boldsymbol{c}_3 = [-4365.1, -36.0, -0.1, 0.3, 0.7, 0.8, 1.0] \\ \boldsymbol{c}_4 = [4367.9, 16.9, -0.1, -0.4, -0.2, 0.3, 1.0] \end{cases}$$

Again set $q_i\overline{T} = 0.01$ and $\varepsilon_i\overline{T} = 0.0001$. In this case, the time-history responses in displacement and velocity of the first four modes are shown in Figs. 5(a) and (b), respectively, where the solid line represents the result with control and the dashed line without control. It shows that using the proposed control method with considering the existence of the time delay, the responses of the first four modes do not suffer from instability and the control effectiveness of the first-order mode is remarkable. However, different with the case of no time delay, the second, third and fourth modes are not precisely controlled, especially for the third-order modal displacement. The result of the third-order modal displacement with control is even larger than that with no control. This is possibly resulted from the reason that the third-order mode is excited by the control forces due to the existence of time delay. We can also observe from Fig. 5 that, the magnitude of the first-order modal responses (displacement and velocity) are much larger than those of the other three modes, so vibration control of the beam means mainly the control to the first-order mode of the beam, which is the same as the case with no time delay. The tip displacement and velocity responses of the beam are shown in Figs. 6(a) and (b), respectively, where the solid line represents the result with control and the dashed line without control. Since the first-order mode of the beam is effectively controlled (as shown in Fig. 5), so it is observed from Fig. 6 that the vibration of the beam is reduced significantly although it is not fully suppressed.

The tip response of the beam when λ = 0.1 is shown in Fig. 6. When λ = 0.1, the matrix $\tilde{\boldsymbol{a}}_i$ in (14) is a (12 × 12) matrix and all the eigenvalues lie inside the unit circle, and rank $[\tilde{\boldsymbol{a}}_i, \tilde{\boldsymbol{d}}_i]$ = 12. The coefficient vectors of the switching surfaces are given as

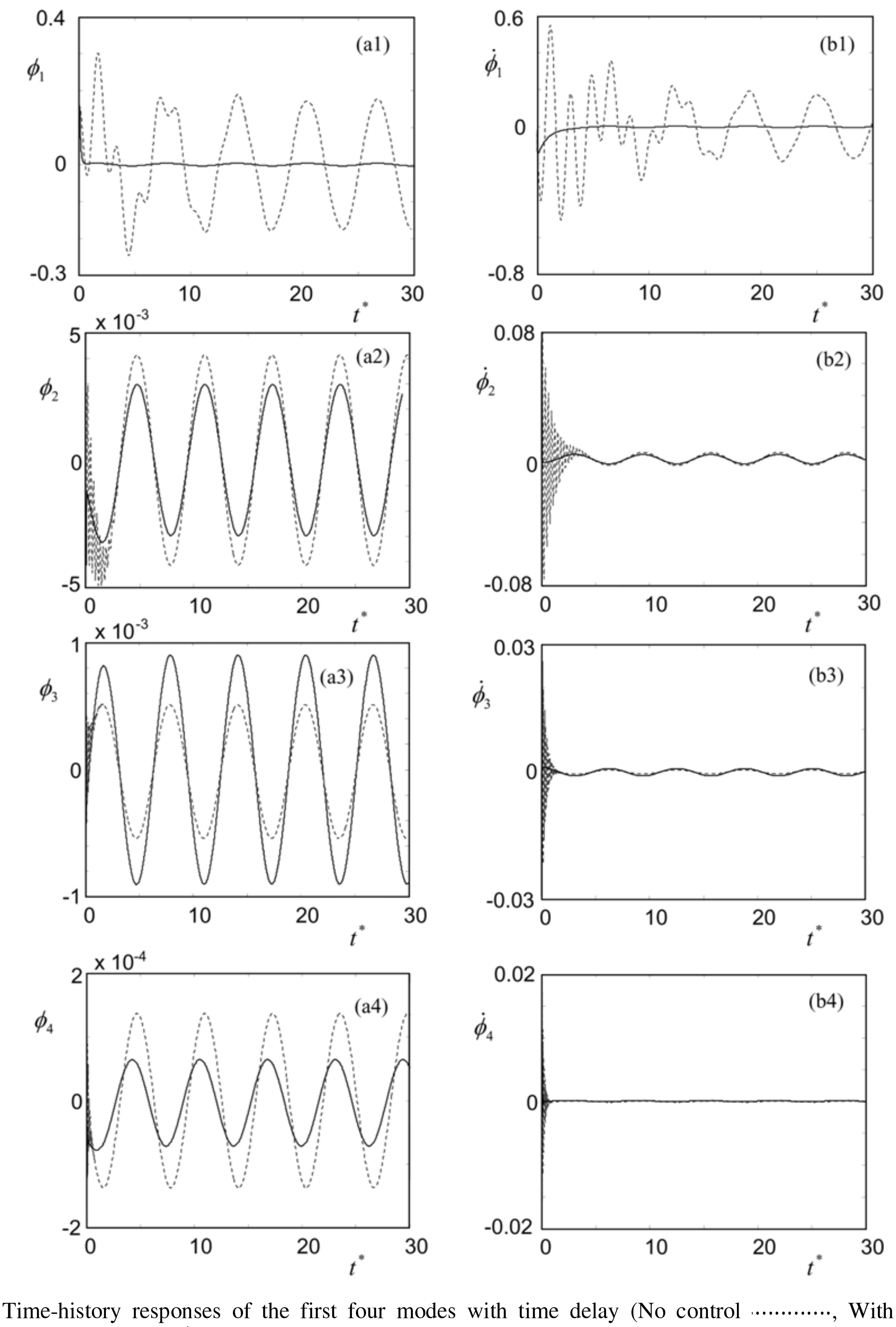

Fig. 5 Time-history responses of the first four modes with time delay (No control ············, With control ($\lambda = 0.05$) ————)

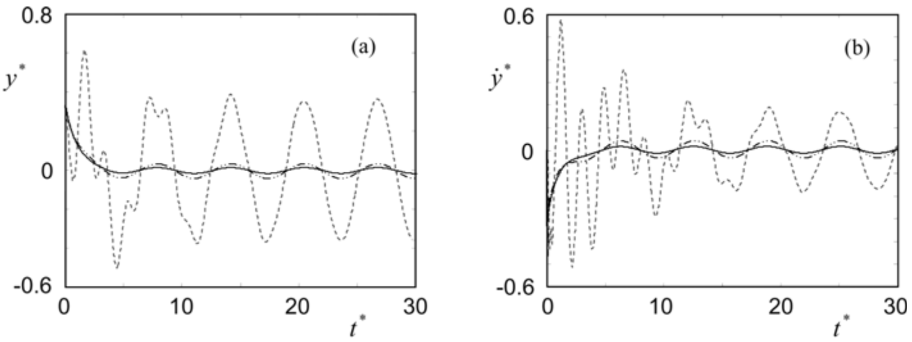

Fig. 6 Time-history response of the tip of the beam with time delay (No control ············, With control (λ = 0.05) ———, With control (λ = 0.1) —··—··—··)

$$\begin{cases} \boldsymbol{c}_1 = [4645.7, 527.3, 5.0, 4.6, 4.1, 3.7, 3.2, 2.7, 2.2, 1.7, 1.2, 1.0] \\ \boldsymbol{c}_2 = [-3259.9, 137.1, 1.5, 1.8, 2.0, 2.1, 2.1, 2.1, 1.9, 1.6, 1.2, 1.0] \\ \boldsymbol{c}_3 = [3838.2, 27.3, 0.1, -0.3, -0.6, -0.7, -0.5, -0.1, 0.3, 0.7, 0.8, 1.0] \\ \boldsymbol{c}_4 = [3459.9, 5.8, -0.1, -0.3, -0.1, 0.3, 0.3, -0.1, -0.4, -0.2, 0.3, 1.0] \end{cases}$$

In the design of switching surfaces, $q_i\overline{T}$, $\varepsilon_i\overline{T}$, $\tilde{\boldsymbol{q}}_i$ and $\tilde{r}_i$ are chosen as the same as those in the case of λ = 0.05. Again, instability occurs when the controller designed in the case of no time delay is used for the beam with λ = 0.1 in control.

5. Conclusions

Time delay exists inevitably in active control systems. If the existence of time delay is neglected in the controller design, the control system is susceptible to dynamic instability. Therefore, the time delay should be analyzed and tackled properly before the controller is used to a dynamic system with time delay. In this paper, the active control of flexible cantilever beam with time delay is investigated using an independent modal space control strategy, where the modal controller is designed using a discrete variable structure control method. The discrete switching surface and the variable structure modal controller are designed for the proposed control algorithm. In addition, a modal filter is designed to estimate the modal coordinates from physical sensor measurements, and the conversion method of the physical control forces of actuators from the modal ones are provided. Simulation results indicate that the proposed control method is effective in suppressing the vibration of the beam. When there exists no time delay in the control system, the response of the beam can be almost fully controlled. Instability may occur if the controller designed by neglecting the time delay is used to control the flexible beam with the existence of time delay. In the proposed control method, time delay term is incorporated into the mathematical model from the very beginning of the control algorithm derivation, without any approximation and hypothesis. Therefore, the system stability and control performance are prone to be guaranteed.

Acknowledgements

This work is supported by the National Science Foundation of China (No. 10472065), the "Shu Guang Plan" of Shanghai and the Science Foundation of Shanghai (No. 03ZR14062), for which the authors are grateful.

References

Abdel-Mooty, M. and Roorda, J. (1991), "Time-delay compensation in active damping of structures", *J. Eng. Mech.*, ASCE, **117**(11), 2549-2570.

Agrawal, A.K., Fujino, Y. and Bhartia, B.K. (1993), "Instability to time delay and its compensation in active control of structures", *Earthq. Eng. Struct. Dyn.*, **22**, 211-224.

Bailey, T. and Hubbard, J.E. (1985), "Distributed piezoelectric polymer active vibration control of a cantilever beam", *Journal of Guidance, Control and Dynamics*, **8**(5), 605-611.

Balas, M.J. (1980), "Enhanced modal control of flexible structures via innovations feedthrough", *Int. J. Control*, **32**(6), 983-1003.

Balas, M.J. (1982), "Trends in large space structure control theory: Fondest hopes, wildest dreams", *IEEE Transactions on Automatic Control*, **27**(3), 522-535.

Baz, A. and Poh, S. (1999), "Experimental implementation of the modified independent modal space control method", *J. Sound Vib.*, **139**(1), 133-149.

Cai, G.P. (2002), "Active control of time-delay systems and its application to seismic-excited building structures", Postdoctoral Research Report, Shanghai Jiaotong University, China.

Cai, G.P. and Huang, J.Z. (2002a), "Optimal control method for seismically excited building structures with time delay in control", *J. Eng. Mech.*, ASCE, **128**(6), 602-612.

Cai, G.P. and Huang, J.Z. (2002b), "Discrete-time variable structure control method for seismic-excited building structures with time delay", *J. Earthq. Eng. Struct. Dyn.*, **31**(7), 1347-1359.

Chait, Y. and Radeliffe, C.J. (1989), "Control of flexible structures using an augmented observer", *Journal of Guidance, Control and Dynamics*, **12**(2), 155-161.

Chen, J. (1995), "On computing the maximal delay intervals for stability of linear indendent of delay systems", *IEEE Transactions on Automatic Control*, **40**(6), 1087-1093.

Chung, L.L., Lin, C.C. and Lu, K.H. (1995), "Time-delay control of structures", *Earthq. Eng. Struct. Dyn.*, **24**, 687-701.

Chung, L.L., Reinhorn, A.M. and Soong, T.T. (1988), "Experiments on active control of seismic structures", *J. Eng. Mech.*, ASCE, **114**(2), 241-256.

Darby, A.P. and Pellegrino, S. (1999), "Modeling and control of a flexible structure incorporating inertial slip-stick actuators", *Journal of Guidance, Control and Dynamics*, **22**(1), 36-42.

Gao, W. (1998), *Theory and Design Method of Variable Structure Control*, Chinese Press of Science and Technology, Beijing, China. (In Chinese)

Gao, W. and Hung, J.C. (1993), "Variable structure control of nonlinear systems: A new approach", *IEEE Transactions on Industrial Electronics*, **40**(1), 45-55.

Gao, W., Wang, Y. and Homaifa, A. (1995), "Discrete-time variable structure control systems", *IEEE Transactions on Industrial Electronics*, **42**(2), 117-122.

Gennaro, S. (1998), "Active vibration suppression in flexible spacecraft attitude tracking", *Journal of Guidance, Control and Dynamics*, **21**(3), 400-408.

Hu, H. (1997), "On dynamics in vibration with time delay", *Chinese J. Vib. Eng.*, **10**(3), 273-279. (In Chinese)

Kwakernaak, H. and Sivan, R. (1972), *Linear Optimal Control Systems*, Wiley-InterScience, New York.

Meirovitch, L. and Baruh, H. (1982), "Control of self-adjoint distributed-parameter systems", *Journal of Guidance, Control and Dynamics*, **5**(1), 60-66.

Meirovitch, L. and Baruh, H. (1985), "The implementation of modal filters for control of structures", *Journal of Guidance, Control and Dynamics*, **8**(6), 707-716.

Qin, Y.X. (1987), *Motion Stability of Dynamic Systems with Time Delay*, Chinese Press of Science and Technology, Beijing, China. (In Chinese)

Schafer, B.E. and Holzach, H. (1985), "Experimental research on flexible beam modal control", *Journal of Guidance*, *Control and Dynamics,* **8**(5), 597-604.

Wang, D.A. and Huang, Y.M. (2003), "Application of discrete-time variable structure control in the vibration reduction of a flexible structure", *J. Sound Vib.*, **261**(3), 483-501.

Wong, K.K.F. (2005), "Predictive optimal linear control of elastic structures during earthquake: Part I", *J. Eng. Mech.*, ASCE, **131**(2), 131-141.

Yang, J.N., Akbarpour, A. and Askar, G. (1990), "Effect of time delay on control of seismic-excited buildings", *J. Eng. Mech.*, ASCE, **116**(10), 2801-2815.

Yang, J.N., Wu, J.C. and Agrawal, A.K. (1995), "Sliding mode control for nonlinear and hysteretic structures", *J. Eng. Mech.*, ASCE, **121**(12), 1330-1339.

Zee, R.E. and Hughes, P.C. (2000), "Mode localization in flexible spacecraft: A control challenge", *Journal of Guidance, Control and Dynamics*, **23**(1), 69-76.

Zhang, X., Liu, H. and Cao, W. (1996), "Active control of flexible mechanism", *Chinese J. Mech. Eng.*, **32**(1), 9-16. (In Chinese)