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Buckling of cylindrical shells with internal ring supports

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Abstract. This paper is concerned with the elastic buckling of cylindrical shells with internal rigid ring supports. The internal supports impose a zero lateral deflection constraint on the buckling modes at their locations. An automated Rayleigh-Ritz method is presented for solving this buckling problem. The method can handle any combination of end conditions and any number of internal supports. Moreover, it is simple to code and can yield very accurate solutions. New buckling results for cylindrical shells with a single internal ring support, and under lateral pressure and hydrostatic pressure, are given in the form of design charts. These results should be valuable to engineering designers.

Key words: buckling; cylindrical shells; internal ring supports; Rayleigh-Ritz method.

1. Introduction

Circular cylindrical shells are widely used in many types of structures, for examples in aircrafts, space vehicles, marine vessels, pressure vessels, reactor vessels and silos. The buckling aspect of these shells under various loading and boundary conditions is an important design consideration.

In the early 1900s, Lorenz (1908) and Timoshenko (1910) pioneered the study on buckling of cylindrical shells under axial pressure. The buckling of these shells under lateral and hydrostatic pressures was subsequently studied by many researchers, such as von Mises (1914, 1929), Batdorf (1947), Nash (1954), Galletly and Bart (1956), Armenakas and Herrmann (1963) and Soong (1967). In these studies, a simple one-term deflection function was used and the problem was solved under the special boundary conditions which can be satisfied by the assumed deflection function. Later, by integrating the basic functions directly, more accurate solutions were obtained by Ho and Cheng (1963), Sobel (1964), Thielmann and Esslinger (1964), Schnell (1965) and Yamaki (1968) under a variety of loading and boundary conditions.

Buckling of cylindrical shells with stiffeners, such as ring stiffeners or longitudinal stiffeners was also studied by Salerno and Levine (1950, 1951), Nash (1953, 1954), Bodner (1957), Baruch and Singer (1963), Ellinas and Croll (1981), Wang and Zeng (1983). Kendrick (1970) presented an extensive survey of the buckling of ring-stiffened cylindrical pressure vessels.

So far, there has been very little work done on the buckling of cylindrical shells with internal rigid ring supports. Prompted by this lack, we present an automated Rayleigh-Ritz method for

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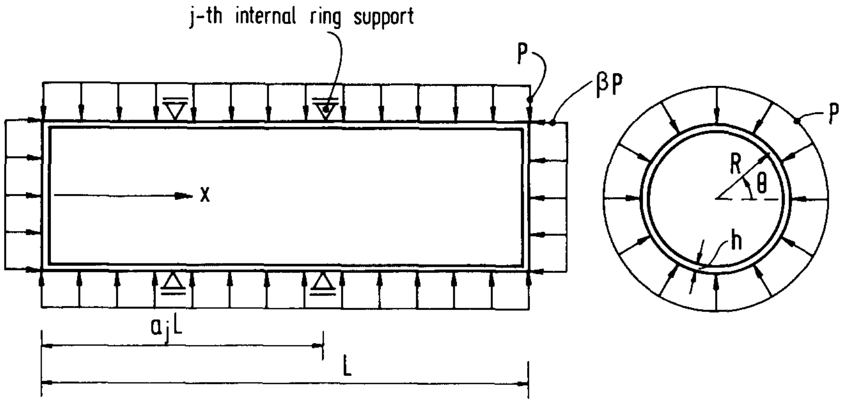

Fig. 1 Externally pressurized cylindrical shell with internal ring supports.

the buckling analysis of such shells under external pressures. The method is simple to code and can furnish very accurate solutions for shells having any combination of support conditions and any number of internal ring supports. Critical lateral loads and hydrostatic loads for cylindrical shells with an internal rigid ring are computed and presented in the form of design charts. These new buckling solutions should be valuable to engineers who are designing cylindrical shell structures.

2. Energy functional for cylindrical shell buckling

Consider a geometrically perfect, closed-ended cylindrical shell of thickness h, radius R, length L, Young's modulus E, Poisson's ratio ν. The ends of the shell may be simply supported or clamped. The shell is also internally supported by r number of rigid ring supports which imposes a zero lateral displacement at the supported positions (see Fig. 1).

There are several thin cylindrical shell theories due to how and when the terms z/R and h/R are to be neglected with respect to unity in the setting up of the constitutive equations or in the enumeration of the force and moment integrals (Markuš (1988). Considering the simplest theory of Donnell (1934) and the more precise theory of Sanders (1959), the strain-displacement relationships may be universally written as

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{1a}$$

$$\varepsilon_\theta = \frac{1}{R}\left[\frac{\partial v}{\partial \theta} - w\right] \tag{1b}$$

$$\varepsilon_{x\theta} = \frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta} \tag{1c}$$

$$\kappa_x = \frac{\partial^2 w}{\partial x^2} \tag{1d}$$

$$\kappa_\theta = \frac{1}{R^2}\left[\frac{\partial^2 w}{\partial \theta^2} + \eta w\right] \tag{1e}$$

$$\kappa_{x\theta}=\frac{1}{R}\left[\frac{\partial^2 w}{\partial x\partial\theta}+\frac{3}{4}\eta\frac{\partial v}{\partial x}-\frac{\eta}{4R}\frac{\partial u}{\partial\theta}\right] \tag{1f}$$

in which x is the longitudinal coordinate; θ is the circumferential coordinate; ε_x, ε_θ, $\varepsilon_{x\theta}$ are the linear strain components of the shell middle surface and κ_x, κ_θ, $\kappa_{x\theta}$ the curvature changes; and w, u and v are the displacements in the lateral, longitudinal and tangential directions, respectively. Depending on the adoption of which of the two shell theories, the scalar indicator η takes the value of

$$\eta=0, \text{ according to Donnell (1934)} \tag{2a}$$

$$\eta=1, \text{ according to Sanders (1959)} \tag{2b}$$

However, it must be noted that Donnell's theory furnishes erroneous buckling solutions when the shell is relatively long and thick (Galletly and Bart 1956, Soong 1967). Sanders' theory does not have this drawback. It is worth noting that Love (1944) and Timoshenko (1961) cylindrical shell theory approximates $\kappa_{x\theta}=(\partial^2 w/\partial x\partial\theta-\partial v/\partial x)/R^2$. This approximation does not lead to much different results from Sanders' buckling solutions.

The elastic strain energy due to in-plane stretching of the shell is given by (Timoshenko and Gere 1961, pp. 443)

$$U_s=\frac{Eh}{2(1-\nu^2)}\int_0^L\int_0^{2\pi}\left[\varepsilon_x^2+\varepsilon_\theta^2+2\nu\varepsilon_x\varepsilon_\theta+\frac{1-\nu}{2}\varepsilon_{x\theta}^2\right]Rd\theta dx \tag{3}$$

while the elastic strain energy due to bending of the middle surface of the shell is given by

$$U_b=\frac{Eh^3}{24(1-\nu^2)}\int_0^L\int_0^{2\pi}\left[\kappa_x^2+\kappa_\theta^2+2\nu\kappa_x\kappa_\theta+2(1-\nu)\kappa_{x\theta}^2\right]Rd\theta dx \tag{4}$$

The potential energy of the lateral pressure p is given by (Wang and Zeng 1983)

$$V_1=-\int_0^L\int_0^{2\pi}\frac{pR}{2}\left\{\frac{1}{R^2}\left[\frac{\partial^2 w}{\partial\theta^2}+\eta w\right]\right\}wRd\theta dx \tag{5}$$

and the potential energy of the axial pressure βp is

$$V_a=-\int_0^L\int_0^{2\pi}\frac{\beta pR}{4}\left[\frac{\partial w}{\partial x}\right]^2Rd\theta dx \tag{6}$$

where β is the ratio between axial pressure to lateral pressure and the considered range of β is $0\leq\beta\leq 1$. The pressure is simply lateral pressure when $\beta=0$ and is hydrostatic when $\beta=1$. Note that this class of cylindrical shell buckling problems is not within the imperfection sensitive group since the axial load is not the dominant load.

In view of (1)-(6), the total potential energy functional is given by

$$F=U_s+U_b+V_1+V_a$$

$$=\int_0^L\int_0^{2\pi}\left\{\frac{Eh}{2(1-\nu^2)}\left[\left[\frac{\partial u}{\partial x}\right]^2+\frac{1}{R^2}\left[\frac{\partial v}{\partial\theta}-w\right]^2+\frac{2\nu}{R}\left[\frac{\partial u}{\partial x}\right]\left[\frac{\partial v}{\partial\theta}-w\right]+\frac{1-\nu}{2}\left[\frac{\partial v}{\partial x}+\frac{1}{R}\frac{\partial u}{\partial\theta}\right]^2\right]\right.$$

$$+\frac{Eh^3}{24(1-\nu^2)}\left[\left[\frac{\partial^2 w}{\partial x^2}\right]^2+\frac{1}{R^4}\left[\frac{\partial^2 w}{\partial\theta^2}+\eta w\right]^2+\frac{2\nu}{R^2}\left[\frac{\partial^2 w}{\partial x^2}\right]\left[\frac{\partial^2 w}{\partial\theta^2}+\eta w\right]\right.$$

$$+\frac{2(1-\nu)}{R^2}\left[\frac{\partial^2 w}{\partial x\partial\theta}+\frac{3}{4}\eta\frac{\partial v}{\partial x}-\frac{\eta}{4R}\frac{\partial u}{\partial\theta}\right]^2\Bigg]$$

$$-\frac{pR}{2}\left[\frac{w}{R^2}\left[\frac{\partial^2 w}{\partial\theta^2}+\eta w\right]+\frac{\beta}{2}\left[\frac{\partial w}{\partial x}\right]^2\right]\Bigg\}R\,d\theta dx \tag{7}$$

For generality and convenience, the following nondimensional terms are adopted:

$$\bar{w}=\frac{w}{R};\ \Theta=\frac{\theta}{2\pi};\ \bar{x}=\frac{x}{L};\ \lambda=\frac{pR(1-\nu^2)}{Eh};\ \bar{u}=\frac{u}{h};\ \bar{v}=\frac{v}{h};$$

$$\alpha=\frac{R}{L};\ \zeta=\frac{h}{R};\ \bar{F}=\frac{F(1-\nu^2)}{2\pi hRLE} \tag{8}$$

Using the foregoing nondimensional terms, the total potential energy functional may be expressed as

$$\bar{F}=\frac{1}{2}\int_0^1\int_0^1\Bigg\{\alpha^2\zeta^2\left[\frac{\partial\bar{u}}{\partial\bar{x}}\right]^2+\left[\frac{\zeta}{2\pi}\frac{\partial\bar{v}}{\partial\Theta}-\bar{w}\right]^2+2\nu\alpha\zeta\left[\frac{\partial\bar{u}}{\partial\bar{x}}\right]\left[\frac{\zeta}{2\pi}\frac{\partial\bar{v}}{\partial\Theta}-\bar{w}\right]$$

$$+\frac{1-\nu}{2}\zeta^2\left[\alpha\frac{\partial\bar{v}}{\partial\bar{x}}+\frac{1}{2\pi}\frac{\partial\bar{u}}{\partial\Theta}\right]^2+\frac{\zeta^2}{12}\Bigg[\alpha^4\left[\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right]^2+\left[\frac{1}{4\pi^2}\frac{\partial^2\bar{w}}{\partial\Theta^2}+\eta\bar{w}\right]^2$$

$$+2\nu\alpha^2\left[\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right]\left[\frac{1}{4\pi^2}\frac{\partial^2\bar{w}}{\partial\Theta^2}+\eta\bar{w}\right]$$

$$+2(1-\nu)\alpha^2\left[\frac{1}{2\pi}\frac{\partial^2\bar{w}}{\partial\bar{x}\partial\Theta}+\frac{3}{4}\eta\zeta\frac{\partial\bar{v}}{\partial\bar{x}}-\frac{\eta\zeta}{8\pi\alpha}\frac{\partial\bar{u}}{\partial\Theta}\right]^2\Bigg]$$

$$-\lambda\left[\bar{w}\left[\frac{1}{4\pi^2}\frac{\partial^2\bar{w}}{\partial\Theta^2}+\eta\bar{w}\right]+\frac{\beta\alpha^2}{2}\left[\frac{\partial\bar{w}}{\partial\bar{x}}\right]^2\right]\Bigg\}d\Theta d\bar{x} \tag{9}$$

The following trigonometric functions in the circumferential coordinate Θ may be used to separate the spatial variables $\bar{x}$ and Θ:

$$\bar{u}(\bar{x},\ \Theta)=\bar{U}(\bar{x})\ \sin\ 2\pi n\Theta \tag{10a}$$

$$\bar{v}(\bar{x},\ \Theta)=\bar{V}(\bar{x})\ \cos\ 2\pi n\Theta \tag{10b}$$

$$\bar{w}(\bar{x},\ \Theta)=\bar{W}(\bar{x})\ \sin\ 2\pi n\Theta \tag{10c}$$

Note that the trigonometric functions satisfy the requirement of periodicity.

By substituting (10) into (9) and then integrating with respect to Θ leads to the following simpler energy functional form

$$\bar{F}=\frac{1}{4}\int_0^1\Bigg\{\alpha^2\zeta^2\left[\frac{d\bar{U}}{d\bar{x}}\right]^2+\left[\zeta n\bar{V}+\bar{W}\right]^2-2\nu\alpha\zeta\left[\frac{d\bar{U}}{d\bar{x}}\right]\left[\zeta n\bar{V}+\bar{W}\right]+\frac{1-\nu}{2}\zeta^2\left[\alpha\frac{d\bar{V}}{d\bar{x}}+n\bar{U}\right]^2$$

$$+\frac{\zeta^2}{12}\Bigg[\alpha^4\left[\frac{d^2\bar{W}}{d\bar{x}^2}\right]^2+\bar{W}^2(\eta-n^2)^2+2\nu\alpha^2\left[\frac{d^2\bar{W}}{d\bar{x}^2}\right]\left[\bar{W}(\eta-n^2)\right]$$

$$+2(1-\nu)\alpha^2\left[n\frac{d\bar{W}}{d\bar{x}}+\frac{3}{4}\eta\zeta\frac{d\bar{V}}{d\bar{x}}-\frac{n\zeta\eta}{4\alpha}\bar{U}\right]^2\Bigg]$$

$$-\lambda\left[\bar{W}^2(\eta-n^2)+\frac{\beta\alpha^2}{2}\left[\frac{d\bar{W}}{d\bar{x}}\right]^2\right]\Bigg\}\ d\bar{x} \tag{11}$$

where n is the circumferential wave number.

3. Geometric boundary conditions

Owing to possible different combinations of in-plane displacement components (u, v), there are four kinds of boundary conditions for simply supported cylinders which are designated as follows (Sobel 1964):

$$S_1:\ w=v=0 \tag{12a}$$

$$S_2:\ w=0 \tag{12b}$$

$$S_3:\ w=u=0 \tag{12c}$$

$$S_4:\ w=u=v=0 \tag{12d}$$

Similarly, the four kinds of clamped condition are:

$$C_1:\ w=\frac{dw}{dx}=v=0 \tag{13a}$$

$$C_2:\ w=\frac{dw}{dx}=0 \tag{13b}$$

$$C_3:\ w=\frac{dw}{dx}=u=0 \tag{13c}$$

$$C_4:\ w=\frac{dw}{dx}=u=v=0 \tag{13d}$$

At the j-th internal ring support, the geometric constraint is such that

$$w=0 \text{ at } x=a_jL \tag{14}$$

where a_jL is the longitudinal coordinate of the j-th internal ring support position (see Fig. 1).

4. Analysis *via* Rayleigh-Ritz method

In view of the foregoing kinematic boundary conditions, the following Ritz functions for cylindrical shells with internal ring supports may be adopted

$$\bar{U}=\left[\sum_{i=1}^{M} c_i\bar{x}^{i-1}\right]\left[\bar{x}\right]^{\Omega_u^0}\left[1-\bar{x}\right]^{\Omega_u^L}=\sum_{i=1}^{M} c_i\bar{U}_i \tag{15a}$$

$$\bar{V}=\left[\sum_{i=1}^{M} d_i\bar{x}^{i-1}\right]\left[\bar{x}\right]^{\Omega_v^0}\left[1-\bar{x}\right]^{\Omega_v^L}=\sum_{i=1}^{M} d_i\bar{V}_i \tag{15b}$$

$$\bar{W}=\left[\sum_{i=1}^{M} e_i\bar{x}^{i-1}\right]\left[\bar{x}\right]^{\Omega_w^0}\left[1-\bar{x}\right]^{\Omega_w^L}\prod_{j=1}^{r}\left[\bar{x}-a_j\right]^{\Lambda_j}=\sum_{i=1}^{M} e_i\bar{W}_i \tag{15c}$$

where a_j is the distance (normalized with respect to the length L) of the j-th internal ring support, r the number of internal ring supports, the power $\Lambda=0$ if there is no ring support and $\Lambda=1$ when there are ring supports and depending on the type of boundary condition (B.C.), the powers Ω take on

B.C.	S_1	S_2	S_3	S_4	C_1	C_2	C_3	C_4
Ω_u	0	0	1	1	0	0	1	1
Ω_v	1	0	0	1	1	0	0	1
Ω_w	1	1	1	1	2	2	2	2

and the superscripts of Ω, i.e. 0 and L, denote the cylindrical shell ends at $x=0$ or $x=L$.

Applying the Rayleigh-Ritz method,

$$\left.\begin{aligned}\frac{\partial\bar{F}}{\partial c_i}&=0;\\ \frac{\partial\bar{F}}{\partial d_i}&=0;\\ \frac{\partial\bar{F}}{\partial e_i}&=0;\end{aligned}\right\}\quad i=1,\ 2,\ \cdots,\ M \tag{16}$$

Substituting (15) into (11) and then into (16) yields

$$\left[\begin{pmatrix}[K_{cc}] & [K_{cd}] & [K_{ce}]\\ & [K_{dd}] & [K_{de}]\\ \text{symmetric} & & [K_{ee}]\end{pmatrix}-\lambda\begin{pmatrix}[0] & [0] & [0]\\ & [0] & [0]\\ & & [G_{ee}]\end{pmatrix}\right]\begin{bmatrix}\{c\}\\ \{d\}\\ \{e\}\end{bmatrix}=\begin{bmatrix}\{0\}\\ \{0\}\\ \{0\}\end{bmatrix} \tag{17}$$

where

$$\{c\}=\begin{Bmatrix}c_1\\ c_2\\ \vdots\\ c_M\end{Bmatrix};\qquad \{d\}=\begin{Bmatrix}d_1\\ d_2\\ \vdots\\ d_M\end{Bmatrix};\qquad \{e\}=\begin{Bmatrix}e_1\\ e_2\\ \vdots\\ e_M\end{Bmatrix} \tag{18}$$

and the elements of $[K]$ and $[G]$ are given by

$$K_{ccij}=2\zeta^2\alpha^2\int_0^1\frac{d\bar{U}_i}{d\bar{x}}\frac{d\bar{U}_j}{d\bar{x}}d\bar{x}+\zeta^2n^2(1-\nu)\left(1+\frac{1}{48}\zeta^2\eta\right)\int_0^1\bar{U}_i\bar{U}_j\ d\bar{x} \tag{19a}$$

$$K_{cdij}=\zeta^2\alpha n\left[(1-\nu)\left(1-\frac{1}{16}\zeta^2\eta\right)\int_0^1\bar{U}_i\frac{d\bar{V}_j}{d\bar{x}}d\bar{x}-2\nu\int_0^1\frac{d\bar{U}_i}{d\bar{x}}\bar{V}_jd\bar{x}\right] \tag{19b}$$

$$K_{ceij}=-\alpha\zeta n^2\eta(1-\nu)\frac{\zeta^2}{12}\int_0^1\bar{U}_i\frac{d\bar{W}_j}{d\bar{x}}d\bar{x}-2\zeta\alpha\nu\int_0^1\frac{d\bar{U}_i}{d\bar{x}}\bar{W}_jd\bar{x} \tag{19c}$$

$$K_{ddij}=2\zeta^2n^2\int_0^1\bar{V}_i\bar{V}_jd\bar{x}+\zeta^2\alpha^2(1-\nu)\left[1+\frac{3\eta\zeta^2}{16}\right]\int_0^1\frac{d\bar{V}_i}{d\bar{x}}\frac{d\bar{V}_j}{d\bar{x}}d\bar{x} \tag{19d}$$

$$K_{deij}=2\zeta\eta\int_0^1 \bar{V}_i\bar{W}_j d\bar{x}+\eta\alpha^2 n\frac{\zeta^3}{4}(1-\nu)\int_0^1\frac{d\bar{V}_i}{d\bar{x}}\frac{d\bar{W}_j}{d\bar{x}}d\bar{x} \tag{19e}$$

$$\begin{aligned}K_{eeij}=&\left(2+\frac{\zeta^2}{6}(\eta-n^2)^2\right)\int_0^1 \bar{W}_i\bar{W}_j d\bar{x}+\frac{\zeta^2}{6}\alpha^4\int_0^1\frac{d^2\bar{W}_i}{d\bar{x}^2}\frac{d^2\bar{W}_j}{d\bar{x}^2}d\bar{x}\\&+\frac{\zeta^2}{6}\nu\alpha^2(\eta-n^2)\left[\int_0^1\frac{d^2\bar{W}_i}{d\bar{x}^2}\bar{W}_j d\bar{x}+\int_0^1\bar{W}_i\frac{d^2\bar{W}_j}{d\bar{x}^2}d\bar{x}\right]\\&+\frac{\zeta^2}{3}(1-\nu)\alpha^2 n^2\int_0^1\frac{d\bar{W}_i}{d\bar{x}}\frac{d\bar{W}_j}{d\bar{x}}d\bar{x}\end{aligned} \tag{19f}$$

$$G_{eeij}=2(\eta-n^2)\int_0^1 \bar{W}_i\bar{W}_j d\bar{x}+\beta\alpha^2\int_0^1\frac{d\bar{W}_i}{d\bar{x}}\frac{d\bar{W}_j}{d\bar{x}}d\bar{x} \tag{19g}$$

where $i, j=1, 2, \cdots, M$.

The elastic buckling load λ is obtained by solving the generalized eigenvalue problem defined in (17). The integration and differentiation of the polynomial functions were carried out in an exact manner.

5. Numerical results

5.1. Convergence and comparison case studies

Convergence studies were carried out to establish the number of polynomial terms M required for accurate solution. Table 1 presents typically the convergence behaviour of the buckling solutions λ for simply supported (S_1-S_1) cylindrical shells under lateral pressure for $\nu=0.3$ and various dimensions, $\alpha=R/L=1/10$, 1/40, 1, 2; $\zeta=h/R=1/100$, 1/500, 1/1100, 1/1900 and with no internal ring support ($a=0$) or with a single ring support at the mid-span ($a=0.5$). It can be observed that the Rayleigh-Ritz method gives monotonic convergence. The number of polynomial terms for converged results is found to be $M=10$ and this value has been used to generate all subsequent results.

Moreover, it can be seen from Table 2 that the computed results for cylindrical shells without internal ring supports agree with those obtained by Sobel (1964) and Soong (1967). Nash (1954) results for clamped shells are approximate upper bound solutions.

5.2. Buckling results for cylindrical shells with an internal ring support

Although the method can handle any number of internal supports and any combination of end support conditions, only results for S_1-S_1, C_1-C_1, C_1-S_1 shells with a single internal support located at a distance aL from one end are presented due to space limitation.

Fig. 2 presents the variations of the critical lateral pressure ($\beta=0$) with respect to the radius to length ratio, R/L for various internal support positions. The integers on the curves indicate the number of circumferential waves n of the buckling mode. Owing to a relatively higher buckling load for shorter and thicker shells, this value of n increases with respect to increasing R/L and

Table 1 Convergence study: Critical lateral pressure parameter λ ($\times 10^4$) for simply supported (S_1-S_1) cylindrical shell according to Sanders' theory

Polynomial terms	$h/R=1/500$				$h/R=1/100$			
	$R/L=1$		$R/L=1/10$		$R/L=1$		$R/L=1/10$	
M	No internal ring support	Internal ring support at $a=0.5$	No internal ring support	Internal ring support at $a=0.5$	No internal ring support	Internal ring support at $a=0.5$	No internal ring support	Internal ring support at $a=0.5$
2	0.8288(13)	2.7258(18)	0.07908(4)	0.2813(6)	10.0625(8)	27.7455(11)	0.8487(3)	7.3876(4)
4	0.7925	1.8309	0.07346	0.1687	9.5864	23.2788	0.8165	1.8733
6	0.7924	1.6900	0.07345	0.1500	9.5864	22.2339	0.8164	1.6684
8	0.7924	1.6877	0.07345	0.1498	9.5864	22.2137	0.8164	1.6655
10	0.7924(13)	1.6877(18)	0.07345(4)	0.1498(6)	9.5864(8)	22.2136(11)	0.8164(3)	1.6655(4)
Soong(1967)	0.7920(13)		0.07347(4)		9.5886(8)		0.8151(3)	
Polynomial terms	$h/R=1/1100$				$h/R=1/1900$			
	$R/L=2$		$R/L=1/40$		$R/L=2$		$R/L=1/40$	
M	No internal ring support	Internal ring support at $a=0.5$	No internal ring support	Internal ring support at $a=0.5$	No internal ring support	Internal ring support at $a=0.5$	No internal ring support	Internal ring support at $a=0.5$
2	0.5189(22)	1.5634(30)	0.006185(3)	0.3929(4)	0.2255(25)	0.7418(35)	0.002517(3)	0.3860(4)
4	0.4964	1.1624	0.006052	0.01272	0.2143	0.4977	0.002383	0.005826
6	0.4963	1.0815	0.006051	0.01181	0.2143	0.4570	0.002383	0.004924
8	0.4963	1.0811	0.006051	0.01180	0.2143	0.4570	0.002383	0.004912
10	0.4963(22)	1.0811(30)	0.006051(3)	0.01180(4)	0.2143(25)	0.4570(35)	0.002383(3)	0.004912(4)
Soong(1967)	0.4962(22)		0.006058(3)		0.2142(25)		0.002377(3)	

Table 2 Comparison study: Critical pressure parameter λ ($\times 10^4$) for cylindrical shell

$h/R=$ 1/100	Lateral pressure ($\beta=0$)				Hydrostatic pressure ($\beta=1$)							
	Simply supported ends (S_1-S_1)									Clamped ends (C_1-C_1)		
	Donnell's theory		Sander's theory		Donnell's theory			Sander's theory		Donnell's theory		
R/L	Soong (1967)	Present analysis	Soong (1967)	Present analysis	Sobel (1964)	Soong (1967)	Present analysis	Soong (1967)	Present analysis	Nash (1954)	Sobel (1964)	Present analysis
2.000	22.2891	22.2878	22.2228	22.2129	19.1646	19.1608	19.1619	19.0837	19.0752	26.6630	24.1332	24.1368
1.000	9.6437	9.6433	9.5886	9.5864	8.9526	8.9553	8.9529	8.8892	8.8900	12.6308	10.0464	10.0498
0.500	4.4666	4.4653	4.4005	4.4063	4.3170	4.3179	4.3173	4.2518	4.2562	6.3436	4.5709	4.5711
0.250	2.2195	2.2219	2.1864	2.1882	2.1795	2.1809	2.1798	2.1424	2.1441	3.2832	2.2559	2.2733
0.100	0.8867	0.8855	0.8151	0.8164	0.8807	0.8812	0.8807	0.8096	0.8114	1.5260	0.8896	0.9091
0.050	0.4229	0.4229	0.3674	0.3679	–	0.4219	0.4217	0.3657	0.3663	–	–	0.4865
0.025	0.3398	0.3395	0.2578	0.2580	–	0.3393	0.3393	0.2578	0.2578	–	–	0.3554

h/R ratios. For a given shell dimension n is slightly increased with the introduction of an internal support since the buckling load is also raised. The increase in the elastic buckling capacity can be about two to three times by introducing a single internal ring support. For such symmetrical boundary conditions, it is obvious that the optimal location for the internal support is at the mid-span (i.e. $a=0.5$). Fig. 3 shows similar curves for S_1-S_1 cylindrical shells under hydrostatic pressure ($\beta=1$), but with a corresponding lower value for λ when compared to those

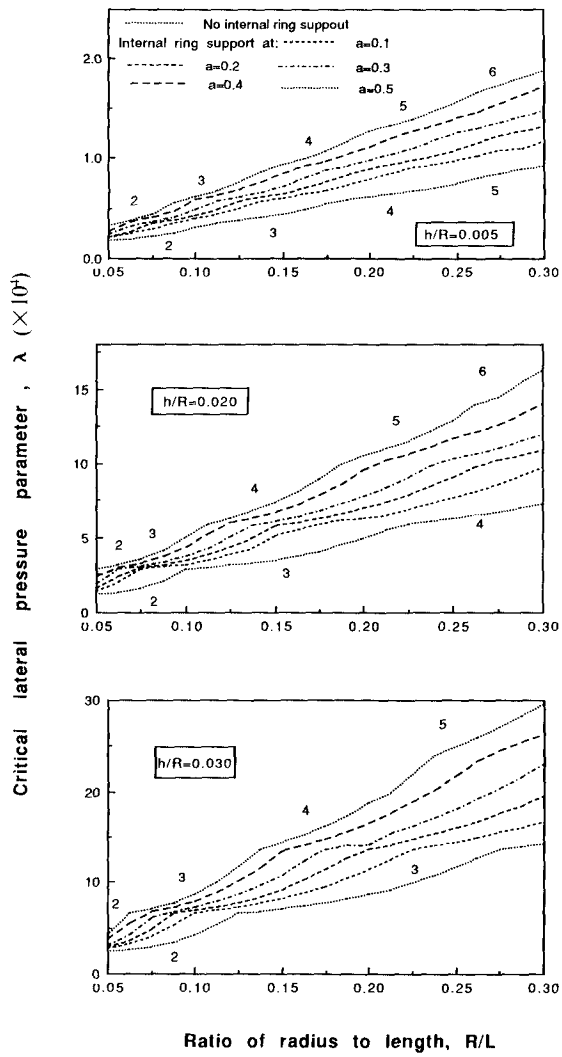

Fig. 2 Critical lateral pressure λ for S_1-S_1 cylindrical shells with an internal support at various positions aL.

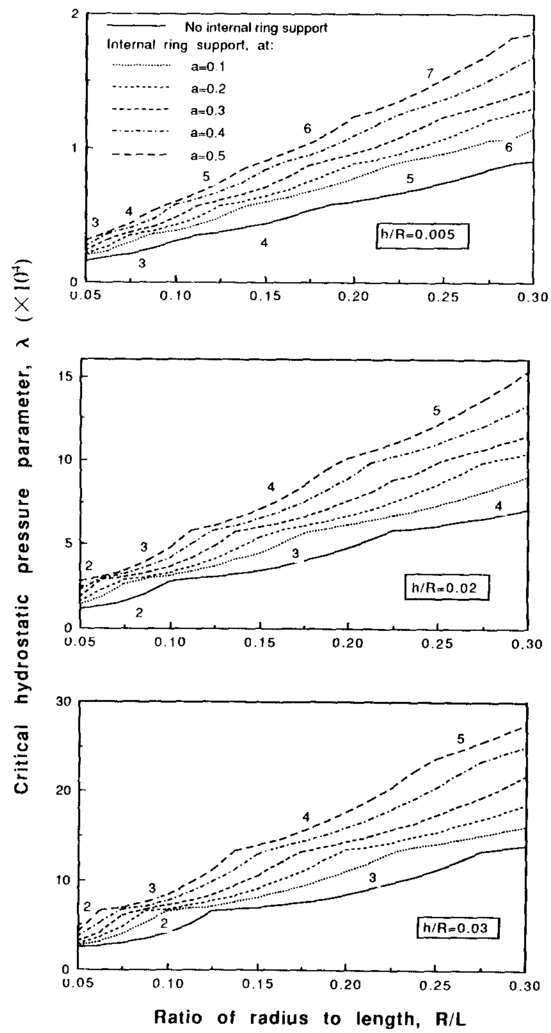

Fig. 3 Critical hydrostatic pressure λ for S_1-S_1 cylindrical shells with an internal support at various positions aL.

in Fig. 2.

Figs. 4 and 5, respectively, present the critical lateral ($\beta=0$) and hydrostatic pressures ($\beta=1$) for cylindrical shells with C_1-C_1 end conditions. The maximum increase of the critical pressures is also about two to three times when $a=0.5$.

Figs. 6 and 7 present the solutions for C_1-S_1 shells. Similar trends were observed. Figs. 8 and 9 show more clearly the influence of the internal ring support location on the buckling load for shells under lateral pressure and hydrostatic pressure, respectively. The buckling load is more sensitive to the effect of internal support position when the shell is relatively short ($R/L=1.0$) and thick ($h/R=0.03$). Also, it was observed that even though one end is clamped and the other is simply supported, the optimal location of the internal support is near the midspan.

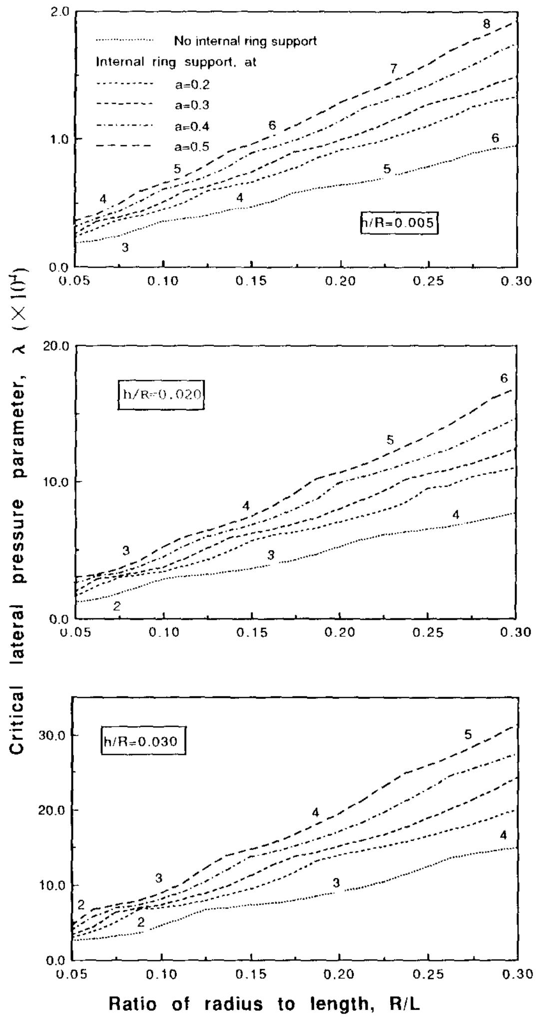

Fig. 4 Critical lateral pressure λ for C_1-C_1 cylindrical shells with an internal support at various positions aL.

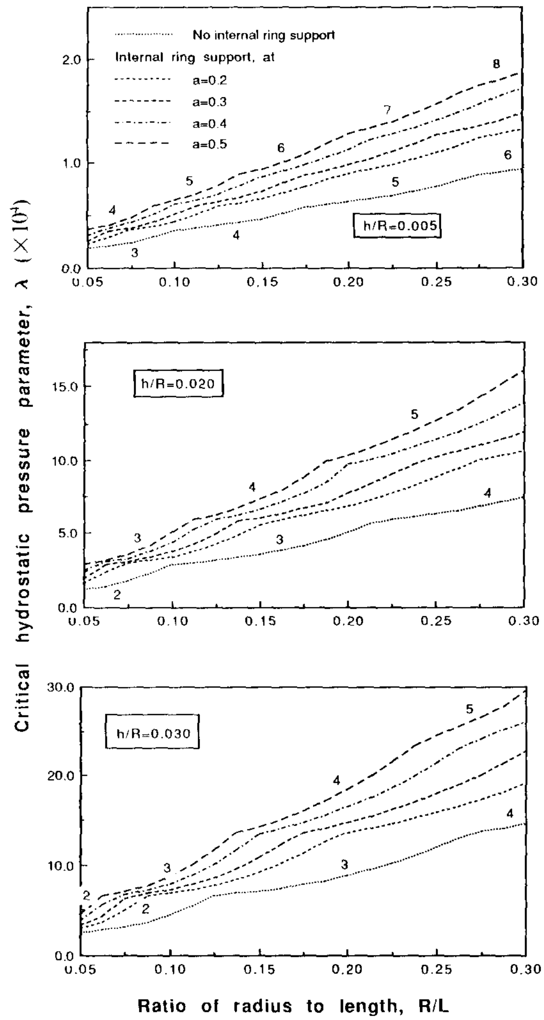

Fig. 5 Critical hydrostatic pressure λ for C_1-C_1 cylindrical shells with an internal support at various positions aL.

There is only a slight shift of this optimal location towards the simply supported end when the shell is short and thick.

6. Conclusions

An automated Rayleigh-Ritz method is presented for the elastic buckling analysis of externally pressurized cylindrical shells with internal ring supports. Close agreement of buckling results with those obtained by previous researchers for shells without an internal support shows the validity and high accuracy of the method.

Based on this method, generic buckling design charts for cylindrical shells under lateral ($\beta=0$) or hydrostatic ($\beta=1$) pressure and having an internal ring support are obtained. These new buckling results should be valuable to engineering designers.

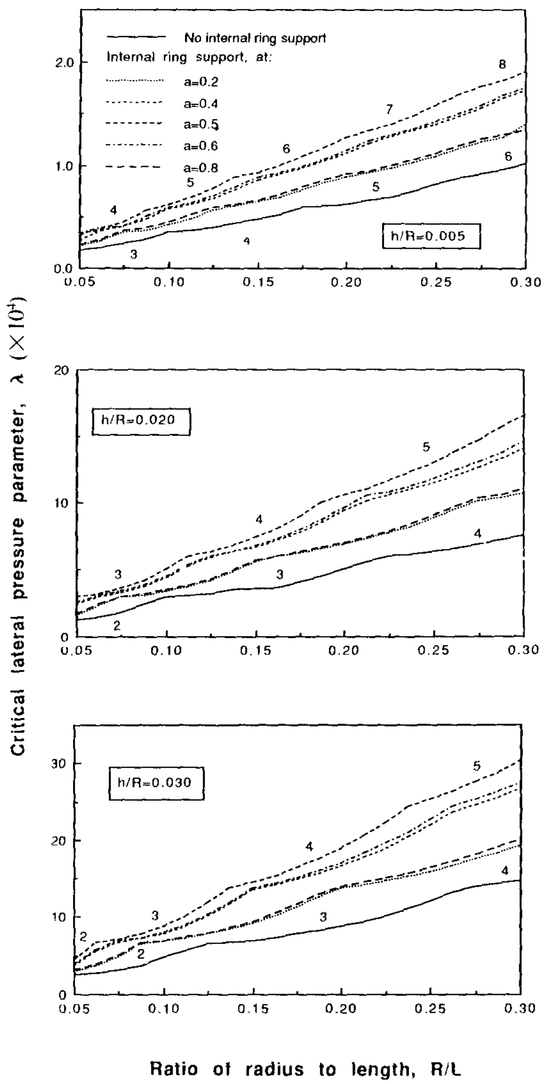

Fig. 6 Critical lateral pressure λ for C_1-S_1 cylindrical shells with an internal support at various positions aL.

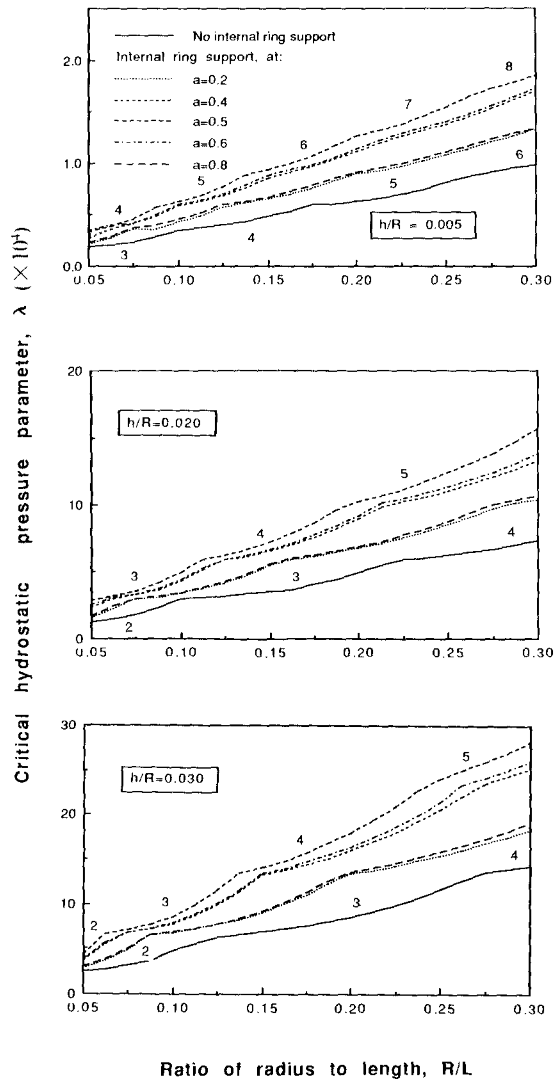

Fig. 7 Critical hydrostatic pressure λ for C_1-S_1 cylindrical shells with an internal support at various positions aL.

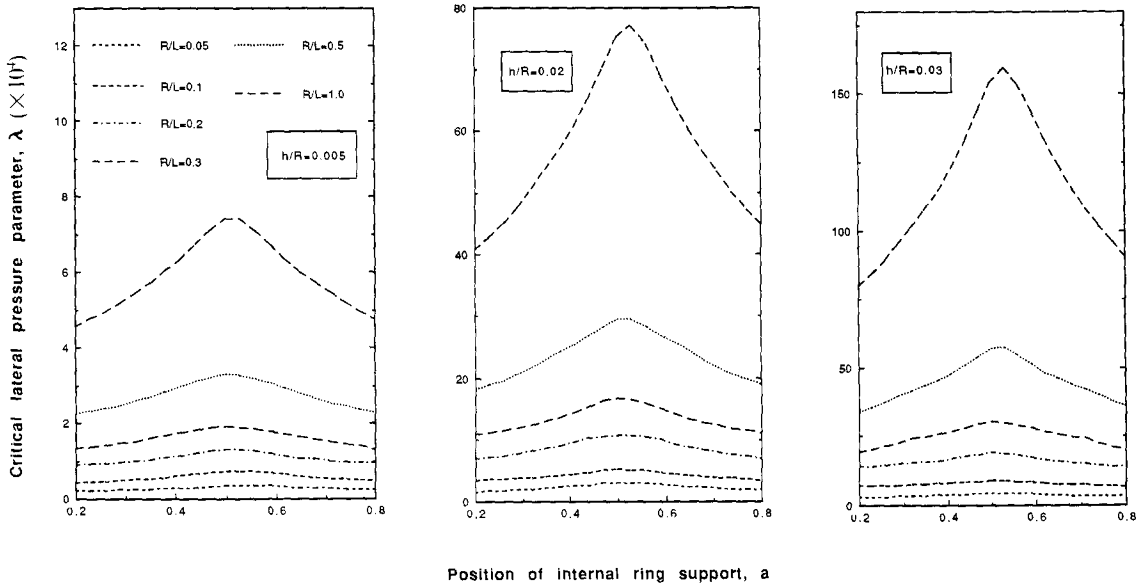

Fig. 8 Influence of internal ring support position on critical lateral pressure for C_1-S_1 cylindrical shells.

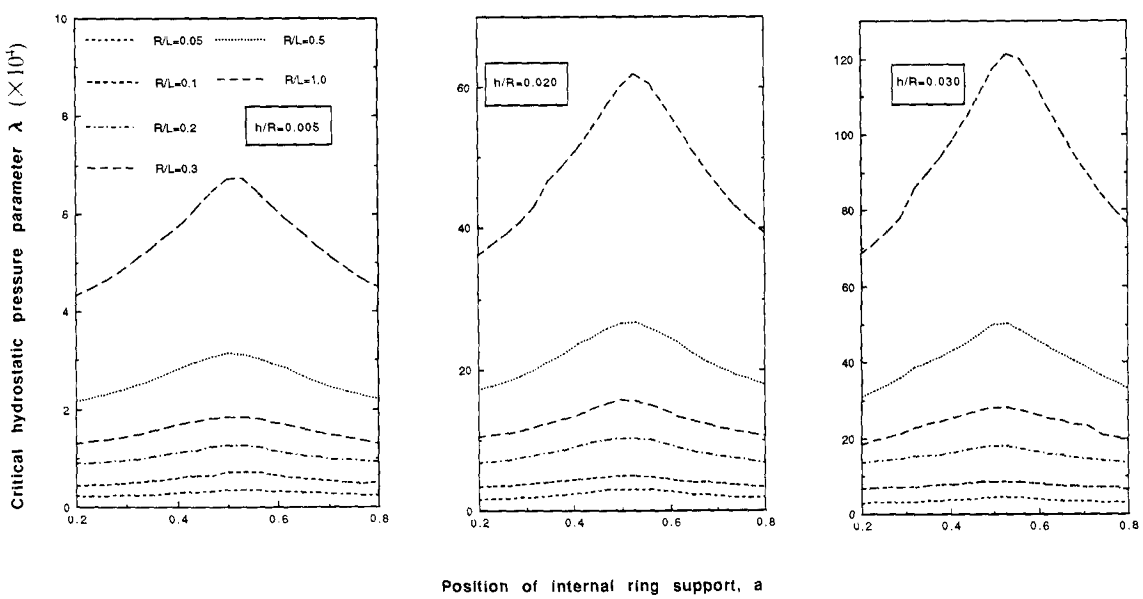

Fig. 9 Influence of internal ring support position on critical hydrostatic pressure for C_1-S_1 cylindrical shells.

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