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Failure mechanisms of a rigid-perfectly plastic cantilever with elastic deformation at its root subjected to tip pulse loading

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Abstract. In this paper, the effect of material elasticity was evaluated through a simple model as proposed by Wang and Yu (1991), for yield mechanisms of a cantilever beam under tip pulse loading. The beam was assumed rigid-perfectly plastic but instead of the usual fully clamped constraints at its root, an elastic-perfectly plastic rotational spring was introduced there so the system had a certain capacity to absorb elastic energy. Compared with a rigid-perfectly plastic beam without a spring root, the present beam-spring model showed differences in the initial plastic hinge position and the minimum magnitude of the dynamic force needed to produce a plastic failure. It was also shown that various failure responses may happen while the hinge travels along the beam segment towards the root, rather than a unique response mode as in a rigid perfectly plastic analysis.

Key words: pipe whip; plastic hinge; impact; elasticity.

1. Introduction

The response of structures under intense dynamic loading is of practical interest in many engineering areas, such as crachworthiness, collision protection, etc. The structures usually undergo large inelastic deformation under large impact loads and this poses significant problems in theoretical investigations. Due to the complexity caused by the combination of elastic and plastic deformations with moving boundaries between these two regions, only few are available for complete solutions. In most cases, the influence of material elasticity is usually neglected under the assumption that the input energy is many times larger than the maximum elastic energy which can be stored in the structures. This idealization often allows the development of simple solutions which are satisfactory for design purpose, such as those done by Lee and Symonds (1952), Parkes (1955), Jones (1989a), (1989b), etc.

However, the appearance of finite element codes makes it possible to investigate the elastoplastic solutions numerically and the results given by Symonds and Fleming (1984), Reid and Gui (1987) using ABAQUS indicate that elastic deformation can have significant effects on the details of dynamic plastic response of beams, even change the structural response modes in some cases. This strongly suggests that further investigations of the effect of elastic deformation are necessary. Nevertheless, if an elastic-plastic constitutive relation is employed for the beam, we have to

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solve a dynamic elastoplastic problem with floating boundaries. Alternatively, here we intend to provide a preliminary analysis of the effect of elastic deformation by adopting a structural model recently proposed by Wang and Yu (1991), which is quite similar to a rigid plastic one, except that the model has a certain capacity for storing elastic energy.

With the energy consideration in mind, the model assumes that the structural model consists of a rigid-plastic cantilever beam and an elastic-plastic rotational spring at the beam root. In other words, the beam itself has no capacity to store any elastic strain energy, but the root constraint is elastic-plastic, instead of the usual fully clamped one.

This beam-spring model was first examined by Wang and Yu (1991) in connection with the Parkes' problem, an impulsively loaded cantilever beam carrying a tip mass. It was found the response of the beam to be significantly different from Parkes' (1955) in certain cases. In this paper, a straight beam under a transverse pulse force load at its tip was treated by the above approach. Results of the analysis indicate that several modes of responses may exist which are different from the corresponding rigid-plastic solution as presented in the Appendix where the response mode is unique.

The following assumptions are made

1) The beam is rigid-perfectly plastic, and the rotational spring at the beam root is elastic-perfectly plastic;
2) Both the beam and the spring are made of a rate-independent material, and the dynamic fully plastic bending moment M_o of the beam and the spring are equal to each other.
3) The beam segment has uniform section and density;
4) The shear force Q is neglected in the yield condition;
5) The cross-section of the beam possesses an axis of symmetry parallel to the direction of the load F so that the deformation all takes place within a fixed plane.

2. Governing equations

Consider a straight cantilever beam subjected to a dynamic force load F applied at its tip. Instead of a fixed end at the root of the beam, it is assumed that there is a elastic-perfectly plastic rotational spring, as shown in Fig. 1. The cantilever has a tip mass G, mass per unit length μ, and a total length L. The beam is made from a rigid-perfectly plastic material and the rotational spring at the beam root is an elastic-perfectly plastic one. The spring becomes plastic when the moment on it reaches M_o, exactly the same value of the dynamic plastic moment at the cross-section of the beam. In other words, the bending moment at the beam root should be

$$M=\begin{cases} K\,\vartheta_e & \text{elastic or unloading} \\ \pm M_o & \text{otherwise} \end{cases}$$

where K is the elastic constant of the root spring, $\vartheta_e=\vartheta-\vartheta_p$ is the elastic rotation angle of the spring and ϑ_p is the plastic rotation angle, see Fig. 2.

The elastic stiffness of the spring is defined on the basis that all of the elastic strain energy capacity of the beam is lumped into the root spring. Thus while as the beam is considered to be rigid, the system as a whole has the same elastic energy storage capacity as that of a fixed-ended elastic plastic beam. This gives,

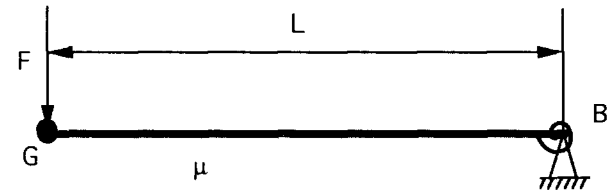

Fig. 1 The beam-spring model.

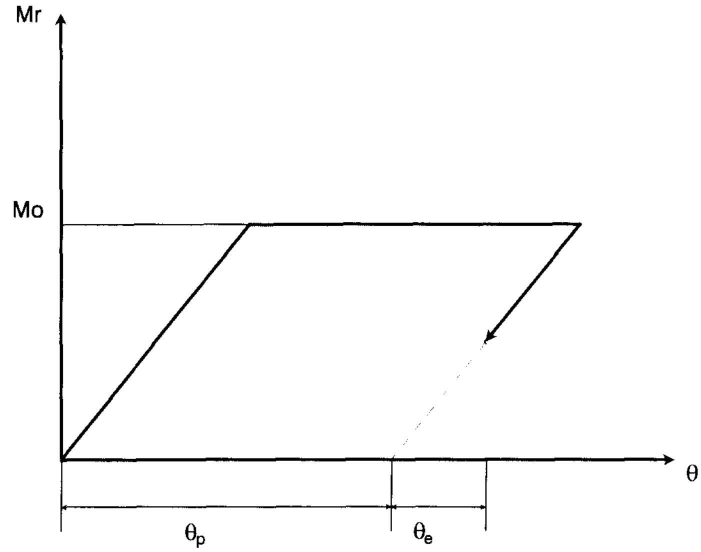

Fig. 2 Characteristics of the rotational spring at the beam root.

$$\frac{M_o^2 L}{2EI}=\frac{M_o^2}{2K}$$

where EI is the flexural rigidity of the beam if its elastic properties were not ignored. Therefore, the elastic constant of the spring can be defined as $K=EI/L$.

Static analysis shows that when

$$F\leq F_o=\frac{M_o}{L} \tag{1}$$

there will be no failure anywhere. And when $F>F_o$, a plastic hinge will appear either at root B or in the beam.

Assume that a hinge H is formed at a distance x from the tip in the cantilever, Fig. 3 shows the bending moment distribution and free-body diagram of the segments of the beam. $\dot{\vartheta}$ is the angular velocity of the rotational spring at the root, $\dot{\alpha}$ is the angular velocity of segment AH at the hinge H relative to segment HB.

For an arbitrary point ζ in AH, its transverse velocity in z direction is

$$V_\zeta=(L-\zeta)\dot{\vartheta}+(x-\zeta)\dot{\alpha} \tag{2}$$

Then its acceleration is

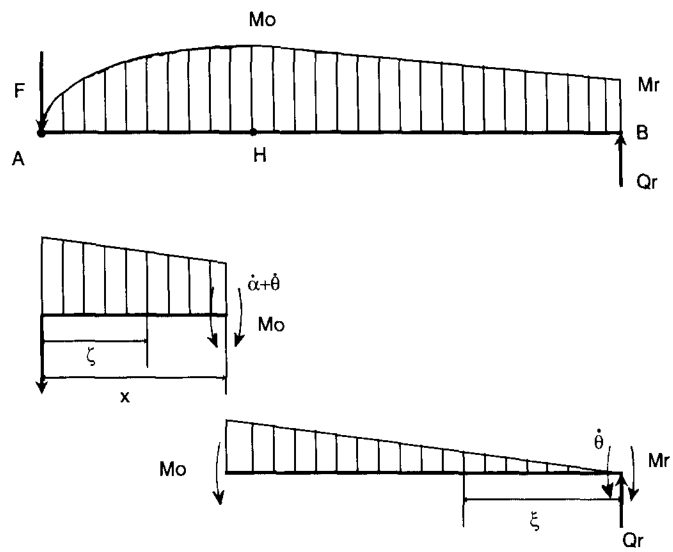

Fig. 3 The bending moment distribution in the beam and the free body diagram of beam segments.

$$a_\zeta=(L-\zeta)\ddot{\vartheta}+(x-\zeta)\ddot{\alpha}+\dot{x}\dot{\alpha} \tag{3}$$

Similarly, for an arbitrary point ξ in HB, it has

$$V_\xi=\xi\dot{\vartheta} \text{ and } a_\xi=\xi\ddot{\vartheta} \qquad (4) \text{ and } (5)$$

Employing d'Alembent's principle, the governing equation of translational motion of AH is,

$$\begin{aligned} F&=G\left[L\ddot{\vartheta}+x\ddot{\alpha}+\dot{x}\dot{\alpha}\right]+\mu\int_o^x\left[(L-\zeta)\ddot{\vartheta}+(x-\zeta)\ddot{\alpha}+\dot{x}\dot{\alpha}\right]d\zeta \\ &=\left[GL+\mu x\left(L-\frac{x}{2}\right)\right]\ddot{\vartheta}+\left(G+\frac{\mu}{2}x\right)x\ddot{\alpha}+(G+\mu x)\dot{x}\dot{\alpha} \end{aligned} \tag{6}$$

The equation of rotational motion of AH about B is

$$\begin{aligned} FL-M_o&=G\left[L\ddot{\vartheta}+x\ddot{\alpha}+\dot{x}\dot{\alpha}\right]L+\mu\int_o^x\left[(L-\zeta)\ddot{\vartheta}+(x-\zeta)\ddot{\alpha}+\dot{x}\dot{\alpha}\right](L-\zeta)\mathrm{d}\zeta \\ &=\mu\left[L^2+xL+\frac{x^2}{3}\right]x\ddot{\vartheta}+\frac{\mu}{2}\left(L-\frac{x}{3}\right)x^2\ddot{\alpha}+\mu\left(L-\frac{x}{2}\right)x\dot{x}\dot{\alpha}+G\left[L\ddot{\vartheta}+x\ddot{\alpha}+\dot{x}\dot{\alpha}\right]L \end{aligned} \tag{7}$$

For segment HB, we have

$$-Q_r=\mu\int_o^{L-x}\xi\ddot{\vartheta}\,d\xi=\frac{\mu}{2}(L-x)^2\ddot{\vartheta} \tag{8}$$

$$M_o-M_r=\mu\int_o^{L-x}\xi\ddot{\vartheta}\xi d\xi=\frac{\mu}{3}(L-x)^3\ddot{\vartheta} \tag{9}$$

Multiplying eqn. (6) with $-L$ then adding to (7) produces

$$M_o=\mu\left(\frac{L}{2}-\frac{x}{3}\right)x^2\ddot{\vartheta}+\frac{\mu}{6}x^3\ddot{\alpha}+\frac{\mu}{2}x^2\dot{x}\dot{\alpha} \tag{10}$$

Eqns. (6), (9) and (10) form the set of equations for unknowns x, $\dot{\alpha}$ and $\dot{\vartheta}$.
At $t=0$, all velocities are zero, the above equations become

$$F=\left[GL+\mu x\left(L-\frac{x}{2}\right)\right]\ddot{\vartheta}+\left(G+\frac{\mu}{2}x\right)x\ddot{\alpha} \tag{11}$$

$$M_o=\mu\left(\frac{L}{2}-\frac{x}{3}\right)x^2\ddot{\vartheta}+\frac{\mu}{6}x^3\ddot{\alpha} \tag{12}$$

$$M_o=\frac{\mu}{3}(L-x)^3\ddot{\vartheta} \tag{13}$$

Note that at early stages when the rotational spring at the root remains elastic, $M_r=K\vartheta$, thus at $t=0$, $M_r=0$ (with $\vartheta=0$). From eqns. (12) and (13), we have

$$\ddot{\vartheta}\Big|_{t=0}=\frac{3M_o}{\mu(L-x)^3} \tag{14}$$

$$\ddot{\alpha}\Big|_{t=0}=\frac{6}{\mu x^3}\left[M_o-\left(\frac{L}{2}-\frac{x}{3}\right)\frac{3M_o x^2}{(L-x)^3}\right] \tag{15}$$

Substituting the above into eqn. (11) yields the relationship between the magnitude of F and the initial hinge position x at $t=0$,

$$F=\left[GL+\mu x\left(L-\frac{x}{2}\right)\right]\frac{3M_o}{\mu(L-x)^3}+\left(G+\frac{\mu}{2}x\right)\frac{3M_o}{\mu x^2}\left[1-\left(\frac{3L}{2}-x\right)\frac{x^2}{(L-x)^3}\right] \tag{16}$$

It should be noted that $\ddot{\vartheta}|_{t=0}$, the angular acceleration at the beam root, is not zero at the first instant of the impact. This is an important distinction from a fully clamped rigid-perfectly plastic beam model in which the root remains undisturbed until the travelling hinge reaches there.

Putting $\ddot{\alpha}|_{t=0}=0$ gives the transition hinge position at which a hinge will appear in the beam (rather than at the root) with the least magnitude of the applied force F. Indicating the position by $\bar{x}$, from eqn. (15), we have

$$\frac{1}{3}(L-\bar{x})^3=\left(\frac{L}{2}-\frac{\bar{x}}{3}\right)\bar{x}^2, \text{ or } \bar{x}=\left[1\pm\frac{1}{\sqrt{3}}\right]L$$

Therefore, this transition position of the hinge is obtained as

$$\bar{x}=\left[1-\frac{1}{\sqrt{3}}\right]L\approx 0.423L \tag{17}$$

This is the furthest possible position an initial hinge may form in the beam from the tip (except at the root). The corresponding magnitude of force is

$$F_1=\left[GL+\mu\bar{x}\left(L-\frac{\bar{x}}{2}\right)\right]\frac{3M_o}{\mu(L-\bar{x})^3}$$

or

$$F_1=\left(G+\frac{\mu}{3}L\right)\frac{3\sqrt{3}}{\mu L}\frac{3M_o}{L} \tag{18}$$

This implies that when $F_o \leq F < F_1$, a hinge will appear at the beam root some time later after the force is applied, i.e., when the rotational spring has undergone a finite rotation up to its plastic state. With $F \geq F_1$, a hinge will immediately appear somewhere in the beam no more than $0.423L$ from the tip, and the root spring begins to undergo an elastic rotation. This is another distinction from the rigid-plastic analysis which shows that the initial hinge position can be anywhere in the beam depending on the magnitude of the dynamic load.

A rigid-perfectly plastic solution of a cantilever beam without root spring is seen in the Appendix which gives the force magnitude-hinge position relationship as

$$F'=\frac{6M_o}{\mu x^2}\left[G+\frac{\mu}{2}x\right] \tag{19}$$

For easy comparison, let the tip mass $G=0$, eqns. (16) and (19) become

$$F\big|_{G=0}=\frac{3M_o}{x}-\frac{3xM_o}{2(L-x)^2} \tag{16a}$$

and

$$F'\big|_{G=0}=\frac{3M_o}{x} \tag{19a}$$

These indicate that if a hinge is formed at the same position in the beam, the rigid-perfectly plastic model requires a higher magnitude of force than the beam-spring model does. Calculations also show that when a dynamic force F of the same magnitude is applied, the hinge in the spring-beam model will always appear at a position closer to the tip than in the other model.

With $G=0$, the minimum magnitude of force required to produce a hinge in the beam-spring model (not at the root) is, from eqn. (18),

$$F_1\big|_{G=0}=\frac{3\sqrt{3}M_o}{L}$$

and the hinge is at $0.423L$ from the tip.

For a rigid-perfectly plastic model, since the hinge may appear at any position, the minimum magnitude of force to produce a hinge in the beam is

$$F'\big|_{G=0}=\frac{3M_o}{L}$$

with the hinge appearing in the section adjoining the root, i.e. $x=L$.

Thus in the beam-spring model, the lowest magnitude of force required to produce an initial yield inside the beam is increased by a factor of $\sqrt{3}$ compared with the rigid-perfectly plastic analysis. And hinge positions are different.

If a hinge in the rigid-plastic model is also formed at x given by eqn. (17), the required force is

$$F'\big|_{G=0,\ x=x}=\frac{3M_o}{\left[L-\dfrac{L}{\sqrt{3}}\right]}$$

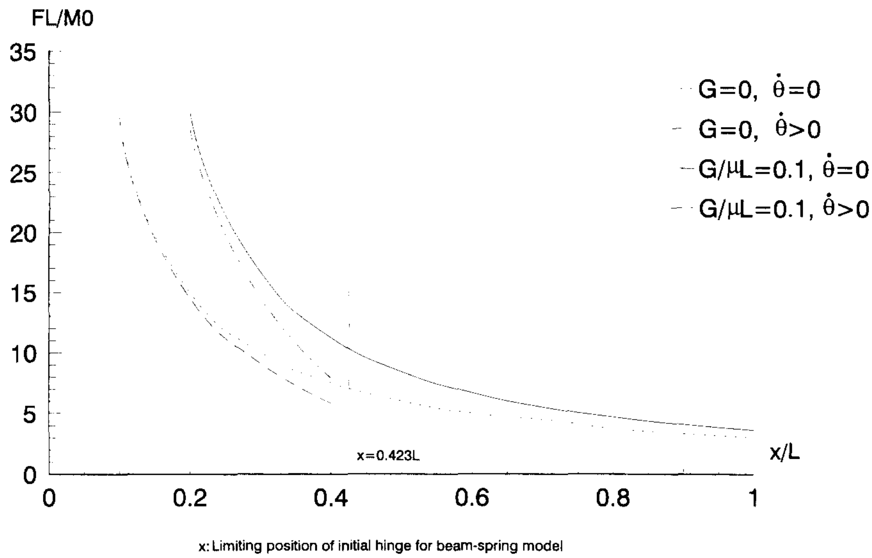

Fig. 4 Initial hinge position versus load magnitude for both beam-spring and rigid-plastic models.

That gives

$$\frac{F_1}{F'}\bigg|_{G=0,\ x=\bar{x}}=\sqrt{3}-1\approx 0.732$$

Hence, to produce a hinge at the same position, the spring-beam model needs a lower force than the rigid-plastic one does.

Fig. 4 shows the initial position of hinge versus magnitude of force for a beam with and without a tip mass. Results of the rigid-perfectly plastic model are also shown for the purpose of comparison. It is clearly illustrated that for the same value of the force, the beam-spring model always gives a hinge position closer to the beam tip, and this tendency is more obvious when the magnitude of the impact force is lower.

Eqns. (6), (9) and (10) can be solved with standard integration procedures, such as a fourth order Runge-Kutta approach. The initial value of x is obtained from eqn. (16). All velocities at the beginning are set to zero. Numerical results are discussed in the next section.

Due to the small deflection assumption, the tip deflection of the beam can be considered as

$$\delta_A \approx x\alpha + L\vartheta \tag{20}$$

At the end of the pulse, the total input energy of the system is

$$E = F\delta_{A\tau_1} \tag{21}$$

where τ_1 is the duration of the pulse, and

$$\delta_{A\tau_1} = x\alpha_1 + L\vartheta_1$$

α_1 and ϑ_1 being the values of rotation angles at the hinge and beam root respectively, at the moment when the force is removed.

At any time $t>0$, the kinetic energy of the system is

$$E_k=\frac{1}{2}G(x\dot{\alpha}+L\dot{\vartheta})^2+\frac{\mu}{2}\int_o^x\left[\zeta\dot{\alpha}+(L-x+\zeta)\dot{\vartheta}\right]^2 d\zeta+\frac{\mu}{2}\int_o^{L-x}(\xi\dot{\vartheta})^2 d\xi$$

$$=\frac{G}{2}(x\dot{\alpha}+L\dot{\vartheta})^2+\frac{\mu x^3}{3}\dot{\alpha}^2+\mu\left[L^2-Lx+\frac{x^2}{3}\right]x\dot{\vartheta}^2+\frac{\mu(L-x)^3}{6}\dot{\vartheta}^2+\frac{\mu x^2}{2}\left(L-\frac{x}{3}\right)\dot{\alpha}\dot{\vartheta} \tag{22}$$

The elastic deformation energy stored in the rotational spring at the beam root is

$$E_e=\frac{1}{2}K\vartheta_e^2 \tag{23}$$

The energy dissipated by plastic deformation including that at the travelling hinge and that due to plastic rotation at the root is

$$E_p=E-E_k-E_e \tag{24}$$

The plastic energy dissipated by the root spring when it is in a plastic state is

$$E_{pr}=M_o\int_o^t|\dot{\vartheta}_p|dt \tag{25}$$

And the plastic energy dissipated by the hinge is

$$E_{ph}=E_p-E_{pr} \tag{26}$$

3. Numerical solutions

In all numerical examples, the structural parameters of the cantilever are set at $M_o=350.0\,Nm$, $G=0.3$ kg, $\mu=0.971$ kg/m, $L=2.0\,m$, $K=774.0\,Nm/rad$, this corresponds to a 25.4 mm bore mild steel tubular beam with wall thickness 1.5 mm, and the maximum elastic energy able to be stored in the system is $E_{e,\ max}\approx 79.1$ J, which corresponds to a maximum elastic rotation of the spring of 0.452 rad, or 25.9°. Different combinations of force magnitude and pulse duration were considered. Numerical results reveal that when the elastic deformation is introduced by a rotational spring at the beam root, the dynamic behavior of the system can be significantly different from that given by a rigid-perfectly plastic approach.

Calculations show that the development of hinge velocity is slow and if the duration of a pulse loading is short, say 10 ms, the distance the hinge travels during the pulse can virtually be neglected, therefore the hinge can be treated as a stationary one. This means that elasticity of the system does not influence the hinge velocity development much. In fact the influence of the elasticity remains mainly in deciding the initial hinge position through the change of bending moment distribution in the beam. But for a long pulse ($\tau_1>30\,ms$), the hinge does move and its velocity developed during the pulse cannot be neglected. At the end of the pulse, the hinge starts to travel towards the root at a much faster speed. Depending on the input energy, the hinge may or may not reach the root. In total, there exists three possible response modes after the dynamic force is removed.

Mode I. A travelling hinge moves towards the beam root, but before arriving there, the rotational spring at the root enters its plastic state and becomes another hinge. The indicates that two hinges appear in the beam simultaneously and in this case, the travelling hinge always reaches the root and the remaining kinetic energy is dissipated merely in the fixed hinge at the root.

Mode II. A travelling hinge moves towards the root, at a certain internal point in the beam, the hinge becomes inactive, and during which time the root spring remains elastic. After that, depending on the amount of remaining kinetic energy, the root spring may or may not enter a plastic state, as denoted by Mode II_a and II_b, respectively.

Mode III. A travelling hinge moves towards the root and reaches there, before that the root spring always remains elastic. This is similar to the solution of a rigid-perfectly plastic model.

The following examples show the above different response modes under different level of input.

[CASE 1] $F=1500\,N$ $\tau_1=8\,ms$

A hinge is formed at $\lambda_o=0.815\,m$ from the tip when the force is applied. During the pulse the hinge travels a distance of $5\,mm$. After the pulse, the hinge quickly moves towards root B and then becomes inactive at $\lambda_{TD}=1.11\,m$ at $t_{TD}=9\,ms$. Rotation at the root spring is only 0.027 (0.55°) at this moment. After that, the spring never enters its plastic state.

Note that subscript *TD* means "**T**ravelling hinge **D**isappearing", *TR* means "**T**ravelling hinge arriving the **R**oot", and *RP* and *RD* indicate "**R**oot spring entering its **P**lastic state" and "**R**oot hinge **D**isappearing", respectively.

[CASE 2] $F=3000\,N$ $\tau_1=10\,ms$

Initially the hinge appears at $\lambda_o=0.606\,m$, and at the end of the pulse, it has travelled a distance of $1\,mm$. Then it disappears at $\lambda_{TD}=1.19\,m$ at $t_{TD}=24.3\,ms$. After that the rotational spring becomes plastic at the time $t_{RP}=45\,ms$. The remaining kinetic energy is dissipated by the root hinge until $t_{RD}=200\,ms$ when the root hinge disappears and the spring returns to its elastic state.

[CASE 3] $F=1500\,N$ $\tau_1=30\,ms$

This is a relatively long pulse loading. A hinge forms at $\lambda_o=0.815\,m$ and moves to $\lambda_{\tau_1}=0.893\,m$ when the force drops to zero. It is very clear that the hinge is moving during the pulse. At the moment $t=\tau_1$, rotation at the hinge and the rotational spring are 0.281 and 0.270, respectively. The hinge reaches the root at $t_{TR}=40\,ms$ and the spring becomes a hinge. The spring returns to elastic at $t_{RD}=277\,ms$.

For all the above three cases, the rotational spring remains elastic during the beam deformation process and only after the beam ceases to deform, may the spring enter its plastic state, or in the case of low input energy, may not. The next case shows that if the input energy is high enough, there could be a double-hinge mechanism.

[CASE 4] $F=1500\,N$ $\tau_1=50\,ms$

Again a hinge is formed at $\lambda_o=0.815\,m$, it has reached $\lambda_{\tau_1}=1.01\,m$ when the force is removed. At this moment, rotation at the hinge and the root are 0.993 and 0.626, respectively. The hinge reaches the root at $t_{TR}=85\,ms$, and before that, at $t_{RP}=40.5\,ms$, the rotational spring at the root enters its plastic state, thus there are two plastic hinges appearing simultaneously in the system from $40.5\,ms$ to $85.0\,ms$. After the travelling hinge arrival at the root, there exists only one hinge at the root in the whole system until an elastic state is regained at $t_{RD}=281\,ms$.

Let p_1 be the proportion of the total plastic energy dissipated to the input energy, p_2 be the proportion of the plastic energy dissipated by the root hinge to the total input energy, p_3 be the plastic energy dissipated by the hinge in the beam to the input energy, (obviously, $p_1=p_2+p_3$).

If β is defined as the ratio of the elastic energy stored in the rotational spring to the input energy, the following table gives a summary of the above four cases.

It shows that CASEs 1 and 2 correspond respectively to the Modes II_b and II_a, with the value of β not too small. CASE 3 has a travelling hinge moving towards the root, and only at the arrival of the hinge at the root does the spring becomes plastic. This is Mode III and similar to the known result from the commonly adopted rigid-perfectly plastic analysis. CASE 4 is Mode I, with a small value of β, i.e. a lower capability to store elastic energy, there appears a double-hinge mechanism.

Figs. 5-8 show the hinge positions, velocities, forces and bending moments at the beam root for CASEs 2 and 3. Values of p_1, p_2 and p_3 are shown in Figs. 9 and 10 for the above two cases.

Table 1 Summary of CASE 1 to 4

	F N	τ_1 ms	λ_o m	$\lambda_{\tau 1}$ m	λ_{TD} m	t_{TD} ms	t_{RP} ms	t_{RD} ms	P_2 %	P_3 %	β %	E J
1	1500	8	.815	.820	1.11	9	/	/	0	9.49	90.51	83.91
2	3000	10	.606	.607	1.19	24	45	200	60.88	29.57	9.55	828.48
3	1500	30	.815	.893	2.0	40	40	277	82.32	11.01	6.67	1185.98
4	1500	50	.815	1.01	2.0	85	41	281	77.14	20.51	2.35	3367.50

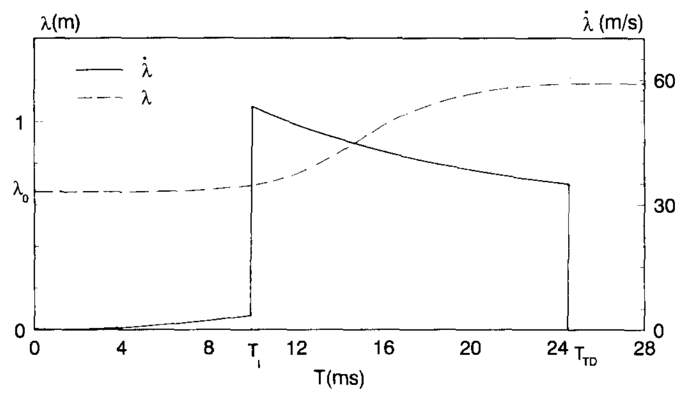

Fig. 5 History of hinge position and velocity for Case 2.

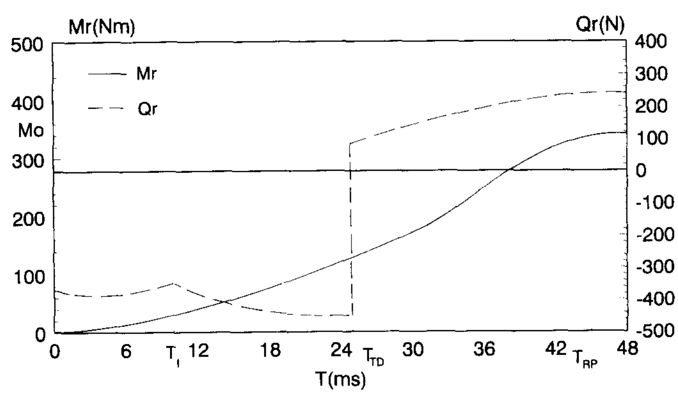

Fig. 6 History of bending moment and force at the root for Case 2.

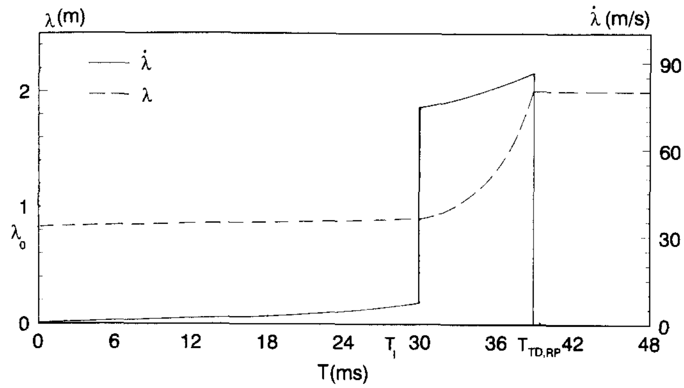

Fig. 7 History of hinge position and velocity for Case 3.

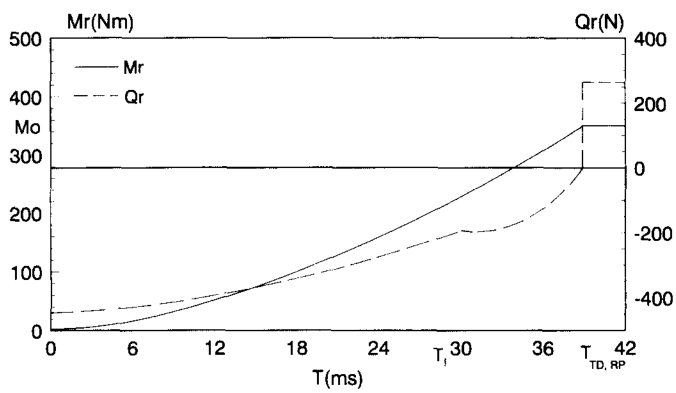

Fig. 8 History of bending moment and force at the root for Case 3.

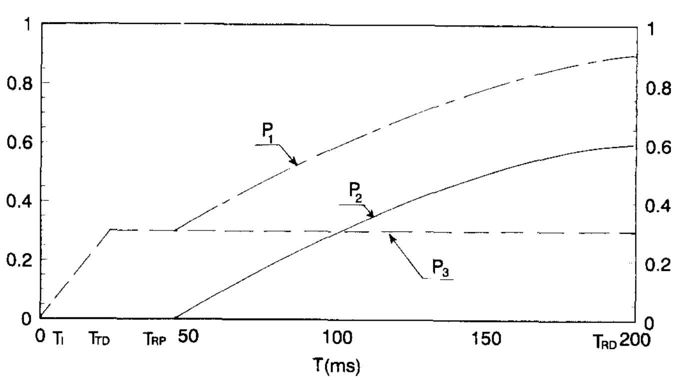

Fig. 9 Values of P_1, P_2 and P_3 for Case 2.

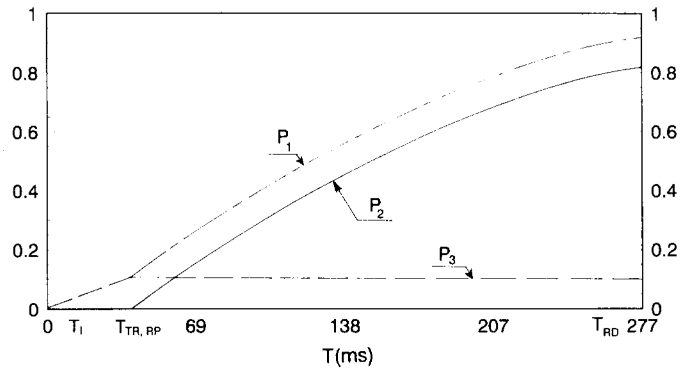

Fig. 10 Values of P_1, P_2 and P_3 for Case 3.

4. Summary

A structural model comprised of a simple spring-beam system has been introduced to investigate the effect of elastic deformation on the response of a straight beam subjected to a dynamic force load of finite duration at its tip. The introduction of the rotational spring at the beam root means that the system has a certain capacity to store elastic energy, which is ignored in the other rigid-perfectly plastic approach. Numerical results confirm that the capacity to store elastic deformation energy has a significant effect on the response mode of the system. Some interesting results which are different from the rigid-plastic solution have been obtained.

1) The present structural model has three different response modes, instead of a single, unique one as in the rigid-perfectly plastic analysis. These are: (a) for low levels of input energy, the plastic hinge travels through only part of the beam, and the root spring may never enters its plastic state and the following response remains elastic, or after the travelling hinge disappears inside the beam, the root spring eventually becomes plastic and the motion finishes with a root hinge rotation; (b) for moderate levels of input energy the travelling hinge will go through the beam and reach the root and then become a root hinge; (c) for high levels of input energy, while the hinge is still moving along in the beam the root spring enters its plastic state, this implies a double-hinge mechanism and in this case the travelling hinge always reaches the root.

2) The existence of different modes indicates that a single plastic hinge travelling through the beam to the root is not the unique response mode. The root of the beam can enter its plastic state before the travelling hinge arrives there, so that a root hinge and a travelling hinge appear simultaneously. Also, after the travelling hinge disappears, the root of the beam could enter its plastic state and create a new hinge there. It is interesting to recall that in the elastoplastic finite element solution given by Reid and Gui (1987), the plastic zone at the root also initiates when a plastic region is travelling in the middle part of the beam for Parkes' problem.

3) The present results also indicate that the elastic effect of the structure has a significant influence on the initial position of the hinge. Because of a different distribution of bending moment in the beam, the hinge will appear closer to the tip, compared with its position in a rigid-plastic model. This provides some guidances about the location of the initial hinges expected in experimental tests.

4) In a rigid plastic analysis, the plastic hinge may form at any position in the beam depending on the magnitude of the force, but in the beam-spring model, the hinge may only form in

a certain part of the beam and the minimum magnitude of the dynamic force to produce a yield is increased by a factor of $\sqrt{3}$, compared to a rigid plastic model. However, if the hinge is formed at the same position, the rigid plastic model requires higher magnitude of force.

5) It seems that the elasticity in the structure does not have a strong effect on the development of a travelling hinge in a small deflection analysis where a short pulse load is applied, it is found that the hinge moves only very slightly and the influence of elastic deformation is trivial. However, the influence may become prominent if a long pulse is applied and large deflection is allowed.

6) It is noticed that the energy that can be stored elastically in the system has an important effect on the response mode, as the so-called high or low level of the input energy is always with respect to the beam's capability to store elastic energy.

On the whole, although the beam-spring model is a simple approximation, the small deflection analysis does show that elasticity makes structures respond differently, compared with the rigid-plastic approaches. However an analysis which enables both large deflection and elasticity to be simulated at the same time is desirable as it will offer a better comparison with the knowledge obtained from the laboratory tests which, in most circumstance, involve large deflections.

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Notations

EI	flexural rigidity of the beam
E	energy
F	magnitude of applied force at the tip of the beam
G	concentrated mass at the tip of the beam
K	elastic constant of the rotational spring at the beam end
L	length of the beam
M	bending moment
M_o	dynamic fully plastic bending moment of the beam
P_1	ratio of the total plastic energy dissipated by all plastic hinges to the total input energy
P_2	ratio of the plastic energy dissipated by root spring to the total input energy
P_3	ratio of the plastic energy dissipated by the travelling hinge to the total input energy
Q	supporting force at beam root
t	time
x	hinge position
μ	mass of the beam per unit length
$\dot{\alpha}$	angular velocity of the hinge on the beam
$\dot{\vartheta}$	angular velocity of the spring at the root of the beam
δ_A	deflection of the beam tip
β	ratio of the maximum elastic energy stored in the spring at the root of the beam to the total input energy
λ	hinge position in the beam
τ	duration of pulse load

Subscripts

e	elastic values
h	values at the current hinge
k	values of kinetic energy
p	plastic values
r	values at the root of the beam
RP	values at the moment when the root enters its plastic state
TD	values at the moment when the travelling hinge disappears
TR	values at the moment when the travelling hinge reaches the root
RD	values at the moment when the root hinge disappears

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Appendix

Rigid perfectly plastic analysis of a cantilever beam subjected to a transverse pulse load at its tip

Suppose that a cantilever beam as shown in Fig. 1 but without a rotational spring at its root is subjected to a variable dynamic tip force load $P(t)$, if the magnitude of the load is high enough, a plastic hinge may form at an interior point H, measured λ form the beam tip, the equations of motion of the deformed segment are

$$\left[G+\frac{\mu\lambda}{2}\right]\lambda\ddot{\alpha}+(G+\mu\lambda)\,\dot{\lambda}\dot{\alpha}=P(t) \tag{A1}$$

$$\left[G+\frac{\mu\lambda}{2}\right]\lambda^2\ddot{\alpha}+\left[G+\frac{\mu\lambda}{2}\right]\lambda\,\dot{\lambda}\dot{\alpha}=P(t)\lambda-M_o \tag{A2}$$

$\dot{\alpha}$ being the angular velocity at the plastic hinge. Eqn. (A1) can be written as

$$\frac{d}{dx}\left[\left[G\lambda+\frac{\mu\lambda^2}{2}\right]\dot{\alpha}\right]=P(t)$$

or

$$\dot{\alpha}=\frac{\int_o^t P(t)dt}{\left[G+\frac{\mu\lambda}{2}\right]\lambda} \tag{A3}$$

Eliminating $P(t)$ from eqns. (A1) and (A2) produces

$$\frac{\mu\lambda^3}{6}\ddot{\alpha}+\frac{\mu\lambda^2\dot{\lambda}}{2}\dot{\alpha}=M_o$$

which gives

$$\dot{\alpha}=\frac{6M_o t}{\mu\lambda^3} \tag{A4}$$

With (A3) and (A4), we have

$$\frac{\int_o^t P(t)dt}{\left[G+\frac{\mu\lambda}{2}\right]\lambda}=\frac{6M_o t}{\mu\lambda^3}$$

Solution of the above equation gives the required expression of the hinge position,

$$\lambda=\frac{3M_o\mu t+\sqrt{9M_o^2\mu^2t^2+24M_o\mu Gt\int_o^t P(t)dt}}{2\mu\int_o^t P(t)dt} \tag{A5}$$

If $P(t)$ is a constant pulse of finite duration, $P(t)=\begin{cases} F & 0\leq t\leq\tau \\ 0 & t>\tau \end{cases}$

for $0\leq t\leq\tau$, eqn. (A5) gives

$$\lambda=\frac{3M_o\mu+\sqrt{9M_o^2\mu^2+24M_o\mu GF}}{2\mu F} \tag{A6}$$

This indicates a stationary hinge during the rectangular pulse. It can also be written as,

$$F=\frac{3(2M_oG+\mu M_o\lambda)}{\mu\lambda^2} \tag{A7}$$

Thus the minimum magnitude of the dynamic load required to produce a hinge in the beam is $F'=F(L)$ with the hinge appearing at the root. For $F>F'$, a hinge will form at a position decided by (A6).

For $t>\tau$, we have

$$\lambda=\frac{3M_o\mu t+\sqrt{9M_o^2\mu^2t^2+24M_o\mu GF\tau t}}{2\mu F\tau} \tag{A8}$$

clearly, the hinge will move towards the beam root when the pulse is removed. If there is no tip mass, $G=0$, the hinge moves at a constant velocity.

The time at which the hinge arrives at the root is obtained from (A8),

$$t_r=\frac{F\tau\mu L^2}{3(2G+\mu L)M_o}$$

and the angular velocity of the hinge at this moment is

$$\dot{\alpha}_r=\frac{6M_o t_r}{\mu L^3}=\frac{2F\tau}{(2G+\mu L)L}$$

After the hinge arrives at the root, it remains as a fixed hinge there and the equation of motion becomes,

$$\left[G+\frac{\mu L}{2}\right]L\ddot{\alpha}=-Q_r \tag{A9}$$

where Q_r is the transverse force at the root. From (A2) we have

$$\ddot{\alpha}_r|_{t\geq t_r}=-\frac{3M_o}{(3G+\mu L)^2}$$

The beam will stop after all the remaining kinetic energy is dissipated at the root hinge. And during the root rotation, the transverse force is

$$Q_r=\frac{3M_o(2G+\mu L)}{2L(3G+\mu L)}$$

If E_I, E_{II} and E_{III} denoting the ratio of participation of plastic work to the total input energy during the initial stationary hinge phase; the travelling hinge phase and the final root rotation phase, respectively, it is not difficult to prove that with $G=0$ and under a rectangular pulse, we will have

$$E_I:E_{II}:E_{III}=\frac{1}{3}:\left[\frac{2}{3}-\frac{2M_o}{PL}\right]:\frac{2M_o}{PL}$$

A similar discussion was also given by Reid, et al. (1990).