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Buckling of post-tensioned composite beams

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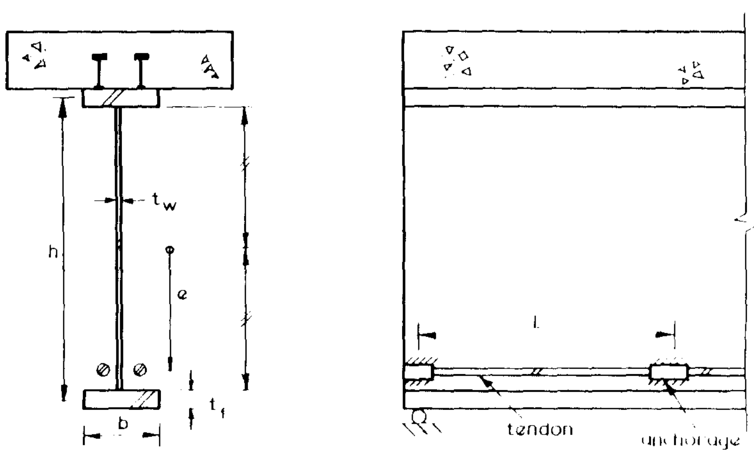
Abstract. A method for computing the elastic buckling prestressing force of a post-tensioned composite steel-concrete tee-beam is presented. The method is based on a virtual work formulation, and incorporates the restraint provided by the concrete slab to the buckling displacements of the steel beam. The distortional buckling solutions are shown to be given by a quadratic equation. The application of the analysis to calculating buckling strengths is given, based on codified rules for beam-columns. Conclusions are then drawn on the importance of distortional buckling when a post-tensioned composite beam is stressed during jacking.

Key words: buckling; composite beams; distortion; post-tensioning; prestressing.

1. Introduction

Existing composite steel-concrete tee-beams may have their strength and stiffness enhanced by the provision of a prestressing cable close to the bottom flange, as shown in Fig. 1. Such structural systems, particularly for bridges, have become quite popular in the U.S. and in Eastern Europe.



(a) Cross-section (b) Part elevation

Fig. 1 Prestressed Composite Beam

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If the limit state under sagging bending is first yield of the bottom flange, the introduction of a prestressing force delays the onset of plastification. The additional load that may be carried has been quantified by Bradford (1991a). Performing the stressing operation in the absence of live loading may induce large compressive stresses in the bottom flange of the steel beam, so that stability considerations are of consequence.

Since the concrete slab restrains the top flange of the steel, lateral deflection and twist of the bottom flange is prevented only by the stiffness of the web. Because of this, the buckling mode must necessarily be lateral–distortional (Bradford and Trahair 1981), in which the web distorts in cross-section. The possibility of instability of the bottom flange, as depicted in Fig. 2, was alluded to in an extensive study by the ASCE (1968) and in a recent book by Troitsky (1990). In both cases, it was suggested that recourse be made to the approximate methods of Bleich (1952).

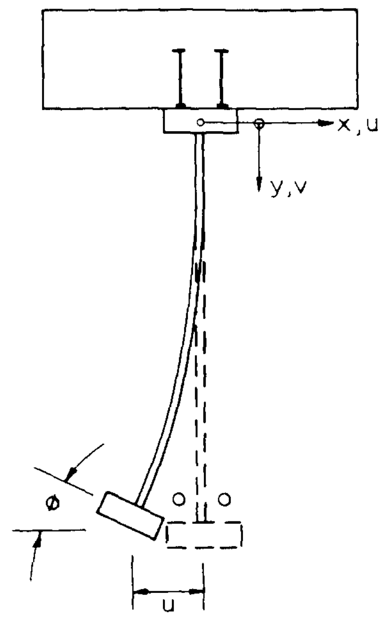

Fig. 2 Lateral–Distortional Buckle

Although studies of lateral–distortional buckling have been relatively extensive (Bradford 1992), the problem of instability under prestress does not appear to have been fully researched, apart from recent approximate studies by the author (Bradford 1991a, b). This paper presents an accurate analysis of the elastic distortional buckling of a composite tee-beam during jacking caused by a tendon of constant eccentricity. Conservatively, dead load is ignored, and all stress in the steel section is assumed to be caused by the prestressing force P. The elastic buckling load is used to calculate a strength, and recommendations are made regarding the influence of post-tensioning on the stability of the bottom flange of the steel beam.

2. Buckling theory

2.1. Displacements

Figure 2 shows the buckling mode for the steel beam under prestress. The shear connection restrains the top flange from deflection and twist, while the bottom flange, because of its relative stockiness, is assumed to twist and displace as a rigid body. Moreover, the flexural displacements of the thin web are assumed to follow a cubic curve in cross-section.

Since the beam is subjected to uniform bending and compression during prestress, its buckling eigenmode is a sine curve (Timoshenko and Gere 1961). Hence the displacements u, ϕ of the bottom flange–web junction can be written in terms of the degrees of freedom $\langle q_1, q_2\rangle^T$ as

$$\begin{Bmatrix} u \\ \phi \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \sin\frac{n\pi z}{L} \tag{1}$$

where L is the length between attachment of the tendons and n is a positive integer representing the harmonic number.

The cubic variation of web flexural displacements u_w which satisfies the end conditions of restraint at $y=0$ is

$$u_w = \langle \eta^2, \eta^3 \rangle \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \sin\frac{n\pi z}{L} \tag{2}$$

where

$$\eta = \frac{y}{h} \tag{3}$$

and $\langle \alpha_1, \alpha_2\rangle^T$ are polynomial coefficients.

By noting that $u_w = u$ and $\partial u_w/\partial y = -\phi$ at $\eta = 1$, the polynomial coefficients may be expressed as

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} 3 & h \\ -2 & -h \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \tag{4}$$

or

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = [C] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \tag{5}$$

2.2. Virtual work formulation

Invoking the principle of virtual work (Hall and Kabaila 1986) produces

$$\delta U_F + \delta U_T + \delta U_W - \delta W = 0 \tag{6}$$

where U_F is the strain energy stored due to flange flexure; U_T is the strain energy stored due to flange twist; U_w is the strain energy stored due to web plate flexure and twist; and W is the work done by the stresses induced by prestress during buckling. Expressions for the variations of these quantities with respect to the degrees of freedom are derived subsequently.

2.3. Strain energies

The term U_F in Eq. 6 may be written as

$$U_F = \frac{1}{2} EI_f \int_0^L (u'')^2 \, dz \tag{7}$$

where primes denote differentiation with respect to z, and where

$$EI_f = \frac{Eb^3 t_f}{12} \tag{8}$$

Substituting Eq. 1 into Eq. 7 produces

$$\delta U_F = \frac{1}{2} \{\delta q\}^T \begin{bmatrix} \dfrac{n^4 \pi^4 EI_f}{2L^3} & 0 \\ 0 & 0 \end{bmatrix} \{q\} \tag{9}$$

where $\{q\} = \langle q_1, q_2 \rangle^T$

Similarly, the term U_T in Eq. 6 may be expressed as

$$U_T = \frac{1}{2} GJ_f \int_0^L (\phi')^2 \, dz \tag{10}$$

where

$$GJ_f = \frac{Gbt_f^3}{3} \tag{11}$$

and G = the shear modulus of elasticity. Hence substituting Eq. 1 into Eq. 10 gives

$$\delta U_T = \frac{1}{2} \{\delta q\}^T \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{n^2 \pi^2 GJ_f}{2L} \end{bmatrix} \{q\} \tag{12}$$

Finally, the term U_w in Eq. 6 may be written as (Bradford 1988)

$$U_w = \frac{1}{2} \int_0^L \int_0^h \{\sigma^*\}^T \quad \{\varepsilon\} \, dy \, dz \tag{13}$$

where $\{\varepsilon\}$ = the vector of infinitesimal buckling strains given by

$$\{\varepsilon\} = \left(\frac{\partial^2 u_w}{\partial y^2}, \frac{\partial^2 u_w}{\partial z^2}, -2 \frac{\partial^2 u_w}{\partial y \partial z} \right)^T \tag{14}$$

and $\{\sigma^*\}$ is the vector of infinitesimal buckling stresses given by

$$\{\sigma^*\} = [D] \{\varepsilon\}$$

where the plane stress–bending property matrix $[D]$ is (Cheung 1976)

$$[D] = \frac{Et^3_w}{12(1-v^2)}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \tag{16}$$

Hence substituting Eqs. 2, 14, 15 and 16 into Eq. 13 produces

$$\delta U_w = \frac{1}{2}\frac{Et^3}{12(1-v^2)}\{\delta q\}^T[C]^T\begin{bmatrix} k_{w11} & k_{w12} \\ k_{w21} & k_{w22} \end{bmatrix}[C]\{q\} \tag{17}$$

where

$$k_{w11} = \frac{2L}{h^3} + \frac{2vn^2\pi^2}{3} + \frac{n^4\pi^4 h}{10L^3} + \frac{4n^2\pi^2(1-v)}{3hL} \tag{18}$$

$$k_{w12} = k_{w21} = \frac{12L}{h^3} + \frac{vn^2\pi^2}{hL} + \frac{n^4\pi^4 h}{12L^3} + \frac{3n^2\pi^2(1-v)}{2hL} \tag{19}$$

$$k_{w22} = \frac{6L}{h^3} + \frac{6vn^2\pi^2}{5hL} + \frac{n^4\pi^4 h}{14L^3} + \frac{9n^2\pi^2(1-v)}{5hL} \tag{20}$$

2.4. Work done

The term W in Eq. 6 can be written as (Bradford 1988)

$$\delta W = \frac{1}{2}\int_A \sigma \int_0^L \left(\frac{\partial\delta u_w}{\partial z}\right)\left(\frac{\partial u_w}{\partial z}\right) + \left(\frac{\partial\delta u}{\partial z}\right)\left(\frac{\partial u}{\partial z}\right) + \left(\frac{\partial\delta v}{\partial z}\right)\left(\frac{\partial v}{\partial z}\right)\, dz dA \tag{21}$$

where A is the area of the steel beam and the flange vertical displacement v is

$$v = x\phi \tag{22}$$

In calculating the stress σ, it is assumed that the eccentric prestress causes tensile stresses in the concrete, so that the effect of the latter in carrying load may be ignored. By denoting σ_T as the stress in the top flange of the steel, and by treating compressive stresses as positive, the concrete slab will be subjected to tension when

$$\sigma_T = \frac{P}{A} - \frac{Peh}{2I_x} < 0 \tag{23}$$

producing the condition

$$e \geq \frac{2I_x}{Ah} \tag{24}$$

where A is the area of the steel beam and I_x is its second moment of area about the beam's centroid. Equation 24 is not particularly restrictive on e, as the prestressing strand tends to be placed towards the bottom flange of the steel beam.

The second term in the integrand in Eq. 21 becomes

$$\frac{1}{2}\int_A \sigma \int_0^L \left(\frac{\partial \delta u}{\partial z}\right)\left(\frac{\partial u}{\partial z}\right) dz\, dA = \frac{1}{2}\{\delta q\}^T \begin{bmatrix} \frac{n^2\pi^2 b t_f \sigma_B}{2L} & 0 \\ 0 & 0 \end{bmatrix} \{q\} \tag{25}$$

while similarly the third term becomes

$$\frac{1}{2}\int_A \sigma \int_0^L \left(\frac{\partial \delta v}{\partial z}\right)\left(\frac{\partial v}{\partial z}\right) dz\, dA = \frac{1}{2}\{\delta q\}^T \begin{bmatrix} 0 & 0 \\ 0 & \frac{n^2\pi^2 b t_f^3 \sigma_B}{24L} \end{bmatrix} \{q\} \tag{26}$$

where

$$\sigma_B = \frac{P}{A} + \frac{Peh}{2I_x} \tag{27}$$

Finally, nothing that σ in the web is given by

$$\sigma = \sigma_T + \eta(\sigma_B - \sigma_T) \tag{28}$$

the first term in Eq. 21 becomes

$$\frac{1}{2}\int_A \sigma \int_0^L \left(\frac{\partial \delta u_w}{\partial z}\right)\left(\frac{\partial u_w}{\partial z}\right) = \frac{1}{2}\{q\}^T [C]^T \begin{bmatrix} g_{w11} & g_{w12} \\ g_{w21} & g_{w22} \end{bmatrix} [C]\{q\} \tag{29}$$

where

$$g_{w11} = \frac{n^2\pi^2 t_w h}{2L}\left(\frac{\sigma_T}{30} + \frac{\sigma_B}{6}\right) \tag{30}$$

$$g_{w12} = g_{w21} = \frac{n^2\pi^2 t_w h}{2L}\left(\frac{\sigma_T}{42} + \frac{\sigma_B}{7}\right) \tag{31}$$

$$g_{w22} = \frac{n^2\pi^2 t_w h}{2L}\left(\frac{\sigma_T}{56} + \frac{\sigma_B}{8}\right) \tag{32}$$

2.5. Buckling solution

By noting that σ_T and σ_B in Eqs. 25, 26, and 30 to 32 are proportional to P, the virtual work equation Eq. 6 becomes

$$\{\delta q\}^T ([k] - P[g])\{q\} = 0 \tag{33}$$

where $[k]$ and $[g]$ are the stiffness and stability matrices of the beam respectively. Since the variations $\{\delta q\}$ in Eq. 33 are arbitrary,

$$([k] - P[g])\{q\} = \{0\} \tag{34}$$

so that for nontrivial buckling displacements $\{q\}$,

$$|[k] - P[g]| = 0 \tag{35}$$

Since $[k]$ and $[g]$ in Eq. 35 are of order two, the latter eigenproblem reduces to a quadratic in P. Although it is theoretically possible to obtain explicit expressions for k_{ij} and g_{ij} in Eq. 35, the pre-and post-multiplication by $[C]$ in Eqs. 17 and 29 is cumbersome, and was performed by computer.

Once the terms in $[k]$ and $[g]$ have been assembled, Eq. 35 reduces to

$$P^2(g_{11}\,g_{22}-g_{12}^2)+P(2k_{12}g_{12}-k_{11}g_{22}-k_{22}g_{11})+(k_{11}k_{22}-k_{12}^2)=0 \tag{36}$$

The quadratic in Eq. 36 was solved on a microcomputer to obtain the elastic critical prestressing force P_{cr}.

3. Design for strength

The Australian limit states structures code AS4100 (SA 1990) provides a useful means by which the strength of a composite beam under prestress may be obtained. According to this code, the limit state of out-of-plane buckling in the absence of load and capacity reduction factors is

$$M \leq M_{bx}\left(1-\frac{N}{N_c}\right) \tag{37}$$

where M is the applied moment, amplified according to the "P–δ effect"; N is the applied axial force; M_{bx} is the bending strength incorporating out-of-plane buckling and N_c is the column strength about the minor axis. It has been shown (Bradford 1991a) that the P–δ effect is small for typical composite beams so that M can be taken as Pe and N as P, where P is the applied prestressing force.

If the composite beam buckles elastically at P_{cr} which is the lowest root of Eq. 36, then according to the AS4100 and the method of "design by buckling analysis" (Trahair and Bradford 1988)

$$M_{bx}=0.6\left\{\left[\left(\frac{M_p}{P_{cr}\,e}\right)^2+3\right]^{\frac{1}{2}}-\frac{M_p}{P_{cr}\,e}\right\}M_p \tag{38}$$

and

$$N_c=\left\{\frac{1+(1+\eta^*)\,P_{cr}\,/\,N_Y}{2}\left(\left(\frac{1+(1+\eta^*)\,P_{cr}\,/\,N_Y}{2}\right)^2-\frac{P_{cr}}{N_Y}\right)^{\frac{1}{2}}\right\}N_Y \tag{39}$$

where the imperfection parameter η^* is

$$\eta^*=0.00326\left(\pi\sqrt{\frac{AE}{P_{cr}}}-13.5\right)\geq 0 \tag{40}$$

in which M_p is the plastic moment and N_Y is the squash load of the steel beam.

If λ represents the strength load factor under a given prestressing force P, then substituting Eqs. 38 to 40 into Eq. 37 produces

$$\lambda Pe = M_{bx}\left(1-\frac{\lambda P}{N_c}\right) \tag{41}$$

so that the strength load factor can be obtained from

$$\lambda = \frac{M_{bx}}{P(e + M_{bx}/N_c)} \tag{42}$$

Of course, the elastic critical prestressing force P_{cr} obtained from Eq. 36 must be used to calculate M_{bx} and N_c.

4. Elastic buckling loads

The elastic lateral–torsional buckling load P_{oc} of the prestressed beam treated as a beam–column is the root of the equation (Trahair and Bradford 1988)

$$\frac{P_{oc}^2 e^2}{r_o^2 N_y N_z} = \left(1 - \frac{P_{oc}}{N_y}\right)\left(1 - \frac{P_{oc}}{N_z}\right) \tag{43}$$

where

$$r_o^2 = \frac{I_y + I_x}{A} \tag{44}$$

$$N_y = \frac{\pi^2 E I_y}{L^2} \tag{45}$$

is the Euler buckling load and

$$N_z = \left(\frac{GJ}{r_o^2}\right)\left(1 + \frac{\pi^2 E I_w}{GJL^2}\right) \tag{46}$$

is the torsional buckling load, in which the warping constant I_w is

$$I_w = \frac{I_y h^2}{4} \tag{47}$$

Equation 36 was solved for the elastic critical prestressing force P_{cr} of the beams, and the normalised buckling loads P_{cr}/P_{oc} are plotted in Figs. 3 and 4 as a function of the dimensionless prestress eccentricity e/h. In these figures, the lowest harmonic ($n = 1$) was used. It can be seen from Fig. 3 that for the more stocky flange ($b/h = 0.1$) the distortional buckling load is approximately twice that considering the steel as an eccentrically loaded beam–column. This is because the unrestrained beam–column is allowed to twist during buckling, while the composite beam has the twist of its bottom flange restrained only by the flexibility of the web. For the more slender flange in Fig. 4 ($b/h = 0.2$), the value of P_{cr} is approximately equal to P_{oc}, indicating that web distortion is less significant.

5. Buckling strength

As an example of application of the buckling solution, a composite beam was considered to be prestressed with two tendons at an eccentricity e/h of 0.4. The stress–strain curve for the

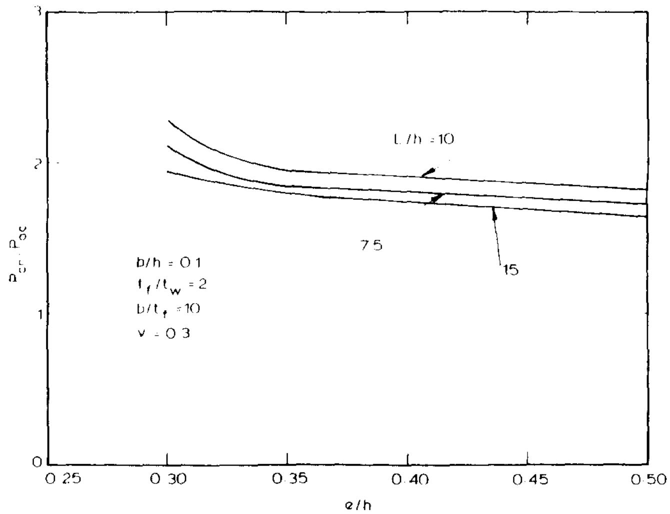

Fig. 3 Buckling Curves for b/h = 0.1

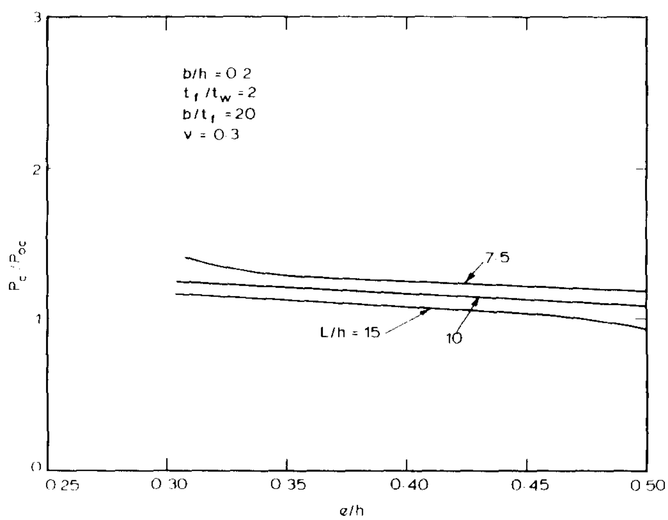

Fig. 4 Buckling Curves for b/h = 0.2

typical 7–wire prestressing strand used is taken from Gilbert and Mickleborough (1990) and shown in Fig. 5. The area of each strand is $A_p = 100mm^2$.

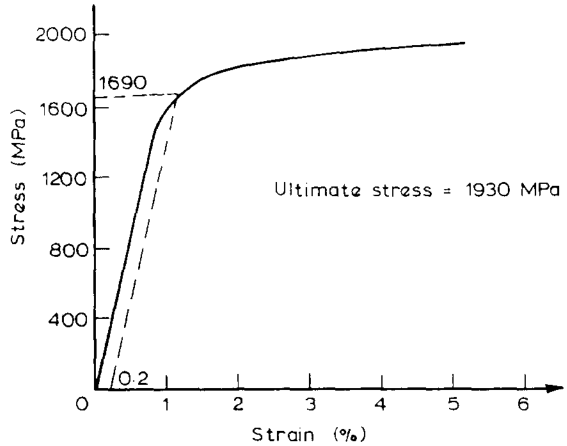

Fig. 5 Typical Stress–Strain Curve for 7–wire Prestressing Strand(after Gilbert and Mickleborough 1990)

Figure 6 shows the strength load factor λ, obtained from Eq. 42, as a function of the stress in the prestressing strand for two different beam geometries. Curves are given for an unrestrained length ratio L/h of 5, 10 and 15. It can be seen that the strength load factor λ drops well below unity when $L/h = 15$, except for very low prestressing stresses. In order to avoid an instability failure, it is thus necessary to place fairly stringent restrictions on the attachment of the tendon to the steel section when the tendon is stressed up to the order of half of its proof stress.

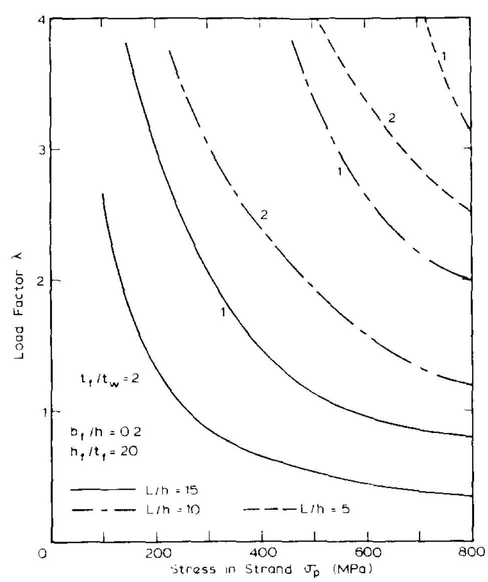

Fig. 6 Buckling Load Factors

6. Conclusions

Based on the principle of virtual work, a quadratic equation was derived to calculate the elastic critical prestressing force to cause instability of a post-tensioned composite beam. Although the solution may be presented explicitly, a computer was deployed to perform the cumbersome numerical manipulations.

The method by which the procedure of "design by buckling analysis" may be used to produce a buckling strength from the elastic buckling solutions was presented. It was shown that for a typical fabricated composite beam, the buckling load factor may drop below unity for moderate strand stresses. Designers should therefore give consideration to the spacing of the attachments of the tendon to the steel beam during the jacking operation.

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