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Determination of the Vlasov foundation parameters −quadratic variation of elasticity modulus− using FE analysis

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Abstract. The objective of this research was to determine the Vlasov soil parameters for quadratically varying elasticity modulus $E_s(z)$ of the compressible soil continuum and discuss the interaction affect between two close plates. Interaction problem carried on for uniformly distributed load carrying plates. Plate region was simulated by Kirchhoff plate theory based (mixed or displacement type) 2D elements and the foundation continuum was simulated by displacement type 2D elements. At the contact region, plate and foundation elements were geometrically coupled with each other. In this study the necessary formulas for the Vlasov parameters were derived when Young's modulus of the soil continuum was varying as a quadratic function of z-coordinate through the depth of the foundation. In the examples, first the elements and the iterative FE algorithm was verified and later the results of quadratic variation of $E_s(z)$ were compared with the previous examples in order to discuss the general behavior. As a final example two plates close to each other resting on elastic foundation were handled to see their interaction influences on the Vlasov foundation parameters. Original examples were solved using both mixed and displacement type plate elements in order to confirm the results.

Key words: Kirchhoff plate; Vlasov foundation; soil parameters; finite elements.

1. Introduction

The analysis of plates on elastic foundation using finite element method is videly used for the analysis of mat foundation Cheung (1978), rigid pavements, etc. The mechanical modeling of structure-foundation interaction problem is mathematically quite complex phenomenon and the response of the subgrade is governed by many factors. The first simplest mechanical model developed by Winkler (1867) which was originally developed for the analysis of railroad tracks. This model was also referred to as the one-parametric model. The first difficulty met in the Winkler foundation was the necessity of determining the modulus of subgrade reaction k. Some pioneering attempts were made to developed empirical relations for determining the value of k such as Biot (1937), Vesic (1961, 1973). Second difficulty was the inconsistent behavior of the Winkler formulation due to the discontinuity of displacements on the boundary of the uniformly loaded area surface since the model was unable to consider the influence of surface displacements which are

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continuing beyond the load region. To overcome this inconsistency some researchers used Boussinesq equations by assuming the soil as a semi-infinite isotropic-homogeneous medium and some others used various two parametric models such as Filonenko-Borodich (1940), Hetényi (1946, 1950), Pasternak (1954), Kerr (1964), Vlasov and Leont'ev (1966). The first parameter can be treated as subgrade reaction and the definition of the second parameter gains the major importance. Vlasov and Leont'ev (1966) proposed a variational method based on an assumed displacement variation with depth in the soil continuum and the soil parameters were found as a function of foundation material constants. A well-documented literature survey on the field of beams resting on elastic foundation has been given by Kerr (1964) and Scott (1981).

Vlasov and Leont'ev (1966) postulated a two-parameter model, using a theoretical approach to represent the soil continuum in order to overcome the difficulty in determining values of k for soil as well as the inconsistent behavior of the Winkler model. Their model considered the neglected shear strain energy and subsequent shear forces on the plate edges due to the soil displacement. Although the model seemed to be eliminating the need to determine experimentally or empirically the values of k, or the shear (second) parameter, which makes the consideration of plate-foundation shear affect. The disadvantage of this model was the introduced parameter γ, whose value must be determined, and no mechanism was provided for computing its value. This parameter was named as mode shape parameter and used to define the decay of the vertical displacement, by means of the mode shape function $\phi(z)$, in the subsoil (see Fig. 1). Exact solutions of this model are rather complex and no mechanism was given for computation of the γ parameter (Harr *et al*. 1969, Kameswara-Rao *et al.* 1971). The relation between the surface displacements and the γ parameter was obtained experimentally by Jones and Xenophontos (1977) but no numerical value was given for γ, Nogami and Lam (1987) developed another two parametric model for slab on elastic foundation, which was limited to plane strain conditions. Finally, Vallabhan and Das (1988) proposed an iterative algorithm for determining γ numerically. The aim was to make the two Vlasov parameters (C_1, C_2) and the γ parameter to be independent from each other and unique for a given beam/slab-elastic foundation interaction. Analytical studies on the field of two-parameter models are

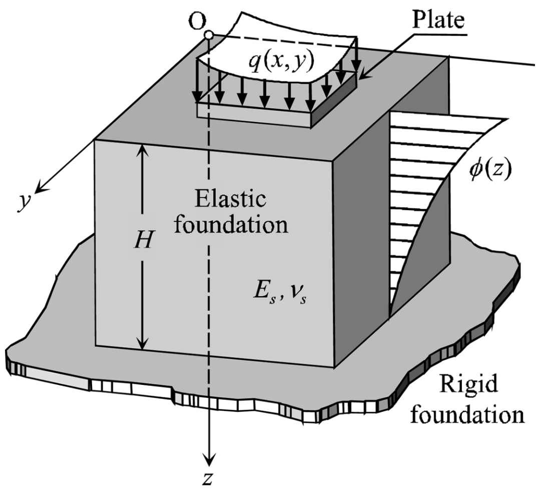


Fig. 1 Plate and foundation continuum

limited due to the complexity of the mathematical formulation and for that reason various numerical methods were preferred. In the literature FE and finite difference studies for determining the Vlasov soil parameters, when Young's modulus of soil was constant (Vallabhan *et al.* 1991, Straughan 1990, Daloğlu 1992, Çelik and Saygun 1999) and linearly varying (Daloğlu 1992), exist. Using the flexibility method Aydoğan *et al.* (1995) considered the variation of foundation elasticity modulus as an exponential function of compressible soil thickness. Besides rectangular/square plates, circular plates-Vlasov foundation interaction studies based on numerical methods exist in the literature, such as, Vallabhan and Das (1991), Buczkowski and Torbacki (2001), Saygun and Çelik (2003).

A constant value for the soil modulus of elasticity (E_s) through the depth of the foundation is quite a special case. For that reason, in this study a quadratic function of z-coordinate through the foundation depth was considered and the necessary Vlasov soil parameters were derived. Vlasov parameters for elasticity modulus of foundation being constant or linearly varying along the depth of the soil exists in the literature (Straughan 1990, Daloğlu 1992, Çelik and Saygun 1999). To verify the finite elements and iterative algorithm proposed by Vallabhan *et al.* (1991) first these problems were solved and quite satisfactory results were obtained. Results of $E_s(z)$ varying as a quadratic function of z-coordinate were quite reasonable when compared with the results and $E_s(z)$ was a linear function of z. For confirmation of results of the original problem two different plate elements were used, one was mixed (MFE) the other one was displacement type Çelik and Saygun (1999) (DFE[1]). Both of the elements were 16 DOF. Explicit form of the plate functional required for the MFE formulation exist in Omurtag *et al.* (1997), when the terms containing the Pasternak foundation parameters were excluded. Formulation of DFE[1] was given by Çelik and Saygun (1999). As an original example, two plates close to each other resting on elastic foundation were handled to see their interaction influences on the Vlasov foundation parameters. Investigation was carried on for weak, semi-stiff and stiff elastic foundations and results were given in tabular form. For this purpose, an extended soil surface around the plate (Ω_s/Ω_p) was considered as well as those under the plate (Ω_p) as shown in Fig. 2.

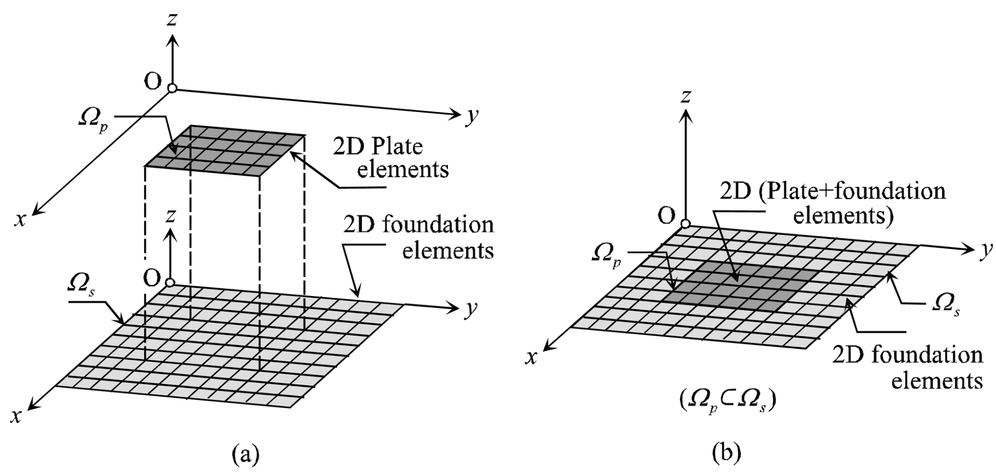


Fig. 2 Finite element simulation, (a) 2D plate and foundation elements, (b) 2D finite element simulation of a plate resting on an elastic foundation continuum

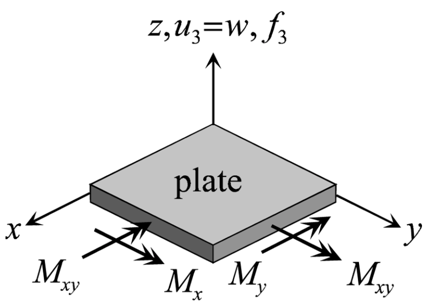


Fig. 3 Positive directions of the components for u_3, M_x, M_y, M_{xy} and f_3 on the plate

2. Field equations and functionals

2.1 Kirchhoff plate

For the sake of simplicity, field equations used for mixed finite elements formulation may be written in the operator form,

$$-\mathbf{D}_f^e \boldsymbol{\sigma}_f - f_3 = 0 \quad \text{and} \quad \boldsymbol{\varepsilon}_f - \mathbf{C}_f \boldsymbol{\sigma}_f = 0 \tag{1}$$

where $\mathbf{D}_f^e = [\partial^2(\ldots)/\partial x^2 \;\; \partial^2(\ldots)/\partial y^2 \;\; 2(\partial(\ldots)/\partial x)(\partial(\ldots)/\partial y)]$ is the equilibrium operator, $\mathbf{D}_f^k = (\mathbf{D}_f^e)^T$ is the kinematic operator, $\boldsymbol{\varepsilon}_f = -z\mathbf{D}_f^k u_3$ is the kinematic relation, u_3 is the vertical displacement vector, $\boldsymbol{\sigma}_f = \{M_x \;\; M_y \;\; M_{xy}\}^T$ is the moment vector, f_3 is the vertical load vector (see Fig. 3) and $\mathbf{C}_f$ is the compliance matrix of a isotropic plate (see Doğruoğlu and Omurtag 2000).

2.2 Vlasov foundation

The displacement vector in the soil continuum is $\bar{\mathbf{u}} = \{\bar{u}_1 \;\; \bar{u}_2 \;\; \bar{u}_3\}^T$. The lateral displacements $\bar{u}_\alpha(x, y, z) \cong 0$ ($\alpha = 1,\ 2$ or $x,\ y$) may be neglected and the vertical displacement in the soil continuum was assumed to be $\bar{u}_3(x, y, z) = u_3(x, y)\phi(z)$, where $u_3(x, y)$ was the deflection of the soil surface and $\phi(z)$ was the mode shape function (see Fig. 1) defining the variation of the vertical displacement $\bar{u}_3(x, y, z)$ in the soil continuum such that $\phi(z=0) = 1$, $\phi(z=H) = 0$. The thickness of the elastic foundation layer (compressible layer thickness) H was supposed to be known. The interaction pressure in the domain of the plate, between the plate and the soil surface at $z = 0$ is,

$$p_3 = C_1 u_3 - 2C_2(u_{3,xx} + u_{3,yy}) \tag{2}$$

where, the two parameters of the Vlasov foundation are,

$$\left.\begin{aligned} C_1 &= \frac{(1-\upsilon_s)}{(1+\upsilon_s)(1-2\upsilon_s)}\int_0^H E_s \phi_{,z}^2 dz \\ 2C_2 &= \int_0^H G_s \phi^2 dz \end{aligned}\right\} \tag{3}$$

where E_s is the elasticity modulus of soil, υ_s is the Poisson’s ratio of soil, $G_s = E_s/[2(1+\upsilon_s)]$ is the

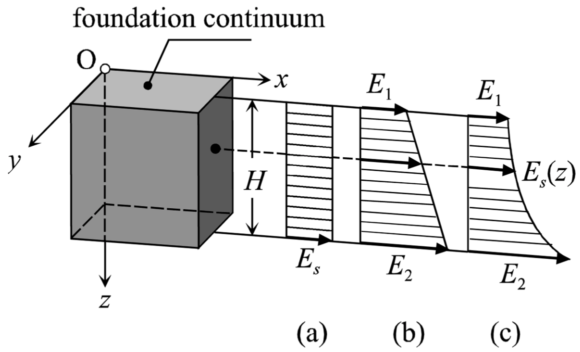


Fig. 4 Elasticity modulus of foundation $E_s(z)$ through the thickness of the soil continuum, (a) constant, (b) linear variation, (c) quadratic variation

shear modulus and the mode function and the mode shape parameter (γ) are,

$$\phi(z) = \frac{\sinh[\gamma(1 - z/H)]}{\sinh(\gamma)} \tag{4}$$

$$\left(\frac{\gamma}{H}\right)^2 = \frac{n}{m} = \frac{(1-2\upsilon_s)}{2(1-\upsilon_s)} \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(u_{3,xx} + u_{3,yy})^2 dxdy}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u_3^2 dxdy} \tag{5}$$

respectively. If the elasticity modulus of the foundation E_s was constant through the thickness of the soil continuum (see Fig. 4) the necessary Vlasov parameters exist in Vallabhan *et al.* (1991). If the elasticity modulus of the foundation was varying linearly through the thickness of the soil continuum (see Fig. 4), such that $E_1 = E_s(z)|_{z=0}$ and $E_2 = E_s(z)|_{z=H}$, elasticity modulus function became $E_s(z) = E_1 + (E_2 - E_1)z/H$ and the Vlasov parameters were given by Vallabhan and Daloğlu (1999).

In this study, a quadratic variation of elasticity modulus through the depth of the compressible soil continuum was considered in the form,

$$E_s(z) = E_1 + (E_2 - E_1)\frac{z^2}{H^2} \tag{6}$$

and in this case the Vlasov parameters given by Eqs. (3), yielded,

$$\left.\begin{aligned} C_1 &= \frac{(1-\upsilon_s)}{(1+\upsilon_s)(1-2\upsilon_s)} \frac{3[E_2 + E_1(2\gamma^2 - 1)]\sinh(2\gamma) + [E_2(2\gamma^2 - 3) + E_1(3 + 4\gamma^2)]2\gamma}{24H\gamma\sinh^2(\gamma)} \\ 2C_2 &= \frac{H}{(1+\upsilon_s)} \frac{3[E_2 + E_1(2\gamma^2 - 1)]\sinh(2\gamma) - [E_2(2\gamma^2 + 3) + E_1(4\gamma^2 - 3)]2\gamma}{48\gamma^3\sinh^2(\gamma)} \end{aligned}\right\} \tag{7}$$

by means of Eqs. (3), (4) and (6).

2.3 Functionals

Functional for MFE the Kirchhoff plate in operator form was,

$$\left.\begin{aligned} I_{Kp} &= -[\mathbf{\Omega}, \boldsymbol{\sigma}_s] - \frac{1}{2}[\mathbf{C}_f\boldsymbol{\sigma}_f, \boldsymbol{\sigma}_f] - [f_3, u_3] \\ &\quad -\langle \boldsymbol{\sigma}_s, (u_3 - u_3)\rangle_\varepsilon + \langle \boldsymbol{\sigma}_f, \hat{\mathbf{\Omega}}\rangle_\varepsilon + \langle \mathbf{\Omega}, (\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_f)\rangle_\sigma - \langle u_3, \hat{\boldsymbol{\sigma}}_3\rangle_\sigma \end{aligned}\right\} \tag{8}$$

where $\boldsymbol{\sigma}_s = \mathbf{D}_m^e\boldsymbol{\sigma}_f = \{(M_{x,x}+M_{xy,y})\ \ (M_{y,y}+M_{xy,x})\}^T$, $\Omega_\alpha = -u_{3,\alpha}$, $(\alpha = x, y)$, $\mathbf{\Omega} = \{\Omega_x\ \ \Omega_y\}^T$ and explicit form of (8) exist in Omurtag *et al*. (1997) if the terms containing the Pasternak foundation parameters were excluded. On the common domain between the plate (Ω_p) and the foundation surface, the vertical equilibrium condition $-\mathbf{D}_f^e\boldsymbol{\sigma}_f - f_3 + p_3 = 0$ must be satisfied, by means of the surface pressure given by (2). Outside the plate domain (Ω_s/Ω_p) the governing foundation field equation is $C_1u_3 - 2C_2(u_{3,xx}+u_{3,yy}) = 0$. The functional for the soil continuum was,

$$I_s = \frac{1}{2}[C_1u_3, u_3] + [C_2\Omega_x, \Omega_x] + [C_2\Omega_y, \Omega_y] + \langle 2C_2\mathbf{\Omega}, (u_3 - \hat{u}_3)\rangle_\varepsilon + \langle 2C_2\hat{\mathbf{\Omega}}, u_3\rangle_\sigma \tag{9}$$

where $\mathbf{\Omega} = \{\Omega_x\ \ \Omega_y\}^T$, $\Omega_\alpha = -u_{3,\alpha}$, $\alpha = x,y$, terms with hat refer to prescribed boundary conditions and $\langle \ldots, \ldots\rangle_\sigma$, $\langle \ldots, \ldots\rangle_\varepsilon$ are dynamic and geometric boundary conditions, respectively. Functionals (8) and (9) were derived by using Gâteaux differential and potential operator requirements (Oden and Reddy 1976) in order to get the appropriate boundary condition terms as stated in Omurtag *et al*. (1997) and Aköz *et al*. (1991).

3. Finite element procedure

In this study, the deflections of the soil surface around the plate were considered as well as those under the plate and for this purpose a limited soil region around the periphery of the plate was considered. Hence, 2D quadrilateral plate and foundation elements were generated independently (see Fig. 2a) and system was constituted as shown in Fig. (2b). The Kirchhoff plate elements had 16 DOF. The nodal unknowns of MFE were u_3 and $\boldsymbol{\sigma}_f$. The nodal unknowns of DFE[1] (Çelik and Saygun 1999) were u_3, $u_{3,x}$, $u_{3,y}$ and $u_{3,xy}$. The soil element had 4 DOF and the nodal unknown was u_3. At the common mesh, plate and foundation elements were geometrically coupled at the nodes only with vertical displacements. Details of the displacement type finite element DFE[1] used for verification of the original problem were given by Çelik and Saygun (1999).

The iterative procedure used in this study for calculating the foundation parameters was as stated by Vallabhan *et al*. (1991). Initially, assuming ${}^{(1)}\gamma = 1$, was used in calculating the initial Vlasov parameters ${}^{(1)}C_1$, ${}^{(1)}C_2$ by means of (7). Using these data the soil surface deflection ${}^{(1)}u_3$ was determined as a first iteration step. The result ${}^{(1)}u_3$ was used in (5) and a new mode shape parameter ${}^{(2)}\gamma$ was calculated. The iteration procedure was continued in this manner until the difference between the two successive values $|{}^{(n-1)}\gamma - {}^{(n)}\gamma| \le \varepsilon$ was less than a small-prescribed value (say ε = 0.001) and the results of the $(n-1)^{\text{th}}$ iteration ${}^{(n-1)}(u_3, C_1, C_2)$ were used as the final values.

4. Numerical examples

In the rest of the text, results of this study were based on mixed and displacement plate elements MFE and DFE[1], respectively.

Example 1: Constant E_s along the depth of the soil continuum

This problem was solved by Straughan (1990) and Daloğlu (1992) using FDM and DFE methods, respectively. The homogenous isotropic material constants for the plate and foundation were $E = 20.685$ GPa, $\nu = 0.2$ and $E_s = 6.894$ MPa, $\nu_s = 0.25$, respectively. Due to the double symmetry, a

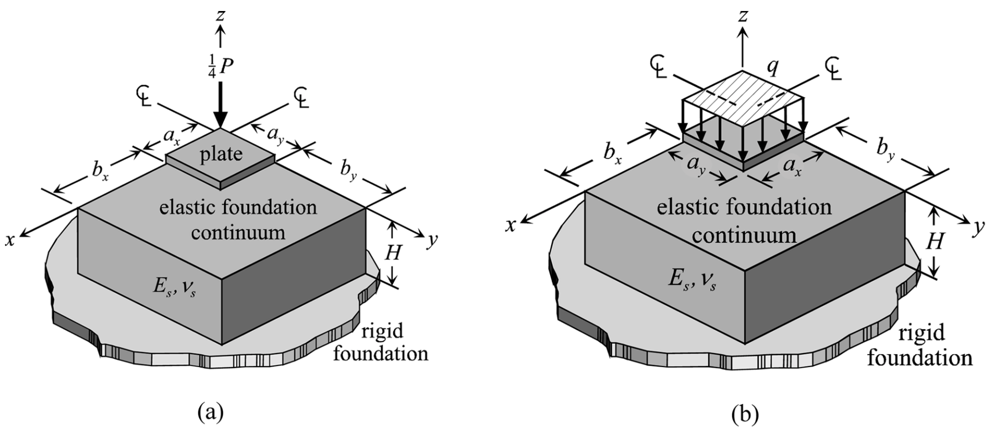


Fig. 5 Quarter of the plate-foundation interaction problem solved in Example 1 and 2. (a) Point load case at the center of the plate, (b) Uniformly distributed load case

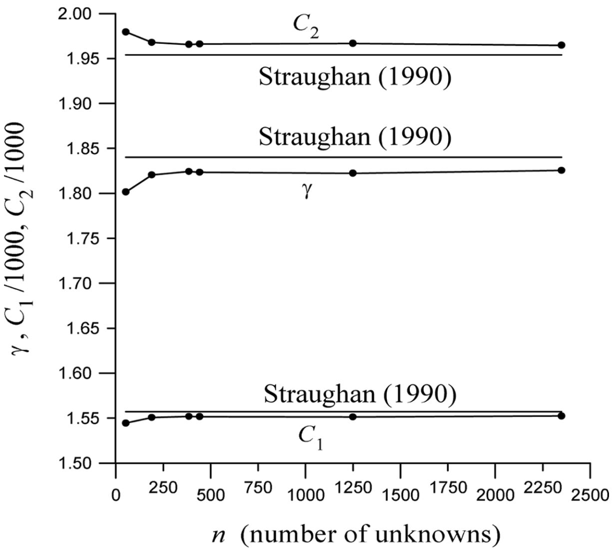


Fig. 6 Convergence of γ, C_1, C_2 compared with Straughan (1990)

Table 1 Comparison of the results obtained in Example 1, for the singular (P) and distributed load (q) cases, by other researches. C_1 [kN/m^3], C_2 [kN/m] are the two foundation parameters, γ is the mode shape parameter and M_x [kNm/m] is the bending moment at the center of the plate. FDM: Straughan (1990), DFE[2]: Daloğlu(1992), MFE and DFE[1]: this study

Load	Researches	C_1 [kN/m^3]	C_2 [kN/m]	γ	M_x [kNm/m]
P	FDM	1557	1954	1.8401	-
	DFE[2]	1557	1954	1.8400	21.13
	DFE[1]	1551	1968	1.8210	34.63
	MFE	1552	1965	1.8255	72.15
q	FDM	1374	2527	0.9017	-
	DFE[2]	1373	2536	0.8845	3.46
	DFE[1]	1376	2520	0.9203	3.61
	MFE	1377	2537	0.9007	3.69

quarter of the whole system domain was analyzed, where the dimensions were $a_x = 4.572$ m, $a_y = 6.096$ m and $b_x = b_y = 9.144$ m (see Fig. 5). Thickness of the plate and the soil stratum were $h = 0.1524$ m and $H = 6.096$ m, respectively. Straughan (1990) and Daloğlu (1992) solved this example under two different load cases were; a point load $P = 133.34$ kN at the center of the plate and uniformly distributed load $q = 23.94$ kN/m^2 all over the plate. A monotonic convergence with respect to Straughan (1990) was observed for parameters γ, C_1, C_2, via MFE plate elements, as given in Fig. 6. Results of this research were compared by Straughan (1990) and Daloğlu (1992) in Table 1 and they were quite satisfactory, except the bending moment results on the midpoint of the plate for the point load case. This is natural, since the bending moment results of FDM and DFE method give average values on an element, whereas, MFE gives nodal values. The bending moment diagram (M_y) along x-(symmetry) axis was as shown in Fig. 7.

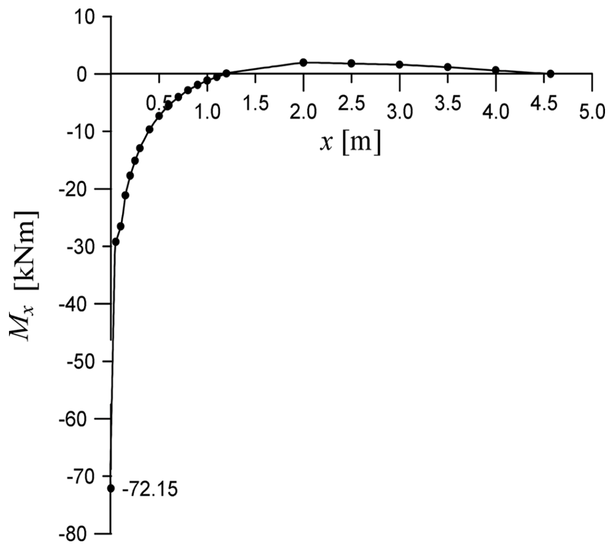


Fig. 7 Bending moment diagram along x axis of the plate solved in example 1 for point load case and H = 15.24 m

Through the analysis it was observed that, if all the material properties (E, υ, E_s, υ_s) and the physical dimensions (a, h, H) of the plate were kept constant, the foundation parameters were not a function of the intensity of the plate load (uniformly distributed or a point load). Hence application of unit loading was enough if the foundation parameters were the only point in question. However if the deflections and bending moment results of the plate were required, one needs to apply the loads with their actual intensity.

Example 2: $E_s(z)$ is a linear and quadratic function of z-coordinate along the depth of the soil continuum

In this problem, variation of Young's modulus of foundation was linear and quadratic function of the depth of the soil stratum (see Fig. 4). Elasticity modulus of the foundation at the top and bottom

Table 2 Comparison of the two foundation parameters C_1 [kN/m^3], C_2 [kN/m] and the mode shape parameter γ results obtained in Ex. 2, for the singular (P) and distributed load (q) cases. DFE^2: Daloğlu (1992), MFE: this study, $\%dif_i = (\mathrm{DFE}^2_i - \mathrm{MFE}_i) \times 100/\mathrm{MFE}_i$, where $i = C_1, C_2, \gamma$

$E_s(z)$	E_2/E_1	Load	Researches	C_1	$\%dif_{C_1}$	C_2	$\%dif_{C_2}$	γ	$\%dif_\gamma$
constant	1	P	DFE^2	1247	–0.06	2330	–0.57	2.5709	0.36
			MFE	1248		2343		2.5617	
		q	DFE^2	937	–0.05	3574	0.28	1.1538	–1.88
			MFE	938		3564		1.1755	
Linear	3	P	DFE^2	1845	0.02	2845	–0.99	2.8370	0.70
			MFE	1845		2873		2.8172	
	5		DFE^2	2398	–0.01	3287	–1.24	3.0274	0.81
			MFE	2398		3328		3.0028	
	10		DFE^2	3697	–0.10	4211	–1.52	3.3669	0.89
			MFE	3700		4275		3.3371	
Quadratic	3	P	MFE	1494	-	2490	-	2.6855	-
	5			1713		2628		2.7806	
	10			2193		2940		2.9601	
Linear	1	q	DFE^2	937	–0.05	3575	0.28	1.1538	–1.88
			MFE	938		3565		1.1757	
	3		DFE^2	1682	0.31	5126	0.43	1.2105	–1.90
			MFE	1676		5104		1.2335	
	5		DFE^2	2413	0.47	6670	0.51	1.2364	–1.92
			MFE	2402		6636		1.2601	
	10		DFE^2	4224	0.65	10520	0.63	1.2658	–1.95
			MFE	4196		10453		1.2905	
Quadratic	3	q	MFE	1382	-	4109	-	1.2365	-
	5			1808		4653		1.2729	
	10			2844		6009		1.3233	

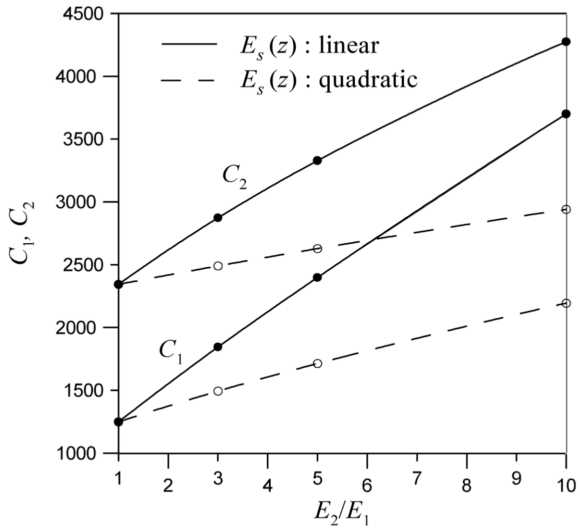


Fig. 8 Soil parameters due to $E_s(z)$ varying as a linear or quadratic function of z-coordinate through the foundation continuum when plate is loaded by a point load and E_2/E_1 = 1, 3, 5, 10

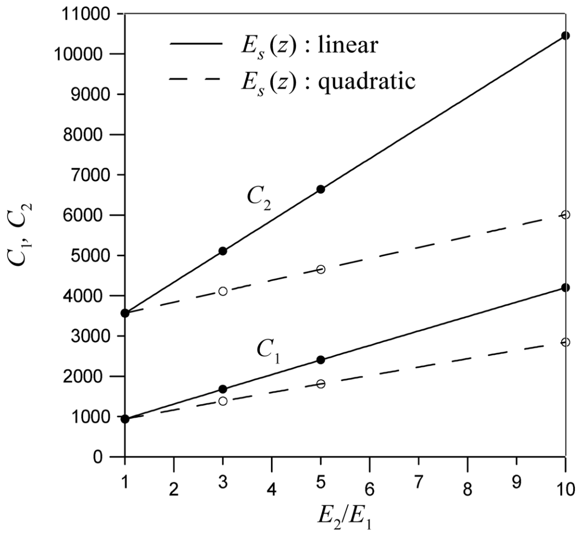


Fig. 9 Soil parameters due to $E_s(z)$ varying as a linear or quadratic function of z-coordinate through the foundation continuum when plate is loaded by a uniformly distributed load and E_2/E_1 = 1, 3, 5, 10

of the soil continuum was $E_1 = E_s|_{z=0} = 6.912$ MPa and $E_2 = E_s|_{z=H}$, respectively, and analysis was carried out for three values of the E_2/E_1 ratio such as 3, 5 and 10. The other material constants for the plate and foundation were, $E = 20.685$ GPa, $\nu = 0.2$ and $\nu_s = 0.25$. A quarter of the whole system domain was analyzed and the dimensions were $a_x = 4.572$ m, $a_y = 6.096$ m and $b_x = b_y = 9.144$ m (see Fig. 5). Thickness of the plate and the soil stratum were $h = 0.1524$ m and $H = 9.144$ m, respectively. Since the only interest was to investigate the foundation parameters, unit loading ($P = 1$ kN or $q = 1$ kN/m^2) was applied on the plate. Numerical results given in Table 2 were obtained from MFE and DFE[1] (the finite element program developed by Çelik and Saygun 1999). Daloğlu (1992) was used for the comparison of the two foundation parameters and the mode shape parameter results when $E_s(z)$ was linear. Referring to the % differences between the two researches one can deduce that results were quite satisfactory. Depending to the E_2/E_1 ratio, variation of the parameters C_1 and C_2 for uniformly distributed and singular load cases were given in Fig. 8 and Fig. 9, respectively. For a constant E_2/E_1 ratio, decrease of the soil parameters in the case of quadratically varying $E_s(z)$ compared to the linearly varying $E_s(z)$ was due to the behavior of the quadratic function of $E_s(z)$.

Example 3: Interaction between two close plates-I

Due to interaction between two close plates and the foundation, a variation on the foundation parameters was expected with some limitations and in this example it was investigated. Since foundations parameters were functions of soil material constants, load type and rigidity of the plate, a large amount of parameters needed to be investigated. First of all, three different foundation types, medium or hard clay, loose or dense sand and gravel, shale, were selected. Soil material constants were collected from Bowles (1988), such that;

Medium clay	E_s = 15: 50 MPa	$\upsilon_s = 0.45$
Hard clay	E_s = 50: 100 MPa	$\upsilon_s = 0.45$
Sand and gravel	E_s = 50: 200 MPa	$\upsilon_s = 0.25$
Shale	E_s = 250: 750 MPa	$\upsilon_s = 0.20$
	E_s = 2500: 5000 MPa	$\upsilon_s = 0.20$

Since the only interest was to investigate the foundation parameters, uniformly distributed unit loading $q = 1$ kN/m^2 was applied on the square plate. In this example singular loading at the mid-span of the plate disregarded, since, results were not sensitive to the extent of the limited soil region around the periphery of the plate region because calculation considerable gradients of surface deflection of the soil continuum took place only close to the point load. Plate material constants were $E = 21$ GPa and $\upsilon = 0.15$. Span of a single plate was $a = 10$ m and thickness was $h/a = 0.01$, 0.02. Plate thickness was changed just to see its influence on the parameters. Three different depth

Table 3 Single square plate (h/a = 0.01) results for Ex. 3. E_s is constant through the soil stratum. DFE[1]: Çelik and Saygun (1999), MFE: Mixed FE formulation in this study

H [m]	E_s [GPa]	υ_s	γ		C_1 [kN/m^3]		C_2 [kN/m]	
			MFE	DFE[1]	MFE	DFE[1]	MFE	DFE[1]
5	15	0.45	0.5387	0.5386	11400	11400	4150	4150
	50	0.45	0.5467	0.5469	38002	38002	13819	13829
	100	0.45	0.5498	0.5502	76007	76008	27626	27624
	50	0.25	0.8325	0.8266	12113	12110	15266	15284
	200	0.25	0.8358	0.8299	48460	48446	61020	61096
	250	0.20	0.8695	0.8612	56170	56149	78913	79048
	750	0.20	0.8703	0.8620	168516	168451	236703	237108
	2500	0.20	0.8705	0.8627	561727	561523	788966	790244
	5000	0.20	0.8706	0.8630	1123458	1123064	1577913	1580373
20	15	0.45	1.3700	1.3607	3006	3002	13850	13886
	50	0.45	1.3742	1.3728	10011	10023	46131	46253
	100	0.45	1.3734	1.3638	20047	20022	92243	92491
	50	0.25	2.1589	2.1440	3709	3695	42095	42293
	200	0.25	2.1605	2.1450	14841	14784	168293	169115
	250	0.20	2.2573	2.2098	17607	17393	212524	215740
	750	0.20	2.2576	2.2104	52825	52187	637507	647095
	2500	0.20	2.2578	2.2113	176089	173994	2124946	2156401
	5000	0.20	2.2578	2.1170	352179	348028	4249859	4312210
40	15	0.45	2.2275	2.1800	1789	1768	21306	21628
	50	0.45	2.2305	2.1831	5969	5897	70952	72023
	250	0.20	4.1091	4.0079	14339	14000	252512	258697
	750	0.20	4.1098	4.0099	43023	42018	757424	775713
	2500	0.20	4.1100	4.0120	143417	140131	2524613	2584389
	5000	0.20	4.1100	4.0128	286837	280310	5049170	5167890

of soil stratum $H = 5$ m, 20 m, 40 m were used. For comparison first we need the single plate results. For this purpose, foundation parameters given in Table 3 and Table 4 are for E_s = constant and in Table 5 and Table 6 are for $E_s(z)$ varying linearly and quadratically. $E_1 = E_s|_{z=0}$ and $E_2 = E_s|_{z=H}$ were the elasticity modulus of the foundation at the top and bottom of the soil continuum, respectively. From these tables it was observed that, within the limits of thin plate theory changing the thickness from $h/a = 0.01$ to $h/a = 0.02$ made immaterial influence on the soil parameters. Hence interaction between two plates were solved only for $h/a = 0.01$. Distance between the two close square plates was $c = \alpha H$ (see Fig. 10a) where $\alpha = 0.1$, 0.5 and 1. It was observed that, as $c \to H$, interaction on the deflections of foundation surface between the two plates becomes negligible. This judgment was verified when the results of Table 3 and Table 7, Table 5 and Table 8 were compared with each other. This conclusion was valid for both constant elasticity modulus and varying elasticity modulus of soil continuum. The necessity of using foundation elements between

Table 4 Single plate results (h/a = 0.02) for Ex. 3. E_s is constant through the soil stratum. DFE[1]: Çelik and Saygun (1999), MFE: Mixed FE formulation in this study

H [m]	E_s [GPa]	υ_s	γ		C_1 [kN/m^3]		C_2 [kN/m]	
			MFE	DFE[1]	MFE	DFE[1]	MFE	DFE[1]
5	15	0.45	0.5263	0.5256	11398	11398	4157	4158
	50	0.45	0.5319	0.5315	37995	37995	13847	13848
	100	0.45	0.5373	0.5371	75995	75995	27674	27674
	50	0.25	0.8186	0.8122	12106	12103	15309	15328
	200	0.25	0.8292	0.8232	48444	48432	61105	61178
	250	0.20	0.8641	0.8558	56156	56135	79001	79136
	750	0.20	0.8680	0.8597	168498	168434	236815	237218
	2500	0.20	0.8697	0.8614	561707	561489	789094	790458
	5000	0.20	0.8702	0.8620	1123437	1123007	1578046	1580730
20	15	0.45	1.3537	1.3430	2999	2995	13913	13954
	50	0.45	1.3656	1.3557	10013	10000	46224	46351
	100	0.45	1.3693	1.3597	20036	20011	92351	92598
	50	0.25	2.1465	2.1393	3697	3683	42259	42464
	200	0.25	2.1568	2.1417	14827	14771	168489	169298
	250	0.20	2.2542	2.2064	17593	17378	212735	215970
	750	0.20	2.2565	2.2086	52801	52163	637731	647466
	2500	0.20	2.2574	2.2097	176073	173926	2125176	2157451
	5000	0.20	2.2576	2.2104	352164	347914	4250090	4313960
40	15	0.45	2.2051	2.1568	1779	1758	21457	21787
	50	0.45	2.2221	2.1747	5956	5885	71140	72214
	250	0.20	4.1029	4.0015	14318	13978	252883	259202
	750	0.20	4.1076	4.0056	43001	41975	757811	776520
	2500	0.20	4.1093	4.0081	143395	14000	2525009	2586847
	5000	0.20	4.1097	4.0096	286815	280098	5049567	5171845

Table 5 Single plate results (h/a = 0.01) for Ex. 3. $E_s(z)$ is varying linearly and quadratically. L: Linear, Q: Quadratic. DFE[1]: Çelik and Saygun (1999), MFE: Mixed FE formulation in this study

H [m]	$E_s(z)$ [GPa]	variation	υ_s	γ		C_1 [kN/m^3]		C_2 [kN/m]	
				MFE	DFE[1]	MFE	DFE[1]	MFE	DFE[1]
5	E_1=15, E_2=50	L	0.45	0.5738	0.5740	24000	23998	6487	6487
		Q		0.5769	0.5770	19609	19609	5058	5058
	E_1=50, E_2=100	L	0.45	0.5665	0.5669	56026	56025	17153	17152
		Q		0.5667	0.5671	49762	49761	15112	15114
	E_1=50, E_2=200	L	0.25	0.8825	0.8804	28127	28134	25880	25894
		Q		0.8910	0.8893	22211	22217	19252	19260
	E_1=250, E_2=750	L	0.20	0.9097	0.9057	105294	105332	115310	115428
		Q		0.9146	0.9109	87089	87121	92504	92587
	E_1=2500, E_2=5000	L	0.20	0.8954	0.8907	808342	808495	971052	972125
		Q		0.8963	0.8916	717463	717603	856924	857837
20	E_1=15, E_2=50	L	0.45	1.4316	1.4203	5565	5568	20583	20664
		Q		1.4382	1.4270	4485	4489	16156	16217
	E_1=50, E_2=100	L	0.45	1.4073	1.3969	13715	13711	55756	55938
		Q		1.4076	1.3972	12176	12172	49417	49575
	E_1=50, E_2=200	L	0.25	2.2370	2.2224	6559	6553	64042	64415
		Q		2.2459	2.2314	5060	5056	48549	48813
	E_1=250, E_2=750	L	0.20	2.3161	2.3010	26303	26258	285033	286654
		Q		2.3189	2.3037	21654	21624	233492	234760
	E_1=2500, E_2=5000	L	0.20	2.2937	2.2790	220060	219511	2487703	2500560
		Q		2.2925	2.2777	196769	196240	2229379	2240510
40	E_1=15, E_2=50	L	0.45	2.2938	2.2463	2832	2819	29826	30376
		Q		2.2979	2.2504	2277	2266	23784	24200
	E_1=250, E_2=750	L	0.20	4.1216	4.0204	17915	17586	311996	321022
		Q		4.1209	4.0200	15259	14948	265911	273011
	E_1=2500, E_2=5000	L	0.20	4.1169	4.0190	161324	158087	2822233	2895796
		Q		4.1161	4.0181	148034	144878	2591652	2656015

the plates comes out in the case of interaction between plates. When foundation was weak, the interaction between two close plates became immaterial if $c > \frac{1}{2}H$ where as in the case of stiff foundation this condition was satisfied if $c \approx H$. The numerical results of this study were also verified by the DFE[1]. Deformation of the foundation surface for a typical problem for the case $0 < c < H$ was given in Fig. 10(b), considering the symmetry of the system with respect to x-axis. In the analysis, as a first iteration step γ was assumed to be 1, and for weak foundations number of iteration steps were between 3 and 5. However in the case of stiff foundations number of iteration steps were 10:12.

Table 6 Single plate results (h/a = 0.02) for Ex. 3. $E_s(z)$ is varying linearly and quadratically. L: Linear, Q: Quadratic. DFE[1]: Çelik and Saygun (1999), MFE: Mixed FE formulation in this study

H [m]	$E_s(z)$ [GPa]	variation	υ_s	γ		C_1 [kN/m^3]		C_2 [kN/m]	
				MFE	DFE[1]	MFE	DFE[1]	MFE	DFE[1]
5	E_1=15, E_2=50	L	0.45	0.5586	0.5579	24029	24030	6504	6504
		Q		0.5633	0.5625	19635	19637	5070	5070.2
	E_1=50, E_2=100	L	0.45	0.5511	0.5508	56066	56066	17194	17195
		Q		0.5510	0.5506	49800	49801	15148	15149
	E_1=50, E_2=200	L	0.25	0.8689	0.8664	28172	28181	25970	25987
		Q		0.8749	0.8728	22262	22269	19329	19339
	E_1=250, E_2=750	L	0.20	0.9047	0.9007	105341	105378	115455	115572
		Q		0.9086	0.9049	87142	87174	92640	92723
	E_1=2500, E_2=5000	L	0.20	0.8947	0.8893	808366	808541	971220	972444
		Q		0.8955	0.8902	717488	717647	857086	858117
20	E_1=15, E_2=50	L	0.45	1.4180	1.4030	5568	5574	20680	20786
		Q		1.4224	1.4098	4490	4494	16241	16309
	E_1=50, E_2=100	L	0.45	1.4007	1.3901	13712	13709	55871	56058
		Q		1.4004	1.3898	12173	12170	49525	49686
	E_1=50, E_2=200	L	0.25	2.2285	2.2143	6556	6550	64260	64623
		Q		2.2352	2.2207	5057	5053	48744	49010
	E_1=250, E_2=750	L	0.20	2.3137	2.2988	26296	26252	285286	286900
		Q		2.3160	2.3010	21647	21607	233727	234991
	E_1=2500, E_2=5000	L	0.20	2.2934	2.2774	220049	219449	2487957	2502023
		Q		2.2922	2.2761	196758	196180	2229622	2241780
40	E_1=15, E_2=50	L	0.45	2.2771	2.2287	2827	2815	30018	30583
		Q		2.2780	2.2293	2273	2262	23956	24385
	E_1=250, E_2=750	L	0.20	4.1166	4.0148	17899	17567	312432	321536
		Q		4.1150	4.0137	15241	14929	266312	273463
	E_1=2500, E_2=5000	L	0.20	4.1163	4.0154	161305	157968	2822665	2898564
		Q		4.1155	4.0144	148013	144756	2592064	2658570

5. Conclusions

In this study, soil continuum was simulated by displacement type 2D foundation elements and plate is simulated by 2D mixed elements and all the results were verified by 2D displacement type plate elements. Vlasov foundation parameters were functions of loading condition, compressible soil thickness, elastic material constants and mode shape parameter. Due to the character of the formulation, mode shape parameter was a function of the soil-surface deformation. Since the soil-surface deformation was not known at the beginning, a value is estimated for the mode shape

Table 7 Interaction of two square plates ($h/a = 0.01$, $c = \alpha H$), for Ex. 3. E_s is constant through the soil stratum. DFE[1]: Çelik and Saygun (1999), MFE: Mixed FE formulation in this study

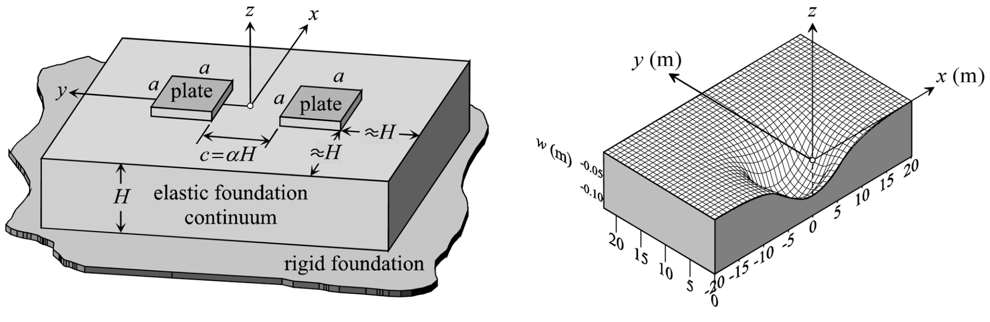
H [m]	E_s [GPa]	υ_s	γ				C_1 [kN/m^3]				C_2 [kN/m]			
			MFE		DFE[1]		MFE		DFE[1]		MFE		DFE[1]	
			$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.1$	$\alpha = 0.5$
5	15	0.45	0.4673	0.5219	0.4638	0.5241	11391	11397	11391	11397	4189	4160	4190	4159
	50	0.45	0.4748	0.5303	0.4691	0.5384	37972	37994	37970	37998	13950	13850	13959	13835
	100	0.45	0.4776	0.5334	0.4713	0.5395	75946	75992	75942	75997	27889	27688	27910	27665
	50	0.25	0.7049	0.7645	0.7033	0.7643	12060	12082	12060	12082	15636	15468	15640	15469
	200	0.25	0.7075	0.7675	0.7041	0.7652	48244	48332	48239	48328	62515	61839	62553	61866
	250	0.20	0.7343	0.7942	0.7294	0.7892	55880	55993	55871	55983	81014	80111	81099	80189
	750	0.20	0.7348	0.7945	0.7289	0.7897	167644	167981	167614	167951	243016	240320	243276	240541
	2500	0.20	0.7350	0.7948	0.7292	0.7901	558817	559941	558719	559843	8100230	801025	810878	801754
	5000	0.20	0.7351	0.7949	0.7293	0.7902	1117635	1119884	1117440	1119690	1620032	1602032	1621738	1603480
20	15	0.45	1.1219	1.3330	1.1486	1.3294	2925	2991	2932	2990	14793	13993	14694	14007
	50	0.45	1.1237	1.3359	1.1479	1.3296	9751	9975	9772	9967	49287	46607	48988	46688
	100	0.45	1.1242	1.3366	1.1483	1.3300	19503	19952	19545	19935	98563	93195	97968	93366
	50	0.25	1.6439	1.9362	1.7901	2.0602	3310	3516	3406	3620	49434	45162	47269	43429
	200	0.25	1.6447	1.9378	1.7907	2.0611	13243	14069	13626	14482	197689	180557	189041	173671
	250	0.20	1.6943	2.0013	1.7464	1.9946	15471	16524	15630	16498	253559	230443	249547	230929
	750	0.20	1.6944	2.0016	1.7467	1.9949	46415	49576	46892	49497	760641	691265	748557	692730
	2500	0.20	1.6945	2.0017	1.7468	1.9950	154717	165256	156309	164998	2535430	2304137	2495100	2308978
	5000	0.20	1.6945	2.0018	1.7468	1.9952	309434	330513	312621	330005	5070841	4608240	4990125	4617775
H	E_s	υ_s	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$
40	15	0.45	2.1863	2.2276	2.1478	2.1830	1771	1789	1754	1769	21585	21305	21850	21608
	50	0.45	2.1893	2.2306	2.1492	2.1839	5907	5969	5847	5899	71881	70949	72800	72004
	250	0.20	3.8856	4.1026	3.7690	4.0010	13591	14317	13206	13976	266554	252906	274452	259129
	750	0.20	3.8863	4.1032	3.7698	4.0018	40781	42957	39625	41937	799525	758602	823195	777238
	2500	0.20	3.8865	4.1034	3.7703	4.0022	135945	143196	132101	139800	2664925	2528538	2743620	2590577
	5000	0.20	3.8865	4.1035	3.7704	4.0022	271894	286396	264211	279604	5329782	5057018	5487056	5181091

parameter and an iterative algorithm was followed through the analysis until it converged to a value. From the results of the first two examples, it was deduced that MFE was quite satisfactory in the sense of engineering precision and convergence was in a monotonic way. In this research it was observed that, if all the material properties (E, υ, E_s, υ_s) and the physical dimensions (a, h, H) of the plate were kept constant, the foundation parameters were not a function of the intensity of the plate load (uniformly distributed or a point load). Only type of loading is important such as point or uniformly distributed. Hence unit loading is enough for the analysis if only the foundation parameters were the point in question. Extent of the limited soil region around the periphery of the plate region was recommended to be proportional to the thickness of the compressible layer thickness H of the soil stratum. When there are two or more plates close to each other and if the distance c between them is less than H, one might expect an interaction on the deformed foundation

Table 8 Interaction of two square plates ($h/a = 0.01$, $c = \alpha H$) for Ex. 3. $E_s(z)$ is varying linearly and quadratically. L: Linear, Q: Quadratic. DFE[1]: Çelik and Saygun (1999), MFE: Mixed FE formulation in this study

H [m]	E_s [GPa]	varia-tion	υ_s	γ MFE		γ DFE[1]		C_1 [kN/m^3] MFE		C_1 [kN/m^3] DFE[1]		C_2 [kN/m] MFE		C_2 [kN/m] DFE[1]	
				$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.1$	$\alpha = 0.5$
5	E_1=15, E_2=50	L	0.45	0.5011	0.5614	0.4952	0.5761	24142	24023	24153	23993	6564	6501	6570	6484
		Q		0.5041	0.5650	0.4988	0.5815	19744	19632	19754	19600	5117	5068	5121	5054
	E_1=50, E_2=100	L	0.45	0.4938	0.5525	0.4867	0.5614	56210	56062	56227	56039	17340	17190	17357	17167
		Q		0.4941	0.5528	0.4871	0.5623	49934	49796	49950	49773	15273	15144	15287	15123
	E_1=50, E_2=200	L	0.25	0.7531	0.8240	0.7552	0.8270	28553	28321	28546	28311	26702	26261	26689	26242
		Q		0.7617	0.8344	0.7653	0.8391	22615	22389	22604	22375	19848	19520	19832	19498
	E_1=250, E_2=750	L	0.20	0.7732	0.8423	0.7719	0.8416	106575	105924	106587	105930	119100	117229	119135	117248
		Q		0.7781	0.8482	0.7774	0.8481	88297	87674	88304	87676	95452	93973	95468	93976
	E_1=2500, E_2=5000	L	0.20	0.7592	0.8250	0.7565	0.8229	813152	810736	813255	810811	1000559	986684	1001117	987139
		Q		0.7602	0.8262	0.7576	0.8242	721909	719666	722000	719731	882230	870291	882639	870660
20	E_1=15, E_2=50	L	0.45	1.1824	1.4056	1.2161	1.4001	5663	5573	5647	5574	22334	20768	22101	20806
		Q		1.1901	1.4134	1.2245	1.4078	4581	4493	4565	4495	17474	16289	17294	16320
	E_1=50, E_2=100	L	0.45	1.1576	1.3769	1.1868	1.3698	13697	13690	13692	13703	60073	56287	59579	56413
		Q		1.1585	1.3774	1.1874	1.3703	12155	13705	12151	12166	53116	49871	52694	49978
	E_1=50, E_2=200	L	0.25	1.7558	2.0565	1.7843	2.0488	6512	6510	6508	6509	77400	68805	76556	69015
		Q		1.7712	2.0711	1.7983	2.0631	5041	5025	5035	5024	58002	51853	57427	52010
	E_1=250, E_2=750	L	0.20	1.7887	2.0987	1.8262	2.0907	25338	25756	25365	25740	347465	309420	342691	310361
		Q		1.7958	2.1032	1.8320	2.0960	20805	21167	20826	21154	281495	252260	277923	252920
	E_1=2500, E_2=5000	L	0.20	1.7542	2.0617	1.7959	2.0542	204864	212231	205652	212010	3004233	2699351	2961502	2706464
		Q		1.7539	2.0605	1.7956	2.0530	182100	189256	182958	189044	2673149	2411717	2636508	2417822
H	E_s	varia-tion	υ_s	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$
40	E_1=15, E_2=50	L	0.45	2.2627	2.2943	2.2248	2.2488	2824	2832	2814	2820	30185	29821	30629	30346
		Q		2.2676	2.2983	2.2308	2.2535	2270	2277	2262	2267	24047	23780	24372	24171
	E_1=250, E_2=750	L	0.20	3.9066	4.1136	3.7944	4.0139	17219	17889	16863	17565	331742	312686	342930	321616
		Q		3.9056	4.1129	3.7933	4.0129	14599	15234	14263	14926	281503	266457	290318	273521
	E_1=2500, E_2=5000	L	0.20	3.8975	4.1086	3.7846	4.0094	154107	161049	150447	157772	2991948	2828334	3086590	2903157
		Q		3.8961	4.1078	3.7830	4.0085	140986	147764	137423	144572	2740422	2597017	2823121	2662460

surface. This distance mainly depends on the material properties of the foundation and loading condition. If plate was uniformly loaded and soil was weak, an interaction was observed when $0 < c < (:\frac{1}{2}H)$ and in the case of stiff foundation this distance was $0 < c < (:H)$. To solve such problems, consideration of foundation elements around the extended region and between the plate elements gave quite satisfactory results.



(a) (b)

Fig. 10 (a) Two square plates close to each other, (b) Deformation of the soil surface with respect to the symmetry axis x

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Notation

a, c	: plate span and the distance between two close plates, respectively
C_1, C_2	: the two Vlasov foundation parameters
$\mathbf{C}_f$	: compliance matrix
$\mathbf{D}_f^e$, $\mathbf{D}_f^k$	: equilibrium and kinematic operators, respectively
E, E_s	: elasticity modulus of plate and soil, respectively
E_1, E_2	: elasticity modulus of the foundation at the top and bottom of the soil continuum, respectively
f_3, P, q	: vertical load, singular and distributed vertical loads on plate, respectively
h, H	: thickness of the plate and compressible layer thickness of the soil, respectively
I_s	: functional of the Vlasov foundation
k	: modulus of subgrade reaction
M_x, M_y, M_{xy}	: bending moments with respect to x and y axis and torsional moment, respectively
$p_3(x, y)$	: vertical surface pressure on foundation
$u_3(x, y)$	: vertical displacement on the middle surface of the plate and on top of the soil continuum, respectively
$\bar{u}_3(x, y, z)$	: vertical displacement in the soil continuum
$\boldsymbol{\varepsilon}_f$	: deformation vector
$\phi(z)$, γ	: mode shape function and mode shape parameter, respectively
υ, υ_s	: Poisson's ratio of plate and foundation, respectively
$\boldsymbol{\Omega}$, Ω_x, Ω_y	: rotation vector and components of rotation with respect to x and y axis, respectively
Ω, Ω_p, Ω_s	: domain, plate domain and foundation domain, respectively
$\boldsymbol{\sigma}_f$	: moment vector

Symbollic representation

$[\ldots,\ldots]$, $\langle\ldots,\ldots\rangle$: inner products for representing the domain and the boundary terms, respectively
$[\ldots,\ldots]_{\sigma}$, $\langle\ldots,\ldots\rangle_{\varepsilon}$: dynamic and geometric boundary conditions, respectively
$(\hat{\ldots})$: prescribed boundary condition term
${}^{(n)}(\ldots)$: nth iteration
$(\ldots)_{,\alpha}$, $(\ldots)_{\alpha\beta}$: partial differentiations $\partial(\ldots)/\partial\alpha$ and $\partial^2(\ldots)/(\partial\alpha\partial\beta)$

Abbreviations

DOF : degrees of freedom
FE, DFE, FDM, MFE : finite elements, displacement type finite elements, finite difference method and mixed finite elements, respectively
2D : two dimensional